Chin. Ann. Math.
25B:2(2004),213–224.

UNCONDITIONAL STABLE DIFFERENCE METHODS WITH INTRINSIC PARALLELISM FOR SEMILINEAR PARABOLIC SYSTEMS OF DIVERGENCE TYPE**

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Abstract

The general finite difference schemes with intrinsic parallelism for the boundary value problem of the semilinear parabolic system of divergence type with bounded measurable coefficients is studied. By the approach of the discrete functional analysis, the existence and uniqueness of the discrete vector solutions of the nonlinear difference system with intrinsic parallelism are proved. Moreover the unconditional stability of the general difference schemes with intrinsic parallelism justified in the sense of the continuous dependence of the discrete vector solution of the difference schemes on the discrete initial data of the original problems in the discrete $W_2^{(2,1)}(Q_\Delta)$ norms. Finally the convergence of the discrete vector solutions of the certain difference schemes with intrinsic parallelism to the unique generalized solution of the original semilinear parabolic problem is proved.

Keywords Difference scheme, Intrinsic parallelism, Parabolic system, Stability, Convergence
2000 MR Subject Classification 65M60, 65M12

§ 1. Introduction

1.1. In [1–10] the finite difference methods with intrinsic parallelism for the boundary value problems of the linear and quasilinear parabolic system of non-divergence type with smooth coefficients are studied, and these general difference schemes having the intrinsic character of parallelism are proved to be stable and convergent unconditionally. In this paper, we shall consider the finite difference methods with intrinsic parallelism for the boundary value problems of the semilinear parabolic system of divergence type with bounded measurable coefficients, and various fundamental behavior of these schemes will be discussed.

This paper is presented as follows. In Section 2 we at first construct general finite difference scheme with intrinsic parallelism for the boundary value problems of the semilinear parabolic systems of divergence type with bounded measurable coefficients. In Section 3 we prove a priori estimates of the discrete vector solution of the general difference scheme with

Manuscript received February 25, 2003. Revised September 15, 2003.
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**Project supported by China "National Key Program for Developing Basic Sciences" (No.G1999032801), the National Natural Science Foundation of China (No.19932010), the Science and Technology Foundation of Chinese Academy of Engineering Physics (No.20020652) and (No.ZX0107).

intrinsic parallelism, so that the existence theorem follows from the fixed point theorem. Then in Section 4 we prove the uniqueness of the discrete vector solution of the general difference scheme with intrinsic parallelism. Moreover, in Section 5 the unconditional stability of the general difference scheme with intrinsic parallelism is justified in the sense of the continuous dependence of the discrete vector solutions on the discrete initial data of the original problems for the semilinear parabolic systems in the discrete H^1 norms. Further, in Section 6 the convergence of the discrete vector solutions to the unique generalized vector solution of the original problem of the semilinear parabolic systems is proved under certain conditions.

§ 2 . Difference Schemes with Intrinsic Parallelism

2.1. Consider the boundary value problems for the semilinear parabolic systems of second order of the form

$$u_t = (A(x,t)u_x)_x + f(x,t,u), \tag{2.1}$$

where $u(x,t) = (u_1(x,t), \cdots, u_m(x,t))$ is the m-dimensional vector unknown function ($m \geq 1$), $u_t = \frac{\partial u}{\partial t}$ and $u_x = \frac{\partial u}{\partial x}$ are the corresponding vector derivatives. The matrix $A(x,t)$ is an $m \times m$ positive definite coefficient matrix, and $f(x,t,u)$ is the m-dimensional vector function. Let us consider in the rectangular domain $Q_T = \{0 \leq x \leq l, 0 \leq t \leq T\}$ with $l > 0$ and $T > 0$, the problem for the systems (2.1) with the boundary value condition

$$u(0,t) = u(l,t) = 0, \tag{2.2}$$

and the initial value condition

$$u(x,0) = \varphi(x), \tag{2.3}$$

where $\varphi(x)$ is a given m-dimensional vector function of variable $x \in [0,l]$.

Suppose that the following conditions are satisfied.

(I) $A(x,t)$ is bounded measurable with respect to $(x,t) \in Q_T$ and Lipschitz continuous with respect to $t \in [0,T]$. There is a constant $\sigma_0 > 0$, such that, for any vector $\xi \in R^m$, and for $(x,t) \in Q_T$,

$$(\xi, A(x,t)\xi) \geq \sigma_0 |\xi|^2.$$

(II) $f(x,t,u)$ is bounded measurable with respect to $(x,t) \in Q_T$ and Lipschitz continuous with respect to $u \in R^m$. Then there is a constant $C > 0$ such that $|f(x,t,u)| \leq |\bar{f}(x,t)| + C|u|$ for $(x,t) \in Q_T$ and $u \in R^m$, where $\bar{f}(x,t) = f(x,t,0)$.

(III) The initial value m-dimensional vector function $\varphi(x) \in H^1[0,l]$ and $\varphi(0) = \varphi(l) = 0$.

2.2. Let us divide the rectangular domain Q_T into small grids by the parallel lines $x = x_j$ $(j = 0, 1, \cdots, J)$ and $t = t^n$ $(n = 0, 1, \cdots, N)$ with $x_j = jh$ and $t^n = n\tau$, where $Jh = l$ and $N\tau = T$, J and N are integers, and h and τ are steplengths of the grids. Denote $Q_j^n = \{x_j < x \leq x_{j+1}, t^n < t \leq t^{n+1}\}$, where $j = 0, 1, \cdots, J-1; n = 0, 1, \cdots, N-1$. Denote $v_\Delta = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ the m-dimensional discrete vector function defined on the discrete rectangular domain $Q_\Delta = \{(x_j, t^n) | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ of the grid points.

Denote

$$\begin{aligned}
v_{j+1}^{n+\lambda_j^n} &= \lambda_j^n v_{j+1}^{n+1} + (1-\lambda_j^n) v_{j+1}^n, \\
v_{j-1}^{n+\mu_j^n} &= \mu_j^n v_{j-1}^{n+1} + (1-\mu_j^n) v_{j-1}^n, \\
\delta v_j^n &= \frac{v_{j+1}^n - v_j^n}{h}, \\
\overset{*}{\delta}{}_A^2 v_j^{n+1} &= \frac{1}{h^2}\big(A_{j+\frac12}^n (v_{j+1}^{n+\lambda_j^n} - v_j^{n+1}) - A_{j-\frac12}^n (v_j^{n+1} - v_{j-1}^{n+\mu_j^n})\big), \\
\delta_A^2 v_j^{n+1} &= \frac{1}{h^2}\big(A_{j+\frac12}^n (v_{j+1}^{n+1} - v_j^{n+1}) - A_{j-\frac12}^n (v_j^{n+1} - v_{j-1}^{n+1})\big).
\end{aligned}$$

In the following the symbol C is a generic positive constant which is independent of h, τ and v_h^τ, and maybe different from line to line.

Let us now construct the finite difference scheme with intrinsic parallelism for the mentioned semilinear parabolic system (2.1)–(2.3) as follows:

$$\frac{v_j^{n+1} - v_j^n}{\tau} = \overset{*}{\delta}{}_A^2 v_j^{n+1} + f_j^{n+1} \qquad (j = 1, 2, \cdots, J-1;\ n = 0, 1, \cdots, N-1); \tag{2.4}$$

$$v_0^n = v_J^n = 0 \qquad (n = 0, 1, \cdots, N), \tag{2.5}$$

$$v_j^0 = \varphi_j \qquad (j = 0, 1, \cdots, J), \tag{2.6}$$

where $\varphi_j = \varphi(x_j)$ $(j = 0, 1, \cdots, J)$, and there are $\varphi_0 = \varphi_J = 0$; and

$$\begin{aligned}
A_{j+\frac12}^n &= \frac{1}{h\tau} \iint_{Q_j^n} A(x,t)\,\omega\Big(\frac{x - x_{j+\frac12}}{h}, \frac{t - t^{n+\frac12}}{\tau}\Big) dxdt, \\
f_j^{n+1} &= \frac{1}{h\tau} \iint_{Q_j^n} f(x, t, \delta^0 v_j^{n+1})\,\omega\Big(\frac{x - x_{j+\frac12}}{h}, \frac{t - t^{n+\frac12}}{\tau}\Big) dxdt,
\end{aligned}$$

where $\omega(x,t) \in C_0^\infty(R^2),\ \omega(x,t) \ge 0,\ \operatorname{supp}\omega \subset B_{\frac12} \equiv \{|x| < \frac12,\ |t| < \frac12\}$ and

$$\iint_{R^2} \omega(x,t) dxdt = 1,$$

and $x_{j+\frac12} = (j + \frac12)h$, $t^{n+\frac12} = (n + \frac12)\tau$. If $\omega(x,t) = \chi_{B_{\frac12}}$, which is the character function of $B_{\frac12}$, the results in this paper also hold.

Define the approximations $\delta^0 v_j^{n+1}$ in the following form

$$\delta^0 v_j^{n+1} = \lambda_j^n \alpha_{1j}^n v_{j+1}^{n+1} + \alpha_{2j}^n v_j^{n+1} + \mu_j^n \alpha_{3j}^n v_{j-1}^{n+1} + \alpha_{4j}^n v_{j+1}^n + \alpha_{5j}^n v_j^n + \alpha_{6j}^n v_{j-1}^n.$$

We assume

(IV) $\frac{\tau}{h^2} \le \Lambda$ with Λ being any fixed positive constant as $\tau, h \to 0$; moreover, for all $1 \le j \le J-1$ and $0 \le n \le N-1$, there hold $0 \le \lambda_j^n, \nu_j^n \le 1$, and

$$\begin{gathered}
\lambda_j^n \alpha_{1j}^n + \alpha_{2j}^n + \mu_j^n \alpha_{3j}^n + \alpha_{4j}^n + \alpha_{5j}^n + \alpha_{6j}^n = 1, \\
|\lambda_j^n \alpha_{1j}^n| + |\alpha_{2j}^n| + |\mu_j^n \alpha_{3j}^n| + |\alpha_{4j}^n| + |\alpha_{5j}^n| + |\alpha_{6j}^n| \le C.
\end{gathered}$$

2.3. For a discrete function $u_\Delta = u_h^\tau = \{u_j^n | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ with $u_0^n = u_J^n = 0$, define the discrete norms as follows:

$$\|u_h^n\|_\infty = \max_{0\le j\le J} |u_j^n|, \qquad \|u_h^n\|_2^2 = \sum_{j=1}^{J-1} |u_j^n|^2 h,$$

$$\|\delta u_h^n\|_2^2 = \sum_{j=0}^{J-1} |\delta u_j^n|^2 h, \qquad \|\delta u_h^n\|_A^2 = \sum_{j=0}^{J-1} A_{j+\frac12}^n |\delta u_j^n|^2 h,$$

$$\|\overset{*}{\delta}{}_A^2 v_h^{n+1}\|_2^2 = \sum_{j=1}^{J-1} |\overset{*}{\delta}{}_A^2 v_j^{n+1}|^2 h, \qquad \Big\|\frac{v_h^{n+1} - v_h^n}{\tau}\Big\|_2^2 = \sum_{j=1}^{J-1} \Big|\frac{v_j^{n+1} - v_j^n}{\tau}\Big|^2 h.$$

The following lemmas will be useful later (see [10, Chapter 1]).

Lemma 2.1. (Discrete Green Formula) *Let u_j and v_j be discrete functions on $\{x_j | j = 0, 1, \cdots, J\}$. Then*

$$\sum_{j=0}^{J-1} u_j(v_{j+1} - v_j) = -\sum_{j=1}^{J-1} (u_j - u_{j-1}) v_j - u_0 v_0 + u_{J-1} v_J.$$

Lemma 2.2. (Discrete Gronwall Inequality) *Let $w^n \ge 0$ be a discrete function on $\{t^n | n = 0, 1, \cdots, N\}$ satisfying*

$$w^{n+1} - w^n \le B\tau(w^{n+1} + w^n) + C_n\tau, \qquad n = 0, 1, \cdots, N-1,$$

where B and C_n are nonnegative constants. Then

$$w^n \le \Big(w^0 + \sum_{k=1}^{N} C_k\tau\Big) e^{4BT}, \qquad n = 0, 1, \cdots, N,$$

where τ is small such that $4B\tau \le \frac{N-1}{N}$.

Lemma 2.3. (Interpolation Formulas) *For any discrete function*

$$u_h = \{u_j | j = 0, 1, \cdots, J\}$$

with $Jh = l$ and $u_0 = u_J = 0$, we have

$$\|u_h\|_2 \le l \|\delta u_h\|_2, \qquad \|u_h\|_\infty \le \|\delta u_h\|_2^{\frac12} \|u_h\|_2^{\frac12}.$$

§ 3. A Priori Estimate and Existence

3.1. We are going now to prove the existence of the discrete vector solutions for the finite difference system (2.4)–(2.6). Let us now at first turn to the a priori estimates of these solutions.

Making the scalar product of the vector $(v_j^{n+1} - v_j^n)h$ with the vector finite difference equation (2.4) and summing up the resulting products for $j = 1, 2, \cdots, J-1$, we have

$$\tau \sum_{j=1}^{J-1} \Big|\frac{v_j^{n+1} - v_j^n}{\tau}\Big|^2 h = \tau \sum_{j=1}^{J-1} \Big(\frac{v_j^{n+1} - v_j^n}{\tau}, \overset{*}{\delta}{}_A^2 v_j^{n+1} + f_j^{n+1}\Big) h. \tag{3.1}$$

Since

$$\begin{aligned}
&\tau \sum_{j=1}^{J-1} \Big(\frac{v_j^{n+1} - v_j^n}{\tau}, \overset{*}{\delta}{}_A^2 v_j^{n+1} \Big) h \\
&= -\frac{1}{2} \|\delta v_h^{n+1}\|_A^2 + \frac{1}{2} \|\delta v_h^n\|_A^2 - \frac{1}{2h} \sum_{j=0}^{J-1} A_{j+\frac{1}{2}}^n [|v_{j+1}^{n+1} - v_{j+1}^n|^2 + |v_j^{n+1} - v_j^n|^2] \\
&\quad - \frac{1}{h} \sum_{j=0}^{J-1} A_{j+\frac{1}{2}}^n (1 - \mu_{j+1}^n - \lambda_j^n)(v_{j+1}^{n+1} - v_{j+1}^n, v_j^{n+1} - v_j^n) \\
&\quad + \frac{1}{2} \sum_{j=0}^{J-1} (A_{j+\frac{1}{2}}^n - A_{j+\frac{1}{2}}^{n-1}) |\delta v_j^n|^2 h,
\end{aligned}$$

we have

$$\begin{aligned}
&\|\delta v_h^{n+1}\|_A^2 - \|\delta v_h^n\|_A^2 + 2\tau \sum_{j=1}^{J-1} \Big| \frac{v_j^{n+1} - v_j^n}{\tau} \Big|^2 h \\
&\quad + \frac{1}{h} \sum_{j=0}^{J-1} (1 - |\tau_{j+\frac{1}{2}}^n|) A_{j+\frac{1}{2}}^n (|v_{j+1}^{n+1} - v_{j+1}^n|^2 + |v_j^{n+1} - v_j^n|^2) \\
&\quad + \frac{1}{h} \sum_{j=0}^{J-1} |\tau_{j+\frac{1}{2}}^n| A_{j+\frac{1}{2}}^n |v_{j+1}^{n+1} - v_{j+1}^n + (\operatorname{sgn} \tau_{j+\frac{1}{2}}^n)(v_j^{n+1} - v_j^n)|^2 \\
&= 2\tau \sum_{j=0}^{J-1} \Big(\frac{v_j^{n+1} - v_j^n}{\tau}, f_j^{n+1} \Big) h + \sum_{j=0}^{J-1} (A_{j+\frac{1}{2}}^n - A_{j+\frac{1}{2}}^{n-1}) |\delta v_j^n|^2 h,
\end{aligned}$$

where $\tau_{j+\frac{1}{2}}^n = 1 - \mu_{j+1}^n - \lambda_j^n$; $\operatorname{sgn} \tau_{j+\frac{1}{2}}^n = 1$ if $\tau_{j+\frac{1}{2}}^n \geq 0$, $\operatorname{sgn} \tau_{j+\frac{1}{2}}^n = -1$ if $\tau_{j+\frac{1}{2}}^n < 0$. Then

$$\begin{aligned}
&\|\delta v_h^{n+1}\|_A^2 - \|\delta v_h^n\|_A^2 + 2\tau \sum_{j=1}^{J-1} \Big| \frac{v_j^{n+1} - v_j^n}{\tau} \Big|^2 h \\
&\leq 2\tau \Big| \sum_{j=1}^{J-1} \Big(\frac{v_j^{n+1} - v_j^n}{\tau}, f_j^{n+1} \Big) h \Big| + \sum_{j=0}^{J-1} |A_{j+\frac{1}{2}}^n - A_{j+\frac{1}{2}}^{n-1}| |\delta v_j^n|^2 h. \qquad (3.2)
\end{aligned}$$

For the first term at the right hand side of (3.2), we apply the Cauchy inequality to obtaining

$$\begin{aligned}
\Big| \sum_{j=1}^{J-1} \Big(\frac{v_j^{n+1} - v_j^n}{\tau}, f_j^{n+1} \Big) h \Big| &\leq \sum_{j=1}^{J-1} \Big| \frac{v_j^{n+1} - v_j^n}{\tau} \Big| |f_j^{n+1}| h \\
&\leq \frac{1}{4} \sum_{j=1}^{J-1} \Big| \frac{v_j^{n+1} - v_j^n}{\tau} \Big|^2 h + \sum_{j=1}^{J-1} |f_j^{n+1}|^2 h, \qquad (3.3)
\end{aligned}$$

and for the second term there holds

$$\sum_{j=0}^{J-1} |A_{j+\frac{1}{2}}^n - A_{j+\frac{1}{2}}^{n-1}| |\delta v_j^n|^2 h \leq C\tau \|\delta v_h^n\|_A^2. \qquad (3.4)$$

By the assumption (II) and the definition of f_j^{n+1}, we have

$$\sum_{j=1}^{J-1} |f_j^{n+1}|^2 h \le C\Big(1 + \|v_h^n\|_2^2 + \|v_h^{n+1}\|_2^2 + \|\delta v_h^n\|_2^2 + \|\delta v_h^{n+1}\|_2^2 + \tau\Lambda\Big\|\frac{v_h^{n+1} - v_h^n}{\tau}\Big\|^2\Big). \quad (3.5)$$

By combining (3.2)–(3.5) we obtain

$$\begin{aligned}&\|\delta v_h^{n+1}\|_A^2 - \|\delta v_h^n\|_A^2 + \frac{3\tau}{2}\Big\|\frac{v_h^{n+1} - v_h^n}{\tau}\Big\|_2^2 \\ &\le C\tau\Big(1 + \|\delta v_h^{n+1}\|_2^2 + \|\delta v_h^n\|_2^2 + \|v_h^{n+1}\|_2^2 + \|v_h^n\|_2^2 + \tau\Lambda\Big\|\frac{v_h^{n+1} - v_h^n}{\tau}\Big\|^2\Big).\end{aligned}$$

By using Lemma 2.3, and taking $C\tau\Lambda \le \frac{1}{2}$, we get

$$\|\delta v_h^{n+1}\|_A^2 - \|\delta v_h^n\|_A^2 + \tau\Big\|\frac{v_h^{n+1} - v_h^n}{\tau}\Big\|_2^2 \le C\tau(1 + \|\delta v_h^{n+1}\|_A^2 + \|\delta v_h^n\|_A^2). \quad (3.6)$$

From the above inequality and Lemma 2.2, we have the estimates

$$\max_{n=0,1,\cdots,N} \|\delta v_h^n\|_A \le C(\|\delta\varphi_h\|_A + 1), \quad \max_{n=0,1,\cdots,N} (\|\delta v_h^n\|_2, \|v_h^n\|_2) \le C(\|\delta\varphi_h\|_2 + 1), \quad (3.7)$$

and also have

$$\Big(\sum_{n=0}^{N-1} \| \overset{*}{\delta}{}_A^2 v_h^{n+1}\Big\|_2^2\tau\Big)^{\frac{1}{2}}, \quad \Big(\sum_{n=0}^{N-1} \|\frac{v_h^{n+1} - v_h^n}{\tau}\Big\|_2^2\tau\Big)^{\frac{1}{2}} \le C. \quad (3.8)$$

It follows that, by using the estimates (3.7) and Lemma 2.3,

$$\max_{n=0,1,\cdots,N-1} \|v_h^{n+1}\|_\infty \le C. \quad (3.9)$$

3.2. By the Brouwer's fixed point theorem and the estimate above, we can obtain the existence of the discrete solution $v_\Delta = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ for the difference scheme (2.4)–(2.6).

Theorem 3.1. *Assume that the conditions* (I)–(IV) *hold, and there is a positive constant* τ_0 *depending only on the given data and* Λ *such that* $\tau \le \tau_0$. *Then the general finite difference scheme* (2.4)–(2.6) *with intrinsic parallelism corresponding to the original problem* (2.1)–(2.3) *has at least one discrete solution* $v_\Delta = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$.

§ 4. Uniqueness

4.1. Given the values $\{v_j^n | j = 0, 1, \cdots, J\}$ of the difference scheme (2.4)–(2.6) on the n-th layer. Let $\{v_j^{n+1} | j = 0, 1, \cdots, J\}$ and $\{\bar{v}_j^{n+1} | j = 0, 1, \cdots, J\}$ be the two solutions of the difference scheme (2.4)–(2.6) on the $(n+1)$-th layer, i.e.,

$$\begin{aligned}\frac{v_j^{n+1} - v_j^n}{\tau} &= \overset{*}{\delta}{}_A^2 v_j^{n+1} + f_j^{n+1} \qquad (j = 1, 2, \cdots, J-1), \\ v_0^{n+1} &= v_J^{n+1} = 0, \\ \frac{\bar{v}_j^{n+1} - v_j^n}{\tau} &= \overset{*}{\delta}{}_A^2 \bar{v}_j^{n+1} + \bar{f}_j^{n+1} \qquad (j = 1, 2, \cdots, J-1), \\ \bar{v}_0^{n+1} &= \bar{v}_J^{n+1} = 0,\end{aligned}$$

where $\bar{f}_j^{n+1}$ $(j = 1, 2, \cdots, J-1)$ are obtained from f_j^{n+1} $(j = 1, 2, \cdots, J-1)$ respectively by replacing v_j^{n+1} $(j = 0, 1, \cdots, J)$ with the corresponding $\bar{v}_j^{n+1}$ $(j = 0, 1, \cdots, J)$. Then the difference $w_j \equiv v_j^{n+1} - \bar{v}_j^{n+1}$ satisfies

$$w_j = \tau \overset{*}{\delta}{}_A^2 w_j + \tau R_j^n \qquad (j = 1, 2, \cdots, J-1), \tag{4.1}$$

$$w_0 = w_J = 0, \tag{4.2}$$

where

$$\overset{*}{\delta}{}^2 w_j = \overset{*}{\delta}{}_A^2 v_j^{n+1} - \overset{*}{\delta}{}_A^2 \bar{v}_j^{n+1} = \frac{1}{h^2}(A_{j+\frac{1}{2}}^n(\lambda_j^n w_{j+1} - w_j) - A_{j-\frac{1}{2}}^n(w_j - \mu_j^n w_{j-1}),$$
$$R_j^n = f_j^{n+1} - \bar{f}_j^{n+1}.$$

4.2. Now making the scalar product of the vectors $w_j h$ with the vector equation (4.1) and summing up the resulting products for $j = 1, 2, \cdots, J-1$, and proceeding the similar calculation as that in Section 3, we have

$$\tau\|\delta w_h\|_A^2 + 2\sum_{j=1}^{J-1}|w_j|^2 h + \frac{\tau}{h}\sum_{j=0}^{J-1}(1 - |\tau_{j+\frac{1}{2}}^n|)A_{j+\frac{1}{2}}^n(w_{j+1}^2 + w_j^2)$$
$$+ \frac{\tau}{h}\sum_{j=0}^{J-1}|\tau_{j+\frac{1}{2}}^n|A_{j+\frac{1}{2}}^n(w_{j+1} + (\operatorname{sgn}\tau_{j+\frac{1}{2}}^n)w_j)^2 = 2\tau\sum_{j=1}^{J-1}(w_j, R_j^n)h.$$

It follows that

$$\tau\|\delta w_h\|_A^2 + \|w_h\|_2^2 \le C\tau^2\sum_{j=1}^{J-1}|R_j^n|^2 h. \tag{4.3}$$

By the assumption (II) and the definitions of f_j^{n+1} and $\bar{f}_h^{n+1}$, we can conclude that

$$\|f_h^{n+1} - \bar{f}_h^{n+1}\|_2^2 \le C\|w_h\|_2^2.$$

So we have

$$\sum_{j=1}^{J-1}|R_j^n|^2 h \le C\|w_h\|_2^2. \tag{4.4}$$

By combining (4.3) with (4.4) we get

$$\tau\|\delta w_h\|_2^2 + \|w_h\|_2^2 \le C\tau^2\|w_h\|_2^2.$$

Choosing $C\tau^2 < 1$, we obtain $w_h \equiv 0$, i.e., $v_h^{n+1} = \bar{v}_h^{n+1}$. The uniqueness of the discrete vector solution for the difference scheme (2.4)–(2.6) is proved.

Theorem 4.1. *Suppose that the conditions* (I)–(IV) *are satisfied. As the meshstep* $\tau \le \tau_0$, *where* τ_0 *is a constant depending only on the given data and* Λ, *the discrete solution* $v_\Delta = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ *of the difference scheme* (2.4)–(2.6) *is unique.*

§ 5. Stability

5.1. Now we turn to consider the theorem of unconditional stability of the general finite difference scheme (2.4)–(2.6) with intrinsic parallelism.

Suppose that the discrete vector function $\tilde{v}_\Delta = \{\tilde{v}_j^n | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ satisfies the finite difference system

$$\frac{\tilde{v}_j^{n+1} - \tilde{v}_j^n}{\tau} = \overset{*}{\delta}{}_A^2 \tilde{v}_j^{n+1} + \tilde{f}_j^{n+1} \qquad (j = 1, 2, \cdots, J-1; n = 0, 1, \cdots, N-1), \tag{5.1}$$

$$\tilde{v}_0^n = \tilde{v}_J^n = 0 \qquad (n = 0, 1, \cdots, N), \tag{5.2}$$

$$\tilde{v}_j^0 = \widetilde{\varphi}_j \qquad (j = 0, 1, \cdots, J), \tag{5.3}$$

where

$$\tilde{f}_j^{n+1} = \frac{1}{h\tau} \iint_{Q_j^n} f(x, t, \delta^0 \tilde{v}_j^{n+1}) \omega\Big(\frac{x - x_{j+\frac{1}{2}}}{h}, \frac{t - t^{n+\frac{1}{2}}}{\tau}\Big) dxdt.$$

The m-dimensional vector function $\widetilde{\varphi}(x)$ is different from $\varphi(x)$ by some errors. The m-dimensional vector function $\widetilde{\varphi}(x)$ also satisfies the condition (III).

Then the difference discrete vector function

$$w_\Delta = v_\Delta - \tilde{v}_\Delta = \{w_j^n = v_j^n - \tilde{v}_j^n | j = 0, 1, \cdots, J;\ n = 0, 1, \cdots, N\}$$

satisfies the system

$$\frac{w_j^{n+1} - w_j^n}{\tau} = \overset{*}{\delta}{}_A^2 w_j^{n+1} + (f_j^{n+1} - \tilde{f}_j^{n+1}), \tag{5.4}$$

$$w_0^n = w_J^n = 0 \qquad (n = 0, 1, \cdots, N), \tag{5.5}$$

$$w_j^0 = \varphi_j - \widetilde{\varphi}_j \qquad (j = 0, 1, \cdots, J). \tag{5.6}$$

The vector equation (5.4) can be rewritten as the following form

$$\frac{w_j^{n+1} - w_j^n}{\tau} = \overset{*}{\delta}{}_A^2 w_j^{n+1} + (f_u)_j^{n+1} \delta^0 w_j^{n+1}, \tag{5.7}$$

where

$$(f_u)_j^{n+1} = \frac{f(x_j, t^{n+1}, \delta^0 v_j^{n+1}) - f(x_j, t^{n+1}, \delta^0 \tilde{v}_j^{n+1})}{\delta^0 w_j^{n+1}}$$

for $\delta^0 w_j^{n+1} \neq 0$, and $(f_u)_j^{n+1} = 0$ for $\delta^0 w_j^{n+1} = 0$.

5.2. Making the scalar product of the vector $\frac{w_j^{n+1} - w_j^n}{\tau} h\tau$ and the vector equation (5.7), and then summing up the resulting products for $j = 1, 2, \cdots, J-1$, we have

$$\tau \sum_{j=1}^{J-1} \Big| \frac{w_j^{n+1} - w_j^n}{\tau} \Big|^2 h$$
$$= \tau \sum_{j=1}^{J-1} \Big(\frac{w_j^{n+1} - w_j^n}{\tau}, \overset{*}{\delta}{}_A^2 w_j^{n+1} \Big) h + \tau \sum_{j=1}^{J-1} \Big(\frac{w_j^{n+1} - w_j^n}{\tau}, (f_u)_j^{n+1} \delta^0 w_j^{n+1} \Big) h.$$

By the same argument as that in Section 3, we have the following inequality

$$\|\delta w_h^{n+1}\|_A^2 - \|\delta w_h^n\|_A^2 + 2\tau \sum_{j=1}^{J-1} \Big|\frac{w_j^{n+1} - w_j^n}{\tau}\Big|^2 h$$
$$\leq 2\tau \Big| \sum_{j=1}^{J-1} \Big(\frac{w_j^{n+1} - w_j^n}{\tau}, (f_u)_j^{n+1} \delta^0 w_j^{n+1}\Big) h\Big|.$$

The above inequality can be reduced to the following inequality

$$\|\delta w_h^{n+1}\|_A^2 - \|\delta w_h^n\|_A^2 + \tau \Big\|\frac{w_h^{n+1} - w_h^n}{\tau}\Big\|_2^2 \leq \tau \sum_{j=1}^{J-1} |(f_u)_j^{n+1} \delta^0 w_j^{n+1}|^2 h \leq C\tau \|\delta w_h^{n+1}\|_A^2.$$

Then, by Lemma 2.2(ii) we obtain, for $n = 0, 1, \cdots, N-1$,

$$\|\delta w_h^{n+1}\|_A^2 + \sum_{k=0}^{n} \Big\|\frac{w_h^{k+1} - w_h^k}{\tau}\Big\|_2^2 \tau \leq C\|\delta w_h^0\|_A^2, \tag{5.8}$$

which gives

$$\|\delta w_h^{n+1}\|_2^2 + \sum_{k=0}^{n} \|\overset{*}{\delta}{}_A^2 w_h^{k+1}\|_2^2 \tau \leq C\|\delta w_h^0\|_2^2. \tag{5.9}$$

5.3. This shows that the discrete solution v_Δ of the finite difference scheme (2.4)–(2.6) in the discrete functional space $W_2^{(2,1)}(Q_\Delta)$ is continuously dependent on the discrete initial vector function $\varphi(x)$ in the discrete functional space of the form H^1. We have proved the following stability theorem.

Theorem 5.1. *Under the conditions of Theorem* 3.1*, the following estimates hold for the difference vector function* $w_\Delta = v_\Delta - \tilde{v}_\Delta = \{w_j^n = v_j^n - \tilde{v}_j^n | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$,

$$\|v_\Delta - \tilde{v}_\Delta\|^2_{W_2^{(2,1)}(Q_\Delta)} \leq C\|\varphi_h - \widetilde{\varphi}_h\|^2_{H_h^1},$$

where C *is a constant independent of steplengths* h *and* τ*; and*

$$\|\varphi_h\|^2_{H_h^1} = \|\varphi_h\|_2^2 + \|\delta\varphi_h\|_2^2,$$
$$\|w_\Delta\|^2_{W_2^{(2,1)}(Q_\Delta)} \equiv \max_{n=0,1,\cdots,N} \|w_h^n\|^2_{H_h^1} + \sum_{n=0}^{N-1} \|\overset{*}{\delta}{}_A^2 w_h^{n+1}\|_2^2 \tau + \sum_{n=0}^{N-1} \Big\|\frac{w_h^{n+1} - w_h^n}{\tau}\Big\|_2^2 \tau.$$

§ 6 . Convergence

6.1. In this section we will discuss the unconditional convergence of the finite difference scheme (2.4)–(2.6) with intrinsic parallelism on the basis of the obtained estimates and the convergence properties of the discrete solutions $v_\Delta = v_h^\tau = \{v_j^n | j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$.

Lemma 6.1. *For the discrete solution of the difference scheme* (2.4)–(2.6)*, there are estimates*

$$\max_{n=0,1,\cdots,N} |v_j^n - v_{j'}^n| \leq C|x_j - x_{j'}|^{\frac{1}{2}}, \qquad 0 \leq j, j' \leq J; \tag{6.1}$$

$$\max_{j=0,1,\cdots,J} |v_j^n - v_j^{n'}| \leq C|t^n - t^{n'}|^{\frac{1}{4}}, \qquad 0 \leq n, n' \leq N, \tag{6.2}$$

where m and s are integers satisfying $0 \le m \le J-1, 0 \le s \le N$; and the constants C are independent of h, τ, m, s and v_h^τ.

Proof. It is easy to see that

$$|v_j^n - v_{j'}^n| \le \|\delta v_h^n\|_2 |x_j - x_{j'}|^{\frac{1}{2}},$$

and then (6.1) follows from the estimate (3.7). By Lemma 2.3 and (3.7)–(3.8), there hold

$$\|v_h^n - v_h^{n'}\|_\infty \le \|\delta(v_h^n - v_h^{n'})\|_2^{\frac{1}{2}} \|v_h^n - v_h^{n'}\|_2^{\frac{1}{2}} \le C\|v_h^n - v_h^{n'}\|_2^{\frac{1}{2}},$$

$$\|v_h^n - v_h^{n'}\|_2 \le \Big(\sum_{k=0}^{N-1} \Big\| \frac{v_h^{k+1} - v_h^k}{\tau} \Big\|_2^2 \tau \Big)^{\frac{1}{2}} |t^n - t^{n'}|^{\frac{1}{2}} \le C|t^n - t^{n'}|^{\frac{1}{2}},$$

so the above two inequalities yield (6.2). The proof of Lemma 6.1 is completed.

6.2. Let us define the piecewise constant functions

$$v_h^\tau(x,t) = v_j^{n+1}, \quad v_{xh}^\tau(x,t) = \delta v_j^{n+1}, \quad v_{th}^\tau(x,t) = \frac{v_j^{n+1} - v_j^n}{\tau}$$

for $(x,t) \in Q_j^n$ $(j = 0, 1, \cdots, J-1;\ n = 0, 1, \cdots, N-1)$.

Lemma 6.2. *Assume that the conditions* (I)–(IV) *hold and $\frac{\tau}{h^2} \le \Lambda$ for any given constant $\Lambda > 0$. When $h \to 0, \tau \to 0$ (for some subsequence), there is a function $u(x,t) \in W_2^{(2,1)}(Q_T)$ such that*

(i) *$v_h^\tau(x,t) \to u(x,t)$ uniformly in Q_T;*
(ii) *$v_{xh}^\tau(x,t) \to u_x(x,t)$ weakly in $L^2(Q_T)$;*
(iii) *$v_{th}^\tau(x,t) \to u_t(x,t)$ weakly in $L^2(Q_T)$.*

Proof. Using Lemma 6.1 and the discrete compactness method in [10], we can prove (i). (ii) and (iii) follow from the estimate (3.8). Lemma 6.2 is proved.

6.3. Define the piecewise constant functions, for $(x,t) \in Q_j^n$ $(j = 0, 1, \cdots, J-1; n = 0, 1, \cdots, N-1)$,

$$\bar{v}_h^\tau(x,t) = \delta^0 v_j^{n+1}, \quad A_h^\tau(x,t) = A_{j+\frac{1}{2}}^n, \quad f_h^\tau(x,t) = f_j^{n+1}.$$

Lemma 6.3. *Assume that the same conditions as those in Lemma 6.2 hold. When $h \to 0, \tau \to 0$ (for some subsequence), there are*

(i) *$\bar{v}_h^\tau(x,t) \to u(x,t)$ strongly in $L^2(Q_T)$ and a.e. in Q_T;*
(ii) *$A_h^\tau(x,t) \to A(x,t)$ strongly in $L^2(Q_T)$ and a.e. in Q_T; $f_h^\tau(x,t) \to f(x,t,u(x,t))$ strongly in $L^2(Q_T)$ and a.e. in Q_T.*

Proof. Note that, by the definition of $\delta^0 v_j^{n+1}$ and the estimates (3.7)–(3.8),

$$\|\bar{v}_h^\tau(x,t) - v_h^\tau(x,t)\|_{L^2(Q_T)}^2 \le Ch^2 \sum_{n=0}^{N} \|\delta v_h^n\|_2^2 \tau + C\tau^2 \sum_{n=0}^{N-1} \Big\| \frac{v_h^{n+1} - v_h^n}{\tau} \Big\|_2^2 \tau \le C(h^2 + \tau^2).$$

It follows that (i) can be proved easily by using Lemma 6.2. Now we prove (ii). Note that by (II) we have $f(\cdot,\cdot,u) \in C(R^m)$, and by (3.9) $\max\limits_{0 \le n \le N-1} \|\delta^0 v_h^{n+1}\|_\infty \le C$. There holds

$$\|f_h^\tau(x,t) - f(x,t,u(x,t))\|_{L^2(Q_T)}^2 \to 0 \qquad \text{as } h \to 0,\ \tau \to 0.$$

Lemma 6.3 is obtained.

6.4. Denote by J_0^n the number of j satisfying $\lambda_j^n \neq 1$ or $\mu_j^n \neq 1$, and let $J_0 = \max\limits_{0\leq n\leq N-1} J_0^n$. We introduce the following assumption.

(V) J_0 is any fixed constant for all $h>0$ and $\tau>0$.

Define

$$\begin{aligned} v_{xxh}^{\tau}(x,t) &= \overset{*}{\delta}{}_A^2 v_j^{n+1}, \qquad (x,t)\in Q_j^n \quad (j=1,2,\cdots,J-1;\ n=0,1,\cdots,N-1),\\ v_{xxh}^{\tau}(x,t) &= \overset{*}{\delta}{}_A^2 v_1^{n+1}, \qquad (x,t)\in Q_0^n \quad (n=0,1,\cdots,N-1). \end{aligned}$$

Let $\Phi(x,t)\in C^\infty(Q_T)$ and $\Phi(x,t)=0$ near $x=0$ and $x=l$. Denote $\Phi_j^n=\Phi(x_j,t^n)$. Define the piecewise constant functions $\Phi_h^\tau(x,t)=\Phi_j^{n+1}$, for $(x,t)\in Q_j^n$. When h and τ are small, there hold

$$\begin{aligned} &\iint_{Q_T} v_{xxh}^{\tau}(x,t)\Phi_h^\tau(x,t)dxdt\\ =&\sum_{n=0}^{N-1}\sum_{j=1}^{J-1}\overset{*}{\delta}{}_A^2 v_j^{n+1}\Phi_j^{n+1}h\tau\\ =&\sum_{n=0}^{N-1}\sum_{j=1}^{J-1}\delta_A^2 v_j^{n+1}\Phi_j^{n+1}h\tau\\ &-\sum_{n=0}^{N-1}\sum_{j=1}^{J-1}\frac{(1-\lambda_j^n)A_{j+\frac12}^n(v_{j+1}^{n+1}-v_{j+1}^n)+(1-\mu_j^n)A_{j-\frac12}^n(v_{j-1}^{n+1}-v_{j-1}^n)}{h}\Phi_j^{n+1}\tau\\ \equiv&\ \mathrm{I}+\mathrm{II}. \end{aligned}$$

It is easy to see that

$$\mathrm{I}\to\iint_{Q_T}A(x,t)u_x(x,t)\Phi_x(x,t)dxdt \qquad \text{for}\quad h\to0,\ \tau\to0;$$

and

$$|\mathrm{II}|\leq 2A_0\Lambda\max_{0\leq n\leq N-1}\|\Phi_h^{n+1}\|_\infty\Big(\sum_{n=0}^{N-1}\Big\|\frac{v_h^{n+1}-v_h^n}{\tau}\Big\|_2^2\tau\Big)^{\frac12}(TJ_0\tau)^{\frac12},$$

so

$$\mathrm{II}\to0 \qquad \text{for}\quad h\to0,\ \tau\to0.$$

There holds

$$\begin{aligned} &\iint_{Q_T}[v_{th}^\tau(x,t)-v_{xxh}^\tau-f_h^\tau(x,t)]\Phi_h^\tau(x,t)dxdt\\ =&\sum_{n=0}^{N-1}\sum_{j=1}^{J-1}\Big[\frac{v_j^{n+1}-v_j^n}{\tau}-\overset{*}{\delta}{}_A^2 v_j^{n+1}-f_j^{n+1}\Big]\Phi_j^{n+1}h\tau=0. \end{aligned}$$

By letting $h\to0,\tau\to0$ (for some subsequences), we have

$$\iint_{Q_T}((u_t(x,t)-f(x,t,u))\Phi(x,t)-A(x,t)u_x\Phi_x(x,t))dxdt=0.$$

Moreover, since the subsequence $v_h^\tau(x,t)$ is uniformly convergent to $u(x,t)$ in the rectangular domain Q_T, the limiting m-dimensional vector function $u(x,t)$ satisfies the homogeneous boundary conditions (2.2) and the initial condition (2.3) in classical sense. This means that the m-dimensional vector function $u(x,t) \in H^1(0,T,L^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega))$ is just the generalized solution of the boundary problem with the homogeneous boundary conditions (2.2) and the initial condition (2.3) for the semilinear parabolic system (2.1) of partial differential equations. The uniqueness of the generalized solution for the problem (2.1)–(2.3) can be justified in usual way. By means of the uniqueness of the generalized solution of the homogeneous boundary problem (2.1)–(2.3), we then can obtain the convergence theorem for the finite difference scheme (2.4)–(2.6) with intrinsic parallelism as follows:

Theorem 6.1. *Assume the conditions* (I)–(V) *hold. As the meshsteps h and τ tend to zero, the m-dimensional discrete vector solution $v_\Delta = v_h^\tau = \{v_j^n | j = 0,1,\cdots,J; n = 0,1,\cdots,N\}$ of the finite difference scheme* (2.4)–(2.6) *with intrinsic parallelism converges to the unique generalized solution $u(x,t) \in H^1(0,T,L^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega))$ of the boundary problem* (2.2) *and* (2.3) *for the semilinear parabolic system* (2.1) *of partial differential equations.*

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