Chin. Ann. Math.
25B:2(2004),225–232.

IDEAL STRUCTURE OF UNIFORM ROE ALGEBRAS OVER SIMPLE CORES***

CHEN Xiaoman* WANG Qin**

Abstract

This paper characterizes ideal structure of the uniform Roe algebra $B^*(X)$ over simple cores X. A necessary and sufficient condition for a principal ideal of $B^*(X)$ to be spatial is given and an example of non-spatial ideal of $B^*(X)$ is constructed. By establishing an one-one correspondence between the ideals of $B^*(X)$ and the ω-filters on X, the maximal ideals of $B^*(X)$ are completely described by the corona of the Stone-Čech compactification of X.

Keywords Uniform Roe algebra, Simple core, Ideal, Ultrafilter, Stone-Čech compactification
2000 MR Subject Classification 46L80

§ 1. Introduction and Preliminaries

Let X be a proper metric space with bounded geometry. Associated to X is a C^*-algebra $C^*(X)$, called the Roe algebra (see [8]), which has proved very useful in C^*-approaches to the Novikov conjecture in manifold theory (see [4, 9]). While the operator K-theory of the Roe algebras has been studied in the way of attacking the coarse Baum-Connes conjecture, the algebraic structure of Roe algebras is still far from being well understood so far. In a recent paper [1], the authors investigated some properties of ideal structure of the Roe algebras. We show that countably generated ideals of $C^*(X)$ cannot be associated to a subspace of X, and that there exists ideals in which finite propagation operators are not dense in the ideals. These facts, however, depend heavily on the local infinite dimensionality of the X-module. Therefore, it seems that Roe algebras cannot faithfully reflect the geometric nature of the underlying spaces. In order to be more geometric, one drops the local infinite dimensionality of the X-modules and define a uniform Roe algebra $B^*(X)$ over a discrete metric space X. Recently, uniform Roe algebras have also been studied in relation with exact C^*-algebras and amenable group actions (see [5]). Again, it is natural and interesting to describe the ideal structure of $B^*(X)$.

In this paper we shall characterize the ideal structure of the uniform Roe algebra $B^*(X)$ over simple cores, which are discrete, extremely coarsely disconnected metric spaces X with bounded geometry. We give a necessary and sufficient condition for a principal

Manuscript received February 20, 2003. Revised May 13, 2003.
*Institute of Mathematics, Fudan University, Shanghai 200433, China. **E-mail:** xchen@fudan.edu.cn
**Department of Applied Mathematics, Donghua University, Shanghai 200051, China.
E-mail: qwang@dhu.edu.cn
***Project supported by the 973 Project of the Ministry of Science and Technology of China, the National Natural Science Foundation of China (No.10201007), the Doctoral Programme Foundation of the Ministry of Education of China and the Shanghai Science and Technology Commission (No.01ZA14003).

ideal of $B^*(X)$ to be spatial and construct an example of non-spatial ideal of $B^*(X)$. This answers our early question (see [1]) on the ideal structure of uniform Roe algebras. On the other hand, we establish an one-one correspondence between the ideals of $B^*(X)$ and the ω-filters on X, and consequently we completely determine the maximal ideals of $B^*(X)$ by the corona of the Stone-Čech compactification βX.

Throughout this note, by an ideal we mean a closed two-sided proper ideal of a C^*-algebra.

Let X be a discrete, bounded geometry metric space. Recall that bounded geometry means that for any $R > 0$ the quantity $\sup\limits_{x \in X} \#\{y \in X : \ d(x,y) \leq R\}$ is finite.

Definition 1.1. *Denote by $B(X)$ the algebra of $X \times X$ complex matrices $T = [t_{xy}]$ such that*

(i) $\sup\limits_{x,y} |t_{x,y}| < \infty$*; and*

(ii) $\mathrm{Prg}(T) := \sup\{d(x,y) : \ t_{x,y} \neq 0\} < \infty$.

The algebra $B(X)$ is represented in an obvious way on $\ell^2(X)$ and we denote by $B^*(X)$ the C^*-algebra completion in this representation. This is the uniform Roe algebra associated to the metric space X. And $\mathrm{Prg}(T)$ is called the propagation of the operator T.

Let $Y \subseteq X$ be a subspace of a discrete, bounded geometry metric space X. For any $R > 0$, denote

$$\mathrm{Pen}(Y; R) := \{x \in X : \ d(x,Y) \leq R\}.$$

For an operator $T \in B^*(X)$, the support $\mathrm{Supp}(T)$ of T is defined by

$$\mathrm{Supp}(T) = \{(x,y) \in X \times X : t_{x,y} \neq 0\}.$$

Definition 1.2. *Denote by $B^*(Y;X)$ the C^*-subalgebra of $B^*(X)$ generated by all operators $T \in B(X)$ with $\mathrm{Supp}(T) \subseteq \mathrm{Pen}(Y;R) \times \mathrm{Pen}(Y;R)$ for some $R > 0$ (depending on T).*

It is not difficult to see that $B^*(Y;X)$ is a two-sided closed ideal of $B^*(X)$. We call such ideals spatial ideals because they are supported around a subspace Y. The spatial ideals play an important role in the computation of the K-theory of the Roe algebras (see [2, 6, 8]). Similarly to our early concerns on the ideal structure of the Roe algebras in [1], we naturally ask the following question for the uniform Roe algebras:

Question 1. Are all the ideals of $B^*(X)$ spatial?

And in the same spirit, we ask the following questions:

Question 2. Are the finite propagation operators in any ideal of $B^*(X)$ dense in the ideal?

Question 3. How about the maximal ideals of $B^*(X)$?

We are going to answer these questions in the case of simple cores.

§ 2. Spatial Ideals over Simple Cores

A simple core is a discrete, extremely coarsely disconnected metric spaces X with bounded geometry. Or precisely, we give the following definition:

Definition 2.1. *A discrete, bounded geometry metric space X is called a simple core if for any $R > 0$, there is a compact subset $K \subset X$ such that $d(x,y) > R$ whenever $(x,y) \in X \times X - K \times K$.*

For a general notion of core, see [9]. Denote by $\ell^2(X)$ the Hilbert space on which the uniform algebra $B^*(X)$ acts. Let $\ell^\infty(X)$ be the algebra of all bounded functions on X, and let $c_0(X)$ be the subalgebra of bounded functions vanishing at infinity of X. Denote by $\mathcal{K}(\ell^2(X))$ or just $\mathcal{K}$ the algebra of all compact operators on $\ell^2(X)$. Note that the functions of $\ell^\infty(X)$ can be viewed as multiplication operators on $\ell^2(X)$. The uniform Roe algebras over simple cores are transparent in the following sense:

Proposition 2.1. *$B^*(X) \cong \ell^\infty(X) + \mathcal{K}(\ell^2(X))$.*

Proposition 2.2. *$\mathcal{K}(\ell^2(X)) \subseteq J$ for any ideal J of $B^*(X)$.*

Proposition 2.3. *If I is an ideal of $\ell^\infty(X)$, then $I + \mathcal{K}(\ell^2(X))$ is an ideal of $B^*(X)$. And any ideal of $B^*(X)$ is of this form.*

Proof. The first statement is obvious. For the second, let J be an ideal of $B^*(X)$. For any $T \in B^*(X)$, let $\mathrm{diag}(T) := (t_{(x,x)})$ denote the diagonal of the matrix representation $[t_{x,y}]$ of T. Set

$$I := \mathrm{Diag}(J) := \{\mathrm{diag}(T) : T \in J\}.$$

Then $I \subseteq \ell^\infty(X)$. Since $B^*(X) = \ell^\infty(X) + \mathcal{K}$ and $\mathcal{K} \subseteq J$, there is a compact operator K such that $T = \mathrm{diag}(T) + K$. Hence, $\mathrm{diag}(T) \in J$. This shows that

$$I \subseteq J \quad \text{and} \quad J = I + \mathcal{K}.$$

To complete the proof, it suffices to show that I is an ideal of $\ell^\infty(X)$.

For any $f \in I$ and $g \in \ell^\infty(X)$, we have $fg = gf \in J$ since $f \in I \subseteq J$ and $g \in \ell^\infty(X) \subseteq B^*(X)$. Then

$$fg = gf = \mathrm{diag}(fg) \in I.$$

Hence, I is an ideal of $\ell^\infty(X)$.

It follows from Proposition 2.3 that the correspondence

$$\Phi : I \mapsto I + \mathcal{K}$$

gives a well-defined order preserving surjective map from the collection of ideals of $\ell^\infty(X)$ to the collection of ideals of the uniform Roe algebra $B^*(X)$. The following result is obvious.

Corollary 2.1. *For any two ideals I_1 and I_2 in $\ell^\infty(X)$, $\Phi(I_1) = \Phi(I_2)$ in $B^*(X)$ if and only if $I_1 + c_0(X) = I_2 + c_0(X)$ in $\ell^\infty(X)$.*

Now we first give an affirmative answer to the Question 2 as follows:

Theorem 2.1. *For any ideal J of $B^*(X)$, the finite propagation operators in J are dense in J.*

Proof. It follows from Proposition 2.3 that any ideal J of $B^*(X)$ takes the form $J = I + \mathcal{K}(\ell^2(X))$ for some ideal I of $\ell^\infty(X)$. So the finite propagation operators in J are precisely those operators which are sums of an element of $\ell^\infty(X)$ and a finite matrix. Hence, they are dense in J.

We proceed to study the relation of principal ideals of $B^*(X)$ and the spatial ideals of the form $B^*(Y;\ X)$, where Y is a subspace of X. We aim to construct an example of principal ideal which cannot be of the form $B^*(Y;\ X)$ and in this way we give a negative answer to the Question 1.

Let Y be a subspace of X and denote by χ_{Y^c} the characteristic function of the complement $Y^c = X - Y$. As in the proof of Proposition 2.3, for any $T \in B^*(X)$, the following correspondence

$$\mathrm{diag}(T)(x) = t_{(x,x)}$$

(for all $x \in X$) defines a conditional expectation

$$\mathrm{diag} : B^*(X) \to \ell^\infty(X).$$

Lemma 2.1. $\mathrm{diag}(T) \cdot \chi_{Y^c} \in c_0(X)$ *for any* $T \in B^*(Y;\ X)$.

Let $f \in \ell^\infty(X)$. Note that f is considered as a multiplication operator on $\ell^2(X)$ with $\mathrm{Prg}(f) = 0$, so $f \in B^*(X)$. Denote by $\langle f\rangle_{\ell^\infty(X)}$ and $\langle f\rangle_{B^*(X)}$, respectively, the principal ideals generated by f in $\ell^\infty(X)$ and $B^*(X)$, respectively. Then it is clear that

$$\langle f\rangle_{\ell^\infty(X)} = \{fg : g \in \ell^\infty(X)\}.$$

We also have the following relation:

Proposition 2.4. *For any* $f \in B^*(X)$*, we have*

$$\langle f\rangle_{B^*(X)} = \langle f\rangle_{\ell^\infty(X)} + \mathcal{K}.$$

In [1] we proved that all countably generated ideals of Roe algebra $C^*(X)$ cannot be of the form $C^*(Y;X)$ for a subspace Y. In contrast with this phenomenon, all ideals $B^*(Y;\ X)$ of $B^*(X)$ associated to a subspace Y are principal ideals.

Proposition 2.5. *Let* $Y \subseteq X$ *be a subspace and let* χ_Y *be the characteristic function of* Y*. Then* $B^*(Y;\ X) = \langle \chi_Y\rangle_{B^*(X)}$.

Moreover, we have the following necessary and sufficient condition characterizing the relation of principal ideals and the spatial ideals.

Theorem 2.2. *Let* $Y \subseteq X$ *be a subspace and let* $f \in \ell^\infty(X)$*. Then the following statements are equivalent:*

(1) $\langle f\rangle_{B^*(X)} = B^*(Y;\ X)$.

(2) *f is exactly bounded below on Y. Or precisely, the following two conditions hold:*

- (i) $f\chi_{Y^c} \in c_0$*; and*
- (ii) $\sup\limits_{K\subset X:\ \text{compact}} \inf\{|f(y)| : y \in Y - K\} \geq b > 0$ *for some* $b > 0$ *(depending on f).*

Proof. (1)$\Rightarrow$(2). Suppose

$$\langle f\rangle_{B^*(X)} = B^*(Y;\ X).$$

Since $f \in B^*(Y;\ X)$, it follows from Lemma 2.1 that $f\chi_{Y^c} \in c_0(X)$. On the other hand, since $\chi_Y \in B^*(Y;\ X)$, it follows from Proposition 2.5 that there exist $g \in \ell^\infty(X)$ and $g_0 \in c_0(X)$ such that

$$\chi_Y = fg + g_0.$$

Set $b = \frac{1}{2\|g\|_\infty}$. Then $b > 0$, $fg = \chi_Y - g_0$ and there exists a compact subset M_0 such that, whenever $y \in Y - M_0$, we have

$$|f(y)| \cdot \|g\|_\infty \geq |f(y)g(y)| = |1 - g_0(y)| > \frac{1}{2}.$$

It follows that for all $y \in Y - M_0$,

$$|f(y)| \geq \frac{1}{2\|g\|_\infty} = b,$$

which implies (ii).

(2)⇒(1). Suppose f is exactly bounded below on Y. Note that

$$B^*(Y;\ X) = \langle \chi_Y \rangle_{B^*(X)} \quad \text{and} \quad f = f\chi_Y + f\chi_{Y^c}.$$

It is also clear that

$$f\chi_Y \in B^*(Y;\ X) \quad \text{and} \quad f\chi_{Y^c} \in c_0(X) \subseteq \mathcal{K}.$$

Thus, we have $f \in B^*(Y;\ X)$ and consequently

$$\langle f \rangle_{B^*(X)} \subseteq B^*(Y;\ X).$$

On the other hand, it follows from (ii) that there exist a compact subset M and $b > 0$ such that

$$|f(y)| \geq b \qquad \text{for all} \ \ y \in Y - M.$$

Then

$$\chi_Y = fg\chi_{M^c} + \chi_M \qquad \text{for some} \ \ g \in \ell^\infty(X)$$

with

$$g(y) = \frac{1}{f(y)} \qquad \text{for all} \ \ y \in Y - M.$$

Hence

$$\chi_Y \in \langle f \rangle_{\ell^\infty(X)} + \mathcal{K} = \langle f \rangle_{B^*(X)}.$$

This implies that

$$B^*(Y;\ X) \subseteq \langle f \rangle_{B^*(X)}.$$

The proof is complete.

Now we are ready to construct a counterexample to the Question 1.

Example 2.1. Let $X = |N|$ be the natural numbers equipped with a metric such that

$$d(i, j) > i + j.$$

The $|N|$ is a model of simple core. Let $f \in \ell^\infty(X)$ be defined as follows:

$$f(2^m(2n+1)) = \begin{cases} 0, & \text{if} \ \ m = 0, \\ \dfrac{1}{m}, & \text{if} \ \ m \neq 0, \end{cases}$$

where m, n run over all non-negative integers. (Note that any natural number has unique expression $2^m(2n+1)$.) Then there is no subspace $Y \subseteq X$ satisfying

$$\langle f \rangle_{B^*(X)} = B^*(Y;\ X).$$

Indeed, suppose on the contrary $Y \subseteq X$ meets the need. It follows from Theorem 2.2 that there exist natural numbers m_0 and k_0 such that

$$|f(y)| \geq \frac{1}{m_0} \qquad \text{for all } \ y \in Y \ \text{ with } \ y > k_0.$$

Set

$$W = \{2^m(2n+1) : m = 1, 2, \cdots, m_0; \ n = 0, 1, 2, 3, \cdots\},$$
$$F = \{y \in Y : 1 \leq y \leq k_0\}.$$

Then

$$Y - F \subseteq W \quad \text{and} \quad \chi_{W^c} \leq \chi_{(Y-F)^c}.$$

Since $f\chi_{Y^c} \in c_0$, we should have that $f\chi_{W^c} \in c_0$. But this is impossible because, for any given $m \geq m_0 + 1$, we have $2^m(2n+1) \in W^c$ for all $n \in \mathbf{Z}^+$ and

$$(f\chi_{W^c})(2^m(2n+1)) = f(2^m(2n+1)) = \frac{1}{m} > 0.$$

The contradiction shows that $\langle f \rangle_{B^*(X)}$ cannot be associated to any subspace of X.

§ 3. ω-Filters and Maximal Ideals

Let X be a simple core. It is well known that $\ell^\infty(X) = C(\beta X)$, the algebra of continuous functions on the Stone-Čech compactification βX, and the algebraic structure of a ring of continuous functions on a completely regular Hausdorff space can be characterized by certain filters on the space (see [3]). Therefore, it is natural to relate ideals of $B^*(X)$ with filters on X. We shall do this in this section. We introduce a notion of ω-filter on X, which can be corresponded to ideals of $B^*(X)$. We show that ω-ultrafilters are precisely free ultrafilters on X. Consequently, the maximal ideals of $B^*(X)$ are in one-one onto correspondence with the points of the corona $\beta X - X$.

To begin with, recall that a nonempty family $\mathcal{F}$ of subsets of X is called a filter on X provided that

(i) $\emptyset \notin \mathcal{F}$;
(ii) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$; and
(iii) if $A \in \mathcal{F}$, $B \subseteq X$ and $A \subseteq B$, then $B \in \mathcal{F}$.

A filter is said to be free or fixed according to that the intersection of all its members is empty or nonempty. And by a ultrafilter on X is meant a maximal filter. The readers are referred to [3] for notions of filters and ultrafilters. We make the following definition:

Definition 3.1. *A filter $\mathcal{F}$ on X is called an ω-filter provided that*
(iv) *$\mathcal{F}$ contains all cofinite subsets of X.*

Here, a subset $D \subseteq X$ is called cofinite if $X - D$ is a finite set. For any $f \in \ell^\infty(X)$ and $\varepsilon > 0$, we define

$$E_\varepsilon(f) = f^{-1}([-\varepsilon, \varepsilon]) = \{x \in X : |f(x)| \leq \varepsilon\}.$$

For $J \subseteq B^*(X)$, we define

$$\Omega(J) = \{E_\varepsilon(\mathrm{diag}(T)) : T \in J, \ \varepsilon > 0\}$$

and for a family $\mathcal{F}$ of subsets of X, we define

$$\Omega^{-1}(\mathcal{F}) = \{T \in B^*(X) : E_\varepsilon(\mathrm{diag}(T)) \in \mathcal{F}, \ \ \forall \varepsilon > 0\}.$$

Proposition 3.1. *If J is an ideal of $B^*(X)$, then $\Omega(J)$ is an ω-filter on X.*

Proposition 3.2. *If $\mathcal{F}$ is an ω-filter on X, then $\Omega^{-1}(\mathcal{F})$ is a closed two-sided proper ideal of $B^*(X)$.*

Theorem 3.1. *Let J be an ideal of $B^*(X)$ and let $\mathcal{F}$ be an ω-filter on X. Then*

$$\Omega^{-1}(\Omega(J)) = J \quad \textit{and} \quad \Omega(\Omega^{-1}(\mathcal{F})) = \mathcal{F}.$$

Proof. Firstly, note that an operator $T \in \Omega^{-1}(\Omega(J))$ if and only if for any $\varepsilon > 0$, there exist $S \in J$ and $\delta > 0$ such that

$$E_\varepsilon(\mathrm{diag}(T)) = E_\delta(\mathrm{diag}(S)).$$

Thus, it is ready that $J \subseteq \Omega^{-1}(\Omega(J))$. To prove the converse inclusion, let $T \in \Omega^{-1}(\Omega(J))$. It suffices to show that $f := \mathrm{diag}(T) \in J$. Since $f \in \Omega^{-1}(\Omega(J))$, for any $\varepsilon > 0$, there exist $g \in J \cap \ell^\infty(X)$ and $\delta > 0$ such that $E_\varepsilon(f) = E_\delta(g)$. Denote $W = E_\varepsilon(f) = E_\delta(g)$ and define $h \in \ell^\infty(X)$ by

$$h(x) = \begin{cases} 0, & \text{if } \ x \in W; \\ \dfrac{1}{g(x)}, & \text{if } \ x \in W^c. \end{cases}$$

Then $\chi_{W^c} = gh \in J$ and $f\chi_{W^c} \in J$. But clearly we have

$$\|f - f\chi_{W^c}\| < \varepsilon.$$

This shows that $f \in J$ since J is closed. Therefore, $J \supseteq \Omega^{-1}(\Omega(J))$.

Secondly, note that a subset $A \in \Omega(\Omega^{-1}(\mathcal{F}))$ if and only if there exist $\varepsilon > 0$, and an operator $T \in B^*(X)$ with $E_\delta(\mathrm{diag}(T)) \in \mathcal{F}$ for all $\delta > 0$, such that $A = E_\varepsilon(\mathrm{diag}(T))$. Thus, it is clear that $\Omega(\Omega^{-1}(\mathcal{F})) \subseteq \mathcal{F}$. On the other hand, let $A \in \mathcal{F}$ and consider the characteristic function χ_A of A. Then $\chi_A \in B^*(X)$ and

$$E_\delta(\chi_A) = \begin{cases} A, & \text{if } \ 0 < \delta < 1; \\ X, & \text{if } \ \delta \geq 1. \end{cases}$$

This implies that $A \in \Omega(\Omega^{-1}(\mathcal{F}))$. Hence

$$\Omega(\Omega^{-1}(\mathcal{F})) = \mathcal{F}.$$

The proof is complete.

The following two corollaries are immediate from Theorem 3.1.

Corollary 3.1. *Let J, J' be ideals of $B^*(X)$ and let $\mathcal{F}, \mathcal{F}'$ be ω-filters on X. Then*

$$J \subseteq J' \iff \Omega(J) \subseteq \Omega(J'),$$
$$\mathcal{F} \subseteq \mathcal{F}' \iff \Omega^{-1}(\mathcal{F}) \subseteq \Omega^{-1}(\mathcal{F}').$$

Corollary 3.2. *The correspondence $M \mapsto \Omega(M)$ is one-one from the set of all maximal ideals of $B^*(X)$ onto the set of all ω-ultrafiltes on X.*

Of cause, an ω-filter is a free filter. The converse is not true in general. However, we have the following

Theorem 3.2. *ω-ultrafilters on X are precisely free ultrafilters on X.*

Proof. Note that if $A \cup B = X$, then either A or B belongs to a given ultrafilter. It follows that any free ultrafilter contains all cofinite subsets of X. That is, all free ultrafilters on X are ω-ultrafilters. Conversely, if $\mathcal{A}$ is an ω-ultrafilter on X, it is contained in a ultrafilter, say $\mathcal{F}$, by Zorn's Lemma. Then $\mathcal{F}$ must be free, and coincide with $\mathcal{A}$.

Since the Stone-Čech compactification βX can be constructed by ultrafilters on X in which X coincides with the fixed ultrafilters and the corona $\beta X - X$ coincides with the free ultrafilters (see [3]), we have the following result by Theorem 3.2 and Corollary 3.2.

Theorem 3.3. *The set of all maximal ideals of X is in one-one onto correspondence with the corona $\beta X - X$.*

References

[1] Chen, X. M. & Wang, Q., Notes on ideals of Roe algebras, *Oxford Quarterly J. of Math.,* **52**(2001), 437–446.

[2] Chen, X. M. & Wang, Q., Localization algebras and duality, *J. London Math. Soc.,* **66:**2(2002), 227–239.

[3] Gillman, L. & Jerison, M., Rings of Continuous Functions, GTM 43, Springer-Verlag, 1960.

[4] Higson, N. & Roe, J., On the coarse Baum-Connes conjecture, in Proc. 1993 Oberwolfach Conference in the Novikov Conjecture, S. Ferry, A. Ranicki and J. Rosenberg (eds), London Math. Soc. Lecture Note Series 227, Cambrige University Press, 1995, 227–254.

[5] Higson, N. & Roe, J., Amenable group actions and the Novikov conjecture, *J. Reine Angew. Math.*, **519**(2000), 143–153.

[6] Higson, N., Roe, J. & Yu, G., A coarse Mayer-Vietoris principle, *Math. Proc. Camb. Phil. Soc.*, **114:**85(1993), 85–97.

[7] Lubotzky, A., Discrete groups, expanding graphs and invariant measures, Progress in Mathematics, Vol. 125, Birkhauser Verlag, 1994.

[8] Roe, J., Index theory, coarse geometry and topology of manifolds, CBMS Reginal Conf. Series in Math., No. 90, AMS, 1996.

[9] Yu, G., Coarse Baum-Connes conjecture, *K-Theory*, **9:**3(1995), 199–221.

[10] Yu, G., The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, *Invent. Math.*, **139**(2000), 201–240.