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BIFURCATION OF LIMIT CYCLES FROM A DOUBLE HOMOCLINIC LOOP WITH A ROUGH SADDLE***

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Abstract

This paper concerns with the bifurcation of limit cycles from a double homoclinic loop under multiple parameter perturbations for general planar systems. The existence conditions of 4 homoclinic bifurcation curves and small and large limit cycles are especially investigated.

Keywords Double homoclinic loop, Bifurcation, Limit cycle
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Consider a plane system of the form

$$\dot{x} = f(x,y) + \varepsilon f_0(x,y,\varepsilon,\delta), \quad \dot{y} = g(x,y) + \varepsilon g_0(x,y,\varepsilon,\delta), \tag{1}$$

where $f,\ g,\ f_0$, and g_0 are C^r functions, $r \geq 3$, $\varepsilon > 0$ small and $\delta \in D \subset R^n$ $(n \geq 1)$ with D being compact. Suppose for $\varepsilon = 0$, (1) has a double homoclinic loop $L = L_1 \cup L_2 \cup S_0$ with homoclinic orbits L_1 and L_2 being homoclinic to the hyperbolic saddle S_0. We will establish a Poincaré map near L based on some results of [1–3], and investigate the behavior of the map.

For definiteness we suppose that L is oriented clockwise. Choose points $A_i \in L$ near S_0, $i = 1,2,3,4$ with $A_1,\ A_4 \in L_1$ and $A_2,\ A_3 \in L_2$. Let l_i be a cross section passing through A_i with the direction as follows

$$n_i = \frac{1}{|\ f(A_i),\ g(A_i)\ |}(-g(A_i),\ f(A_i)), \qquad 1 \leq i \leq 4.$$

Then a point B_1 on l_1 can be denoted as $B_1 = A_1 + an_1$. Suppose the positive orbit $\gamma(B_1)$ of (1) starting at B_1 meets l_1 at a point B for the first time (see Fig. 1). Then B can be written as $B = A_1 + P(a,\varepsilon,\delta)n_1$.

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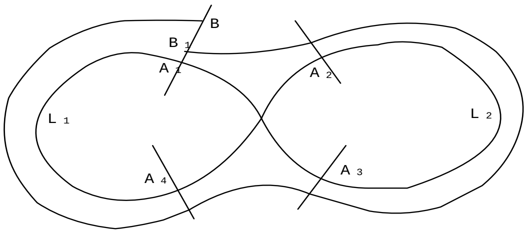

Fig. 1

We call the function P above a Poincaré map of (1). Denote $S(\varepsilon,\delta)$ the saddle point of (1) near S_0. Then by [3], there is a point $B_0 = A_1 + a_0(\varepsilon,\delta)n_1 \in l_1$ on the stable manifold of S such that the domain of P in a is $a > a_0(\varepsilon,\delta)$ and that the following limit exists

$$\lim_{a\to a_0} P(a,\varepsilon,\delta) = P(a_0(\varepsilon,\delta),\varepsilon,\delta).$$

Thus, the domain of P can be extended as $a \geq a_0(\varepsilon,\delta)$. Denote by

$$G : l_1 \to l_2, \qquad \overline{G} : l_3 \to l_4$$

the Dulac maps and by

$$R : l_2 \to l_3, \qquad \overline{R} : l_4 \to l_1$$

the regular maps. Let B_2, B_3 and B_4 be the intersection points of the orbit $\gamma(B_1)$ with the cross section l_2, l_3 and l_4 in turn. They can be written as

$$\begin{aligned} B_2 &= A_2 + G(a,\varepsilon,\delta)n_2 \equiv A_2 + a_2 n_2, \\ B_3 &= A_3 + R(a_2,\varepsilon,\delta)n_3 \equiv A_3 + a_3 n_3, \\ B_4 &= A_4 + \overline{G}(a_3,\varepsilon,\delta)n_4 \equiv A_4 + a_4 n_4, \\ B &= A_1 + \overline{R}(a_4,\varepsilon,\delta)n_1 \equiv A_1 + P(a,\varepsilon,\delta)n_1. \end{aligned} \tag{2}$$

For the sake of convenience we suppose that the saddle point S is always at the origin. Then a C^r local coordinate change of the form

$$T_{\varepsilon,\delta} : \quad \begin{pmatrix} u \\ v \end{pmatrix} = T(\varepsilon,\delta)\begin{pmatrix} x \\ y \end{pmatrix} + O(|x,y|^2) \tag{3}$$

exists which carries (1) into the form

$$\dot{u} = \alpha_1 u(1 + h_1(u,v,\varepsilon,\delta)), \qquad \dot{v} = \alpha_2 v(1 + h_2(u,v,\varepsilon,\delta)), \tag{4}$$

where h_i is C^{r-1} with $h_i(0,0,\varepsilon,\delta) = 0$ and $\alpha_i = \alpha_i(\varepsilon,\delta)$ with $\alpha_1 > 0 > \alpha_2$. For $p > 0$ small, let

$$\begin{gathered} A_1' = (0,p), \qquad A_2' = (p,0), \qquad A_3' = (0,-p), \qquad A_4' = (-p,0), \\ l_1' = \{(u,v)|0 \leq u \leq p, v = p\}, \qquad l_2' = \{(u,v)|0 \leq v \leq p, u = p\}, \\ l_3' = \{(u,v)| -p \leq u \leq 0, v = -p\}, \qquad l_4' = \{(u,v)| -p \leq v \leq 0, u = -p\}. \end{gathered}$$

We suppose $r_0 \neq 1$. That is, the saddle at the origin is rough. Let

$$r = r(\varepsilon,\delta) = -\frac{\alpha_2}{\alpha_1}, \qquad r_0 = r(0,\delta).$$

Denote by $B_2'(p, q(u,\varepsilon,\delta))$ the intersection point of the positive orbit of (4) starting at $B_1'(u,p) \in l_1'$ with l_2', and by $B_4'(-p, -\overline{q}(u_3,\varepsilon,\delta))$ the intersection point of the positive orbit of (4) starting at $B_3'(-u_3,-p) \in l_3'$ with l_4'. Then similarly to Lemma 2.5 in [2], we have

Lemma 1. *The Dulac maps q and $\overline{q}$ of* (4) *satisfy*

$$\begin{aligned} q(u,\varepsilon,\delta) &= p^{1-r}u^r[K(p) + \varphi_0(u,\varepsilon,\delta)], \\ \overline{q}(u,\varepsilon,\delta) &= p^{1-r}u_3^r[\overline{K}(p) + \overline{\varphi}_0(u,\varepsilon,\delta)], \\ q_u(u,\varepsilon,\delta) &= rp^{1-r}u^{r-1}[K(p) + \varphi_1(u,\varepsilon,\delta)], \\ \overline{q}_{u_3}(u_3,\varepsilon,\delta) &= rp^{1-r}u_3^{r-1}[\overline{K}(p) + \overline{\varphi}_1(u_3,\varepsilon,\delta)], \end{aligned}$$

where $K(p), \overline{K}(p) = O(p)$, $\varphi_0, \varphi_1, u\dfrac{\partial \varphi_0}{\partial u} = o(u^k)$ for $0 < u \ll 1$, and $\overline{\varphi}_0, \overline{\varphi}_1, u_3\dfrac{\partial \overline{\varphi}_1}{\partial u_3} = o(u_3^k)$ for $0 < u_3 \ll 1$ and any constant $0 < k < \frac{1}{2}$.

Now we take $A_i = O(p)$ as the pre-image of the A_i' under the transformation (3). Then the following lemma is immediate from Lemma 1.1 in [1].

Lemma 2. *Suppose that the orbit arcs $B_1'B_2'$ and $B_3'B_4'$ of* (4) *lie on the image of the orbit arcs B_1B_2 and B_3B_4 of* (1) *under* (3) *respectively. Then there exist C^r functions*

$$W_i(u,\varepsilon,\delta,) = N_{0i}(\varepsilon,\delta,p) + N_{1i}(\varepsilon,\delta,p)u + O(u^2), \qquad i = 1,2,3,4,$$

where $N_{0i} = O(\varepsilon)$, $\lim\limits_{p\to 0} N_{1i}(0,\delta,p) = \beta_i \sin\theta$ with θ the angle of L at S_0, and

$$\beta_1 = \beta_3 = |T^{-1}(0,\delta)(0,1)^T|, \quad \beta_2 = \beta_4 = |T^{-1}(0,\delta)(0,1)^T|$$

such that

$$\begin{aligned} a &= W_1(u,\varepsilon,\delta), & G(a,\varepsilon,\delta) &= W_2(q(u,\varepsilon,\delta),\varepsilon,\delta), \\ a_3 &= W_3(u_3,\varepsilon,\delta), & \overline{G}(a_3,\varepsilon,\delta) &= W_4(\overline{q}(u_3,\varepsilon,\delta),\varepsilon,\delta). \end{aligned} \tag{5}$$

By the process of deriving (1.1.9) of [1], we have

Lemma 3. *The regular maps R and $\overline{R}$ satisfy*

$$\begin{aligned} R(a_2,\varepsilon,\delta) &= b_0(\varepsilon,\delta,p) + b_1(\varepsilon,\delta,p)a_2 + O(a_2^2), \\ \overline{R}(a_4,\varepsilon,\delta) &= \overline{b}_0(\varepsilon,\delta,p) + \overline{b}_1(\varepsilon,\delta,p)a_4 + O(a_4^2), \end{aligned}$$

where $b_0, \overline{b}_1 = O(\varepsilon)$, $b_1 = K_1(p) + O(\varepsilon)$, $\overline{b}_1 = \overline{K}_1(p) + O(\varepsilon)$ with $K_1 > 0$, $\overline{K}_1 > 0$.

Let

$$M_i(\delta) = \oint_{L_i} \exp\Big(-\int_0^t (f_x + g_y)ds\Big)(fg_0 - gf_0)_{\varepsilon=0}dt. \tag{6}$$

The distance of the intersection points of the two separatrices of the origin near L_1 with l_1 is given by

$$d_i(\varepsilon,\delta,A_j) = \frac{\varepsilon M_i(\delta)}{|f(A_j), g(A_j)|} + O(\varepsilon^2), \tag{7}$$

where $j = 1,4$ for $i = 1$ or $j = 2,3$ for $i = 2$. Define

$$F(a,\varepsilon,\delta) = P(a,\varepsilon,\delta) - a, \quad d(\varepsilon,\delta) = F(a_0(\varepsilon,\delta),\varepsilon,\delta).$$

The function F is called a succession function of (1). We can prove now

Theorem 1. *Suppose $r_0 \neq 1$. Then for $\varepsilon > 0$ small,*

(I) $M_1(\delta)d(\varepsilon,\delta) > 0$ *if* $(r_0 - 1)M_1(\delta) < 0$;
(II) $M_2(\delta)d(\varepsilon,\delta) > 0$ *if* $(r_0 - 1)M_2(\delta) < 0$;
(III) $M_2(\delta)d(\varepsilon,\delta) > 0$ *if* $(r_0 - 1)M_1(\delta) > 0$ *and* $M_2(\delta) \neq 0$;
(IV) $M_1(\delta)d(\varepsilon,\delta) > 0$ *if* $(r_0 - 1)M_2(\delta) > 0$ *and* $M_1(\delta) \neq 0$.

Proof. Let $L_i^{s,u}$ denote the 4 separatrices of (1) near the origin, $i = 1, 2$, where $L_1^s \cup L_2^s$ and $L_1^u \cup L_2^u$ lie on the stable and unstable manifolds of the origin respectively. There are 13 different distributions of the separatrices and 5, 3 and 5 distributions appear respectively for $d(\varepsilon,\delta) < 0$, $= 0$ and > 0. First, consider the case of $d_2 > 0$. Let

$$A_3^{s,u} = L_2^{s,u} \cap l_3, \quad A_i^{s,u} = L_1^{s,u} \cap l_i, \quad B_i^u = L_2^u \cap l_i, \qquad i = 1, 4.$$

Then a possible distribution is as shown in Fig. 2.

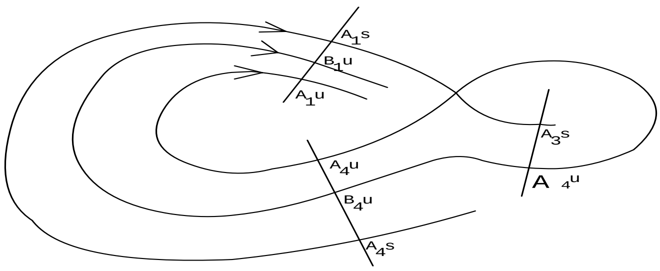

Fig. 2

Let

$$A_i^{s,u} = A_i + a_i^{s,u}(\varepsilon,\delta)n_i, \qquad i = 1, 3, 4,$$
$$B_i^u = A_i + b_i^u(\varepsilon,\delta)n_i, \qquad i = 1, 4.$$

The domain of F is $a \geq a_1^s(\varepsilon,\delta)$ $(= a_0^s(\varepsilon,\delta)$ in this case), and

$$F(a_0(\varepsilon,\delta),\varepsilon,\delta) = b_1^u(\varepsilon,\delta) - a_1^s(\varepsilon,\delta) = d(\varepsilon,\delta).$$

By the definition of $\overline{R}$,

$$b_1^u = \overline{R}(b_4^u,\varepsilon,\delta), \quad a_1^s = \overline{R}(a_4^u,\varepsilon,\delta).$$

Hence it follows from Lemma 3 that

$$b_1^u - a_1^s = \overline{R}(b_4^u,\varepsilon,\delta) - \overline{R}(a_4^u,\varepsilon,\delta) = [\overline{b}_1 + O(\varepsilon)](b_4^u - a_4^s).$$

Thus

$$d(\varepsilon,\delta) = [\overline{b}_1 + O(\varepsilon)](b_4^u - a_4^s). \tag{8}$$

From (7), we have

$$a_4^u - a_4^s = d_1(\varepsilon,\delta,A_4), \quad a_3^u - a_3^s = d_2(\varepsilon,\delta,A_3). \tag{9}$$

The mean value theorem together with (5) and (9) gives

$$\begin{aligned} W_3^{-1}(a_3^u,\varepsilon,\delta) &= W_3^{-1}(a_3^u,\varepsilon,\delta) - W_3^{-1}(a_3^s,\varepsilon,\delta) \\ &= \frac{\partial W_3^{-1}}{\partial u}(O(\varepsilon),\varepsilon,\delta)(a_3^u - a_3^s) \\ &= \frac{1}{N_{13} + O(\varepsilon)} d_2(\varepsilon,\delta,A_3) \equiv d_2^*. \end{aligned}$$

By the definition of $\overline{G}$,

$$b_4^u = \overline{G}(a_3^u, \varepsilon, \delta), \qquad a_4^u = \overline{G}(a_3^s, \varepsilon, \delta).$$

Lemma 2 implies that

$$a_3^s = W_3(0, \varepsilon, \delta) = N_{03}, \qquad a_4^u = W_4(0, \varepsilon, \delta) = N_{04}.$$

Hence we have from (5),

$$\begin{aligned} b_4^u - a_4^u &= (W_4 \circ \overline{q} \circ W_3^{-1})(a_3^u, \varepsilon, \delta) - (W_4 \circ \overline{q} \circ W_3^{-1})(a_3^s, \varepsilon, \delta) \\ &= W_4(\overline{q}(d_2^*, \varepsilon, \delta), \varepsilon, \delta) - W_4(0, \varepsilon, \delta), \\ &= N_{14}\overline{q}(d_2^*, \varepsilon, \delta) + O(|\overline{q}(d_2^*, \varepsilon, \delta)|^2). \end{aligned}$$

Thus we obtain by Lemma 1,

$$b_4^u - a_4^u = N_{14} p^{1-r} \Big(\frac{d_2}{N_{13}}\Big)^r (1 + o(\varepsilon^k)). \tag{10}$$

From (8)–(10), if $r_0 > 1$, we have

$$\begin{aligned} d(\varepsilon, \delta) &= [\overline{b}_1 + o(\varepsilon)][(b_4^u - a_4^u) + (a_4^u - a_4^s)] \\ &= \frac{\varepsilon \overline{b}_1}{|f(A_4), g(A_4)|}\Big[M_1(\delta) + O\Big(\Big(\frac{d_2}{\varepsilon}\Big)^r \varepsilon^{r-1} + \varepsilon\Big)\Big]. \end{aligned} \tag{11}$$

If $r_0 < 1$, then

$$d(\varepsilon, \delta) = \frac{\overline{b}_1 N_{14} p^{1-r} \varepsilon^r}{(N_{13}|f(A_3),\ g(A_3)|)^r}\Big[\Big(\frac{d_2}{\varepsilon}\Big)^{\frac{1}{r}}(1 + O(\varepsilon^k)) + O(\varepsilon^{1-r})\Big]. \tag{12}$$

From the above discussion, if $d_2 = 0$, then formula (11) always holds no matter whether $r_0 - 1$ is positive or not.

If $d_2 < 0$, by using the inverse of G, we know that there exists $N = N(p) > 0$ such that for $\varepsilon > 0$ small,

$$d(\varepsilon, \delta) = \Big(N\varepsilon^{\frac{1}{r}} + O(\varepsilon)\Big)\Big[-\Big(\frac{-d_2}{\varepsilon}\Big)^{\frac{1}{r}}(1 + o(\varepsilon^k)) + O(\varepsilon^{1-\frac{1}{r}})\Big] \tag{13}$$

if $r_0 > 1$, and

$$d(\varepsilon, \delta) = (N(p) + O(\varepsilon))\varepsilon[M_1(\delta) + O(\varepsilon^{\frac{1}{r}-1} + \varepsilon)] \tag{14}$$

if $r_0 < 1$. From (11)–(14), it yields that

$$\begin{aligned} &M_2(\delta) d(\varepsilon, \delta) > 0 \qquad \text{as} \quad (r_0 - 1)M_2(\delta) < 0, \\ &M_1(\delta) d(\varepsilon, \delta) > 0 \qquad \text{as} \quad (r_0 - 1)M_2(\delta) > 0. \end{aligned} \tag{15}$$

On the other hand, we can define another Poincaré map from l_3 to l_3, which gives another succession function $\overline{F}(a, \varepsilon, \delta)$. Let the domain of $\overline{F}$ be $a \geq \overline{a}_0(\varepsilon, \delta)$. It is clear that $d(\varepsilon, \delta)$ and $\overline{d}(\varepsilon, \delta)$ ($\equiv \overline{F}(a_0, \varepsilon, \delta)$) have the same sign. Moreover, similarly to (15), we can prove

$$\begin{aligned} &M_2(\delta)\overline{d}(\varepsilon, \delta) > 0 \qquad \text{as} \quad (r_0 - 1)M_1(\delta) > 0, \\ &M_1(\delta)\overline{d}(\varepsilon, \delta) > 0 \qquad \text{as} \quad (r_0 - 1)M_1(\delta) < 0. \end{aligned} \tag{16}$$

Then the conclusion follows from (15) and (16).

A limit cycle in a neighborhood of L is called large (resp., small) if it surrounds (resp., does not surround) the saddle point S. As was mentioned in [4], if $r_0 \neq 1$ then Equation (1) has at most two limit cycles in a neighborhood of L. In fact, this conclusion can be proved easily by analyzing the stability of limit cycles and the relative positions of separatrices. The following theorem gives some sufficient conditions for the existence of small and large limit cycles.

Theorem 2. *Let $r_0 \neq 1$ and $M_1(\delta)M_2(\delta) \neq 0$ for all $\delta \in D$. Then there exist $\varepsilon_0 > 0$ and a neighborhood U of L such that for $0 < \varepsilon < \varepsilon_0$ and $\delta \in D$,*

(I) *Equation* (1) *has a unique limit cycles in U and it is small as $M_1(\delta)M_2(\delta) < 0$;*
(II) *Equation* (1) *has a unique limit cycles in U and it is large as $M_1(\delta)M_2(\delta) > 0$ and $(r_0 - 1)M_2(\delta) > 0$;*
(III) Equation (1) has a precisely two limit cycles in U and they are small as $M_1(\delta)M_2(\delta) > 0$ and $(r_0 - 1)M_1(\delta) < 0$.

Proof. For definiteness we suppose $r_0 > 1$. In this case L is outer stable while L_1 and L_2 are inner stable. Moreover, any limit cycle near L is stable if it exists. Note that the outer stability of L is equivalent to $F(a, 0, \delta) < 0$ for $a > 0$ small. By $F(a_0, \varepsilon, \delta) = d(\varepsilon, \delta)$, for $\varepsilon > 0$ small F has a unique zero in $a > a_0$ if and only if $d(\varepsilon, \delta) > 0$. This means that a large limit cycle is bifurcated from L if and only if $d(\varepsilon, \delta) > 0$. In the same way, a unique small limit cycle is bifurcated from L_i if and only if $d_i < 0, i = 1, 2$. Then the conclusion follows from Theorem 1. The proof is completed.

Generally, by the Poincaré-Bendison Theorem, it is easy to prove that if $r_0 \neq 1$, then there are at most two limit cycles near L for ε small.

In the following, we treat the case that $M_1(\delta)$ or $M_2(\delta)$ has zeros. This is more interesting. We will suppose $\delta \in R$. If the vector field defined by (1) is central symmetric, then $d_1 = d_2$ and $d = 2d_1$. In this case , the bifurcation diagram of limit cycles is easy to understand. Hence, we suppose (1) is not symmetric. More precisely, we assume

$$M_1(\delta) = \delta - c_1, \quad M_2(\delta) = c_0\delta - c_2, \quad c_0 \neq 0, \quad c_0c_1 - c_2 \neq 0. \tag{17}$$

There are 8 cases to consider as follows:

Case 1. $r_0 > 1,\ c_0 > 0,\ c_1 > \dfrac{c_2}{c_0};$

Case 2. $r_0 < 1,\ c_0 > 0,\ c_1 > \dfrac{c_2}{c_0};$

Case 3. $r_0 < 1,\ c_0 < 0,\ c_1 > \dfrac{c_2}{c_0};$

Case 4. $r_0 > 1,\ c_0 < 0,\ c_1 < \dfrac{c_2}{c_0};$

Case 5. $r_0 > 1,\ c_0 > 0,\ c_1 < \dfrac{c_2}{c_0};$

Case 6. $r_0 < 1,\ c_0 > 0,\ c_1 < \dfrac{c_2}{c_0};$

Case 7. $r_0 > 1,\ c_0 < 0,\ c_1 > \dfrac{c_2}{c_0};$

Case 8. $r_0 < 1,\ c_0 < 0,\ c_1 < \dfrac{c_2}{c_0}.$

Before stating the next theorem we introduce the following definitions.

Definition 1. *Let L_i^u and L_i^s denote the saddle separatrix near L_i and belonging to the unstable and stable manifolds of the origin respectively for $\varepsilon > 0$ small, $i = 1, 2$. If*

$$L_1^u = L_1^s \equiv L_l,$$

we call L_l a left homoclinic loop. If

$$L_2^u = L_2^s \equiv L_r,$$

we call L_r a right homoclinic loop. If

$$L_1^u = L_2^s \equiv L_u,$$

we call L_u an upper homoclinic loop. If

$$L_1^s = L_2^u \equiv L_d,$$

we call L_d a down homoclinic loop.

Definition 2. *We say that the distribution of limit cycles of* (1) *is $m + (i, j)$ if it has m large cycles near L, i small cycles near L_l and j small cycles near L_r.*

By (7) and the implicit function theorem, there exist C^1 functions

$$\delta^*(\varepsilon) = c_1 + O(\varepsilon) \quad \text{and} \quad \delta_r^*(\varepsilon) = c_2/c_0 + O(\varepsilon)$$

such that

$$d_1(\varepsilon, \delta_l^*, A_1) = d_2(\varepsilon, \delta_r^*, A_2) = 0$$

and

$$(\delta - \delta_l^*)d_1 \geq 0, \quad c_0(\delta - \delta_r^*)d_2 \geq 0. \tag{18}$$

Therefore, for $\varepsilon > 0$ small, Equation (1) has a left (right) homoclinic loop if and only if $\delta = \delta_l^*(\varepsilon)$ ($\delta = \delta_r^*(\varepsilon)$). For the existence of an upper or down homoclinic loop and the distribution of limit cycles, we have

Theorem 3. *Let $r_0 \neq 1$ and* (17) *hold. Then*

(I) *There exists a continuous function $\delta_d^*(\varepsilon)$ with*

$$\delta_d^*(0) = \delta_l^*(0) \qquad \textit{for } r_0 > 1$$

and

$$\delta_d^*(0) = \delta_r^*(0) \qquad \textit{for } r_0 < 1$$

such that a down homoclinic loop exists near L for $\delta = \delta_d^*(\varepsilon)$ if and only if one of the cases 1–4 *occurs.*

(II) *There exists a continuous function $\delta_u^*(\varepsilon)$ with*

$$\delta_u^*(0) = \delta_r^*(0) \qquad \textit{for } r_0 > 1$$

and

$$\delta_u^*(0) = \delta_l^*(0) \qquad \textit{for } r_0 < 1$$

such that an upper homoclinic loop exists near L for $\delta = \delta_u^*(\varepsilon)$ if and only if one of the cases 3–6 *occurs.*

(III) *The distributions and bifurcation diagrams of limit cycles for $\varepsilon > 0$ small for all cases are given by Fig.* 3.

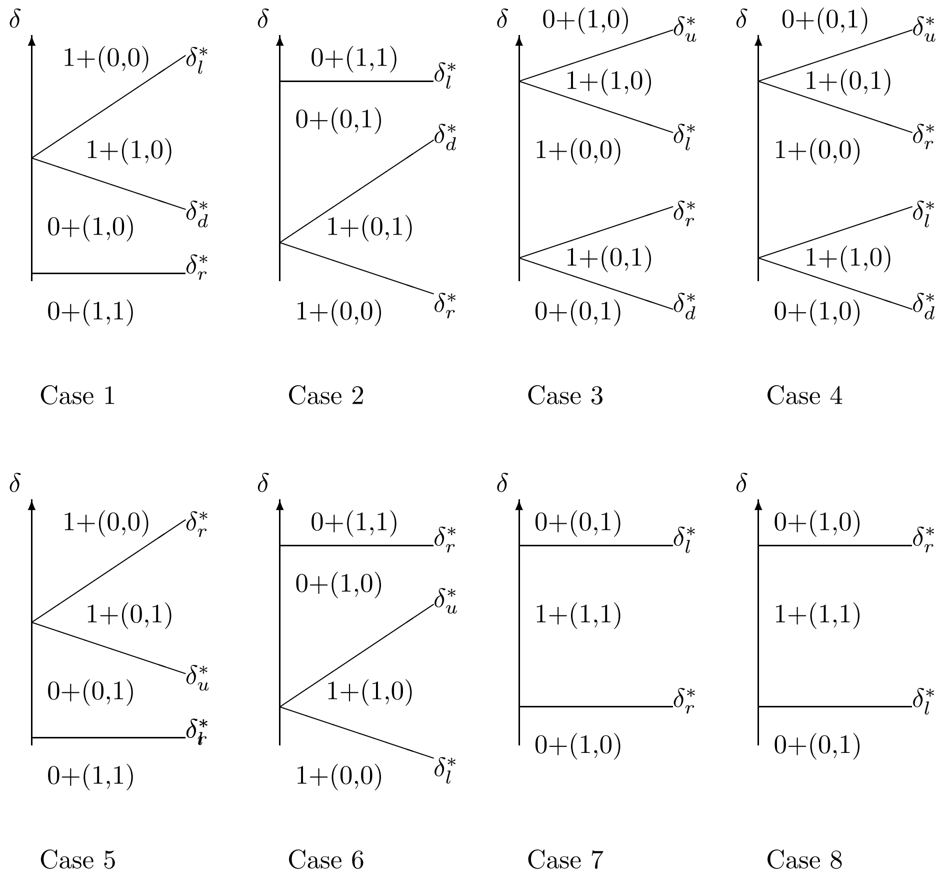

Fig. 3

Proof. We only consider the case 1 and case 4. The other cases are similar. For the case 1, we have

$$r_0 > 1, \quad c_0 > 0, \quad c_1 > c_2/c_0.$$

Hence for $\varepsilon > 0$ small,

$$\delta_l^*(\varepsilon) > \delta_r^*(\varepsilon).$$

Note that L is stable in this case. We have

$$F(a, 0, \delta) < 0 \qquad \text{for} \ \ a > 0 \ \ \text{small}.$$

Thus it follows from (18) that $d_2 > 0$ as $\delta = \delta_l^*$ and $d_1 < 0$ as $\delta = \delta_r^*$. Therefore from (8)–(11), we have

$$d(\varepsilon, \delta_l^*(\varepsilon)) > 0, \qquad d(\varepsilon, \delta_r^*(\varepsilon)) > 0. \tag{19}$$

Since $d(\varepsilon, \delta) = F(a_0, \varepsilon, \delta)$ and $F(a, \varepsilon, \delta_l^*) < 0$ for $0 < \varepsilon \ll a \ll 1$, there exists $a^*(\varepsilon) > a_0(\varepsilon, \delta_l^*)$ such that

$$F(a^*, \varepsilon, \delta_l^*) = 0.$$

This means that Equation (1) has a unique large cycle for $\delta = \delta_l^*$. By the Poincaré-Bendison Theorem, for $\varepsilon > 0$ small and $\delta > \delta_l^*(\varepsilon)$, the large cycle exists with no small cycles. By (19) and the continuity of $d(\varepsilon, \delta)$ there exist $\delta_d^*(\varepsilon) \in (\delta_r^*(\varepsilon), \delta_l^*(\varepsilon))$ such that $d(\varepsilon, \delta_d^*) = 0$. Then by (18) we have

$$d_1(\varepsilon, \delta_d^*(\varepsilon), A_1) < 0, \qquad d_2(\varepsilon, \delta_d^*(\varepsilon), A_2) > 0. \tag{20}$$

Hence, by (11), it yields that $\delta_d^*(0) = \delta_l^*(0) = c_1$ and $\delta_d^*(\varepsilon) = c_1 + O(\varepsilon^{r_0-1} + \varepsilon)$. From (20), Equation (1) has a down homoclinic loop for $\delta = \delta_d^*(\varepsilon)$. Note that Equation (1) has at most one small cycle near L_1 or L_2. The bifurcation diagram in this case follows easily.

Now suppose that the conditions in Case 4 hold. Then $\delta_l^*(\varepsilon) < \delta_r^*(\varepsilon)$. By (18), we have $d_1 > 0$ for $\delta = \delta_r^*$, and then Equation (1) has a right homoclinic loop and a unique large cycle. As δ increase from δ_r^*, the right homoclinic loop is broken and then generates a small cycle. Note that $d_2 < 0$ for $\delta > \delta_r^*$. From (7) and (13) it follows that $d(\varepsilon, \delta) = 0$ (> 0) if and only if

$$-M_2(\delta) + O(\varepsilon + \varepsilon^{r_0-1}) + o(\varepsilon^{kr_0}) = 0 \ (< 0).$$

The above equation has a unique solution

$$\delta = \delta_u^*(\varepsilon) = c_2/c_1 + o(\varepsilon).$$

Hence $d(\varepsilon, \delta_u^*) = 0$. Obviously, $\delta_u^* > \delta_r^*$ since $d(\varepsilon, \delta_r^*) > 0$. In the same way, there exists a function

$$\delta_d^*(\varepsilon) = c_1 + O(\varepsilon + \varepsilon^{r_0-1}) < \delta_l^*(\varepsilon)$$

such that $d(\varepsilon, \delta_r^*) = 0$. Then the bifurcation diagram in this case follows. The proof is completed.

Below we give an application to Theorem 3. From [5] we know that the cubic system

$$\dot{x} = y, \qquad \dot{y} = -x(x^2-1) - (x^2-a)y$$

has an unstable symmetric double homoclinic loop $L = L_1 \cup L_2 \cup O$ and a stable large cycle for some $a > 0$. Consider a perturbed system of the above in the form

$$\dot{x} = y, \qquad \dot{y} = -x(x^2-1) - (x^2-a)y + \varepsilon(bx-1)y. \tag{21}$$

From (6), we have $M_i = B_i b - A_i,\ i = 1, 2, \cdots$, where

$$A_i = \oint_{L_i} y^2 \exp \int_0^t (x^2-a)dsdt, \quad B_i = \oint_{L_i} xy^2 \exp \int_0^t (x^2-a)dsdt.$$

Obviously,

$$A_1 = A_2 > 0, \quad B_1 = -B_2 < 0.$$

Let $\delta = B_1 b$. Then we have

$$r_0 < 1, \quad c_0 = -1 < 0, \quad c_1 - c_2/c_0 = 2A_1 > 0.$$

Hence the case 3 occurs. Thus, Theorem 3 implies the existence of 4 homoclinic bifurcation curves $\delta = \delta_j^*(\varepsilon),\ j = l, r, u, d$, with

$$\delta_u^*(\varepsilon) > \delta_l^*(\varepsilon) > \delta_r^*(\varepsilon) > \delta_d^*(\varepsilon) \qquad \text{for } \ \varepsilon > 0 \ \text{ small}$$

and

$$\delta_u^*(0) = \delta_l^*(0) = A_1, \quad \delta_r^*(0) = \delta_d^*(0) = -A_1.$$

Let

$$b_j^*(\varepsilon) = \frac{1}{B_1}\delta_j^*(\varepsilon), \qquad j = l, r, u, d.$$

Then for the phase portraits of Equation (21) near L, for $\varepsilon > 0$ small we have Fig. 4.

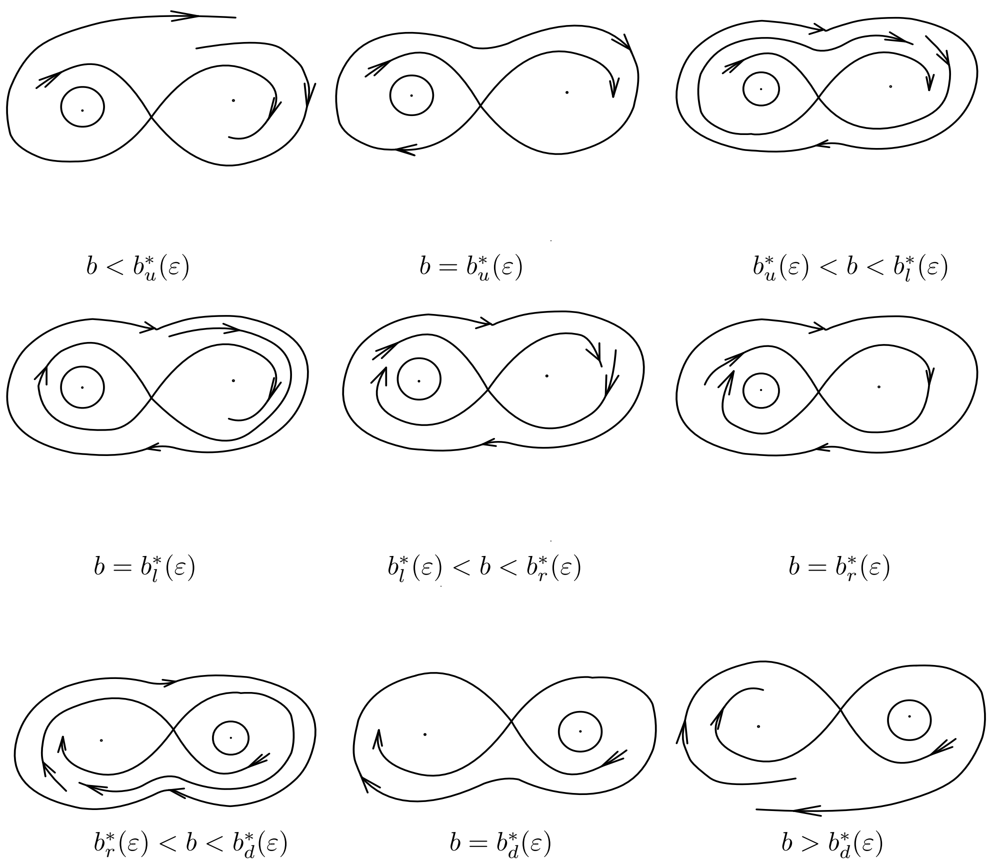

Fig. 4

Note that Equation (21) is invariant under the change $(x, y, b) \to (-x, -y, -b)$. Thus we have

$$b_d^*(\varepsilon) = -b_u^*(\varepsilon) > 0 \quad \text{and} \quad b_r^*(\varepsilon) = -b_l^*(\varepsilon) > 0.$$

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