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NON-CONSTANT POSITIVE STEADY-STATES OF A PREDATOR-PREY-MUTUALIST MODEL***

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Abstract

In this paper, the authors deal with the non-constant positive steady-states of a predator-prey-mutualist model with homogeneous Neumann boundary condition. They first give a priori estimates (positive upper and lower bounds) of positive steady-states, and then study the non-existence, the global existence and bifurcation of non-constant positive steady-states as some parameters are varied. Finally the asymptotic behavior of such solutions as $d_3 \to \infty$ is discussed.

Keywords Predator-prey-mutualist model, Non-constant positive steady-states, Bifurcation, A priori estimates
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§ 1 . Introduction

This paper deals with the positive solutions to the following elliptic system

$$\begin{cases} -d_1\Delta u_1 = \gamma u_1\Big(1-\dfrac{u_1}{L_0+\ell u_2}\Big), & \\ -d_2\Delta u_2 = \alpha u_2\Big(1-\dfrac{u_2}{K}\Big)-\dfrac{\beta u_2 u_3}{1+mu_1}, & \text{in } \Omega, \\ -d_3\Delta u_3 = u_3\Big(-s+\dfrac{c\beta u_2}{1+mu_1}\Big), & \\ \partial_n u_1 = \partial_n u_2 = \partial_n u_3 = 0, & \text{on } \partial\Omega, \end{cases} \tag{I}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, ∂_n is the directional derivative normal to $\partial\Omega$, and all parameters are positive. The non-negative solutions of (I) are in fact the non-negative steady-states (time-independent) of a reaction diffusion system known as a predator-prey-mutualist model (see [7, 10]). In this model, u_1, u_2 and u_3 represent the population densities of the mutualist, mutualist-prey and predator respectively, m and ℓ are the mutualist constants, and $d_i > 0$, $i = 1, 2, 3$, are the diffusion coefficients. For the biological meaning of the other parameters, please see [7]. In paper [10], Zheng studied the corresponding reaction diffusion system of (I) with non-negative initial data (as well as the homogeneous Dirichlet boundary conditions). He first gave the maximum norm

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estimates of non-negative time-dependent solutions, and then discussed the stabilities of non-negative constant solutions.

There are many related works on the non-constant positive steady-states of reaction diffusion systems with homogeneous Neumann boundary conditions, see [2, 4, 5, 9] and the references therein.

We first use the scaling

$$mu_1 \to u_1,\quad u_2/K \to u_2,\quad d_1/\gamma \to d_1,\quad d_2/\alpha \to d_2,\quad d_3/s \to d_3,\quad \beta/\alpha \to b,\quad \alpha c/s \to a,$$

and then denote $L_0^* = mL_0$, $\ell^* = \ell Km$, $a^* = abK$, and omit the super script $*$, the problem (I) can be written as

$$\begin{cases} -d_1\Delta u_1 = u_1\Big(1 - \dfrac{u_1}{L_0+\ell u_2}\Big) \triangleq f_1(u_1,\, u_2,\, u_3), \\ -d_2\Delta u_2 = u_2\Big(1 - u_2 - \dfrac{bu_3}{1+u_1}\Big) \triangleq f_2(u_1,\, u_2,\, u_3), & \text{in} \quad \Omega, \\ -d_3\Delta u_3 = u_3\Big(-1 + \dfrac{au_2}{1+u_1}\Big) \triangleq f_3(u_1,\, u_2,\, u_3), \\ \partial_n u_1 = \partial_n u_2 = \partial_n u_3 = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

We call $\mathbf{u} = (u_1,\, u_2,\, u_3)$ a positive solution of (1.1) provided that $\mathbf{u}$ is a solution of (1.1) and $u_i(x) > 0$ in Ω, $i = 1, 2, 3$. It is easy to verify that the problem (1.1) has positive constant solution if and only if $\ell < a,\ 1 + L_0 < a - \ell$ and this solution is unique and is given by

$$\hat{\mathbf{u}} = (\hat{u}_1,\, \hat{u}_2,\, \hat{u}_3) = \Big(\frac{\ell + aL_0}{a-\ell},\, \frac{1+L_0}{a-\ell},\, \frac{a(1+L_0)(a-1-\ell-L_0)}{b(a-\ell)^2}\Big) \tag{1.2}$$

when it exists. Throughout this paper we assume that $\ell < a,\ 1 + L_0 < a - \ell$.

This paper will be organized as follows: In §2, we first establish a priori positive upper and lower bounds for the positive solutions of (1.1). In §3, we prove the non-existence of non-constant positive solutions for a certain range of the parameters. In §4 and §5 we discuss the global existence and bifurcation of non-constant positive solutions. Finally, in §6, we study the profile of these solutions as $d_3 \to \infty$.

§ 2. A Priori Estimates

We first state one proposition, which is due to Lin, Ni and Takagi [1].

Proposition 2.1. (Harnack Inequality) (cf. [1]) *Let $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution to $\Delta w(x) + c(x)w(x) = 0$ in Ω subject to homogeneous Neumann boundary condition with $c \in C(\overline{\Omega})$. Then there exists a positive constant $C_* = C_*(n, \Omega, \|c\|_\infty)$ such that*

$$\max_{\overline{\Omega}} w \le C_* \min_{\overline{\Omega}} w.$$

Theorem 2.1. *The positive solution $\mathbf{u}$ of (1.1) satisfies*

$$\max_{\overline{\Omega}} u_1 \le L_0 + \ell, \quad \max_{\overline{\Omega}} u_2 \le 1, \quad \max_{\overline{\Omega}} u_3 \le (a/b)(1 + d_2/d_3). \tag{2.1}$$

Proof. By the simple application of the maximum principle, we can get that

$$\max_{\overline{\Omega}} u_1(x) \le L_0 + \ell, \qquad \max_{\overline{\Omega}} u_2(x) \le 1.$$

Let $w = ad_2u_2 + bd_3u_3$, then

$$\begin{cases} -\Delta w = au_2(1-u_2) - bu_3 & \text{in} \quad \Omega, \\ \partial_n w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Let $w(x_0) = \max_{\overline{\Omega}} w(x)$. By the application of the maximum principle, it yields

$$bu_3(x_0) \le au_2(x_0)(1-u_2(x_0)) \le au_2(x_0) \le a.$$

Consequently

$$bd_3 \max_{\overline{\Omega}} u_3 \le \max_{\overline{\Omega}} w = w(x_0) = ad_2u_2(x_0) + bd_3u_3(x_0) \le ad_2 + ad_3,$$

and hence

$$\max_{\overline{\Omega}} u_3 \le (a/b)(1 + d_2/d_3).$$

The proof is completed.

Theorem 2.2. *Let $d > 0$ be fixed. Then there exists a positive constant $\underline{C} = \underline{C}(d) > 0$ such that the positive solution of* (1.1) *satisfies*

$$\min_{\overline{\Omega}} u_i \ge \underline{C}, \qquad i = 1,\, 2,\, 3 \tag{2.2}$$

provided that $d_i \ge d$, $i = 1, 2, 3$.

Proof. We assume on the contrary that the (2.2) does not hold, then there exist sequences $\{d_{1i},\, d_{2i},\, d_{3i}\}_{i=1}^{\infty}$ with $d_{1i},\, d_{2i},\, d_{3i} \ge d$, and the corresponding positive solutions $(u_{1i},\, u_{2i},\, u_{3i})$ of (1.1) such that $\min_{\overline{\Omega}} u_{1i} \to 0$, or $\min_{\overline{\Omega}} u_{2i} \to 0$, or $\min_{\overline{\Omega}} u_{3i} \to 0$ as $i \to \infty$. Moreover

$$\max_{\overline{\Omega}} u_{1i} \le L_0 + \ell, \qquad \max_{\overline{\Omega}} u_{2i} \le 1 \tag{2.3}$$

by (2.1). We divide the discussion into three cases.

Case 1. $\min_{\overline{\Omega}} u_{1i} \to 0$. Proposition 2.1 yields that $u_{1i} \to 0$ uniformly on $\overline{\Omega}$ as $i \to \infty$. Integrating the equation of u_{1i}, we have

$$\int_{\Omega} u_{1i}\Big(1 - \frac{u_{1i}}{L_0 + \ell u_{2i}}\Big)dx = 0, \qquad \forall\, i \ge 1.$$

This is a contradiction since $u_{1i} > 0$ in Ω and $u_{1i} \to 0$ uniformly on $\overline{\Omega}$.

Case 2. $\min_{\overline{\Omega}} u_{2i} \to 0$. Since

$$\int_{\Omega} u_{3i}\Big(-1 + \frac{au_{2i}}{1+u_{1i}}\Big)dx = 0, \qquad u_{3i} > 0 \qquad \text{in} \quad \Omega, \quad \forall\, i \ge 1,$$

we get a contradiction.

Case 3. $\min_{\overline{\Omega}} u_{3i} \to 0$, and $\min_{\overline{\Omega}} u_{1i}$, $\min_{\overline{\Omega}} u_{2i} \ge \delta$, for some $\delta > 0$. Proposition 2.1 yields that $u_{3i} \to 0$ uniformly on $\overline{\Omega}$ as $i \to \infty$. By the regularity of elliptic equation it can be deduced that there exists a subsequence of $(u_{1i},\, u_{2i},\, u_{3i})$, denoted also by itself, and positive functions $\tilde u_1, \tilde u_2$ such that $u_{1i} \to \tilde u_1$, $u_{2i} \to \tilde u_2$ in $C^2(\overline{\Omega})$ as $i \to \infty$. Moreover,

$$\max_{\overline{\Omega}} \tilde u_1 \le L_0 + \ell, \qquad \max_{\overline{\Omega}} \tilde u_2 \le 1 \tag{2.4}$$

by (2.3). Integrating the differential equation of u_{2i}, we have

$$0 = \int_\Omega u_{2i}\Big(1 - u_{2i} - \frac{bu_{3i}}{1+u_{1i}}\Big)dx \to \int_\Omega \tilde u_2(1-\tilde u_2)dx \qquad \text{as} \;\; i \to \infty. \tag{2.5}$$

As $\tilde u_2 > 0$, it follows from (2.4) and (2.5) that $\tilde u_2 \equiv 1$. In a similar way, from the differential equation of u_{1i}, we have $\tilde u_1 \equiv L_0 + \ell$. Since $(1+L_0)/(a-\ell) < 1$ and $u_{1i} \to L_0 + \ell$, $u_{2i} \to 1$, we see that

$$-1 + au_{2i}/(1+u_{1i}) > 0, \qquad \forall\, i \gg 1.$$

It contradicts the fact that

$$\int_\Omega u_{3i}\Big(-1 + \frac{au_{2i}}{1+u_{1i}}\Big)dx = 0, \qquad u_{3i} > 0 \qquad \text{in} \;\; \Omega, \quad \forall\, i \ge 1.$$

The proof is completed.

§ 3. Non-existence of Non-constant Positive Solutions

Let $0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < \cdots$ be the eigenvalues of the operator $-\Delta$ in Ω with the homogeneous Neumann boundary condition.

Theorem 3.1. *Denote $d_0 = [2L_0^2 + \ell(L_0+\ell)^2 + (2a/b)(b+a)L_0^2]/(2\mu_1 L_0^2)$. For any given $d_2 > d_0$, there exists $D > d_2$, such that when $d_3 \ge D$, the problem* (1.1) *has no non-constant positive solution provided that $d_1 > d_0$.*

Proof. Denote $\bar f = \frac{1}{|\Omega|}\int_\Omega f(x)dx$ for $f \in L^1(\Omega)$. Assume that $\mathbf{u} = (u_1,\, u_2,\, u_3)$ is a positive solution of (1.1). We may assume that $d_3 > d_2$. By the third inequality of (2.1) we get $\max_{\overline{\Omega}} u_3 \le 2a/b$. Multiplying the i-th equation of (1.1) by $u_i - \bar u_i$, and integrating the results over Ω, we have

$$\begin{aligned} &d_i \int_\Omega |\nabla(u_i - \bar u_i)|^2 \\ =& \int_\Omega (f_i(u_1,\, u_2,\, u_3) - f_i(\bar u_1,\, \bar u_2,\, \bar u_3))(u_i - \bar u_i) \\ =& \int_\Omega \{f_{iu_i}(\xi_i)(u_i-\bar u_i)^2 + \sum_{j\ne i} f_{iu_j}(\xi_i)(u_i-\bar u_i)(u_j-\bar u_j)\}, \qquad i = 1,\, 2,\, 3, \end{aligned} \tag{3.1}$$

where $\xi_i(x)$ lies between $\mathbf{u}$ and $\bar{\mathbf{u}}$. By a simple calculation, $f_{1u_1} \le 1$, $|f_{1u_2}| \le \ell(L_0+\ell)^2/L_0^2$, $|f_{2u_1}| \le b\max_{\overline{\Omega}} u_3 \le 2a$, $f_{2u_2} \le 1$, $|f_{2u_3}| \le b$, $|f_{3u_1}|, |f_{3u_2}| \le a\max_{\overline{\Omega}} u_3 \le 2a^2/b$, and

$f_{3u_3} \le a-1$. Therefore, by (3.1), we get

$$
\begin{aligned}
&\int_\Omega \{d_1|\nabla(u_1-\bar u_1)|^2 + d_2|\nabla(u_2-\bar u_2)|^2 + d_3|\nabla(u_3-\bar u_3)|^2\}dx\\
\le &\int_\Omega \{|u_1-\bar u_1|^2 + |u_2-\bar u_2|^2 + (a-1)|u_3-\bar u_3|^2\\
&+ (\ell(L_0+\ell)^2/L_0^2 + 2a)|u_1-\bar u_1||u_2-\bar u_2|\\
&+ (2a^2/b)|u_1-\bar u_1||u_3-\bar u_3|) + (b+2a^2/b)|u_2-\bar u_2||u_3-\bar u_3|\}dx\\
\le &\int_\Omega \Big\{\Big(1+\frac{\ell(L_0+\ell)^2}{2L_0^2}+\frac{a(a+b)}{b}\Big)(u_1-\bar u_1)^2\\
&+\Big(1+\frac{\ell(L_0+\ell)^2}{2L_0^2}+\frac{a(a+b)}{b}+\varepsilon\Big)(u_2-\bar u_2)^2+\Big(a-1+\frac{2a^2}{b}+\frac{b^2}{4\varepsilon}\Big)(u_3-\bar u_3)^2\Big\}dx,
\end{aligned}
$$

and using the Poincaré inequality, we then have

$$
\begin{aligned}
&\mu_1\int_\Omega (d_1|u_1-\bar u_1|^2 + d_2|u_2-\bar u_2|^2 + d_3|u_3-\bar u_3|^2)dx\\
\le &\int_\Omega \Big\{\Big(1+\frac{\ell(L_0+\ell)^2}{2L_0^2}+\frac{a(a+b)}{b}\Big)(u_1-\bar u_1)^2\\
&+\Big(1+\frac{\ell(L_0+\ell)^2}{2L_0^2}+\frac{a(a+b)}{b}+\varepsilon\Big)(u_2-\bar u_2)^2\\
&+\Big(a-1+\frac{2a^2}{b}+\frac{b^2}{4\varepsilon}\Big)(u_3-\bar u_3)^2\Big\}dx,
\end{aligned}
\tag{3.2}
$$

where $0<\varepsilon\ll 1$. Since $d_2>d_0$, it follows from (3.2) that $u_1\equiv\bar u_1$, $u_2\equiv\bar u_2$, $u_3\equiv\bar u_3$ provided that $d_1>d_0$ and $d_3\gg 1$. The proof is completed.

§ 4 . Existence of Non-constant Positive Solutions

We shall discuss the existence of non-constant positive solutions to (1.1). Let $\hat{\mathbf{u}}=(\hat u_1, \hat u_2, \hat u_3)$, which is given by (1.2), be the unique positive constant solution of (1.1). For $d_1, d_2, d_3>0$ and $\mu\ge 0$, we define

$$
\begin{aligned}
H(\mu; d_1, d_2, d_3) =&\mu^3 + (d_1^{-1}+d_2^{-1}\hat u_2)\mu^2 + d_2^{-1}\{d_3^{-1}(1-\hat u_2)+d_1^{-1}[\hat u_2-\ell\hat u_2(1+\hat u_1)^{-1}\\
&+\ell\hat u_2^2(1+\hat u_1)^{-1}]\}\mu + (ad_1d_2d_3)^{-1}(a-1-\ell-L_0),\\
\mathcal{A}\overset{\Delta}{=}&\mathcal{A}(d_1, d_2, d_3)=\{i\ge 1\,|\,H(\mu_i; d_1, d_2, d_3)<0\}.
\end{aligned}
\tag{4.1}
$$

Then, for any given $d_1, d_2, d_3>0$, $H(\mu; d_1, d_2, d_3)=0$ has at most two positive roots, and $\mathcal{A}(d_1, d_2, d_3)$ is a finite set. Moreover, as $a>1+\ell+L_0$,

$$H(\mu_0; d_1, d_2, d_3)=H(0; d_1, d_2, d_3)>0.$$

Theorem 4.1. *Suppose $d_1, d_2, d_3>0$ with $H(\mu_i; d_1, d_2, d_3)\ne 0$ for all $i\ge 1$ and $\mathcal{A}=\mathcal{A}(d_1, d_2, d_3)\ne\emptyset$. If $\sum_{i\in\mathcal{A}} m(\mu_i)$ is odd, then the problem (1.1) has at least one non-constant positive solution, where $m(\mu_i)$ is the multiplicity of μ_i.*

Remark 4.1. Note that, for any given $\tilde{d}_1, \tilde{d}_2, \tilde{d}_3 > 0$, $H(\mu; \tilde{d}_1, \tilde{d}_2, \tilde{d}_3) = 0$ has at most two positive real roots, and $\mu_1 < \mu_2 < \cdots < \mu_k \to \infty$ as $k \to \infty$. We conclude that, for any $B(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3; \delta)$, the neighborhood of $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3)$, the intersection

$$\{(d_1, d_2, d_3) \mid H(\mu_i; d_1, d_2, d_3) \neq 0, \ \forall\, i \geq 1\} \cap B(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3; \delta)$$

is an infinite set.

Remark 4.2. Denote $\theta = 1 - \ell(1+\hat{u}_1)^{-1} + \ell\hat{u}_2(1+\hat{u}_1)^{-1}$. If we assume $a\ell > \ell^2 + (1+L_0)(a+\ell)$ (when $\ell > 1$ and $a \gg 1$, this is true), then $\theta < 0$. Note that

$$\begin{aligned}\lim_{d_3\to\infty} H(\mu; d_1, d_2, d_3) =& \mu(d_1d_2)^{-1}[d_1d_2\mu^2 + (d_2 + d_1\hat{u}_2)\mu + \hat{u}_2\theta] \\ \triangleq& (d_1d_2)^{-1}\mu h(\mu; d_1, d_2).\end{aligned} \tag{4.2}$$

For any given $k \geq 1$, there exist suitable $d_1, d_2 > 0$ (d_1, d_2 may be small) such that $h(\mu_k; d_1, d_2) < 0$ and $h(\mu_{k+1}; d_1, d_2) > 0$. Since $h(\mu; d_1, d_2)$ is increasing in μ, it follows that

$$\begin{cases} \mu_i h(\mu_i; d_1, d_2) \leq \mu_i h(\mu_k; d_1, d_2) \leq \mu_1 h(\mu_k; d_1, d_2) < 0, & \forall\, 1 \leq i \leq k, \\ \mu_i h(\mu_i; d_1, d_2) \geq \mu_i h(\mu_{k+1}; d_1, d_2) \geq \mu_1 h(\mu_{k+1}; d_1, d_2) > 0, & \forall\, i \geq k+1. \end{cases} \tag{4.3}$$

From (4.2) and (4.3) we see that there exists d_3^*, which is large and depends on d_1 and d_2, such that for all $d_3 \geq d_3^*$,

$$H(\mu_i; d_1, d_2, d_3) < 0, \quad \forall\, 1 \leq i \leq k, \qquad H(\mu_i; d_1, d_2, d_3) > 0, \quad \forall\, i \geq k+1.$$

Therefore, $\mathcal{A}(d_1, d_2, d_3) = \{1, 2, \cdots, k\}$ for such $d_1, d_2 > 0$ and $d_3 \geq d_3^*$.

Corollary 4.1. *Assume that $a\ell > \ell^2 + (1+L_0)(a+\ell)$. If $m(\mu_i)$, the multiplicity of μ_i, is odd for some $i \geq 1$. Then there exist $d_1, d_2 > 0$ and $d_3^* = d_3^*(d_1, d_2) > 0$ such that, for all $d_3 \geq d_3^*$, the problem* (1.1) *has at least one non-constant positive solution.*

Proof. Let $k \geq 1$ be the smallest one for which $m(\mu_k)$ is odd, Remark 4.2 and Theorem 4.1 conclude our result.

Theorem 3.1 shows that if the diffusion coefficients of the mutualist, mutualist-prey and predator are large, then such a predator-prey-mutualist system has no pattern phenomenon. While, Corollary 4.1 shows that when the diffusion coefficients of the mutualist and mutualist-prey are small, and the diffusion coefficient of the predator is large, then such a predator-prey-mutualist system will have pattern phenomenon.

Proof of Theorem 4.1. We assume on the contrary that the problem (1.1) has no non-constant positive solution. Choose

$$\hat{d}_1, \hat{d}_2 > d_1 + d_2 + [2L_0^2 + \ell(L_0+\ell)^2 + (2a/b)(b+a)L_0^2]/(2\mu_1 L_0^3)$$

such that

$$\begin{aligned}&(\hat{d}_1^{-1} + \hat{d}_2^{-1}\hat{u}_2)\mu_1 + \hat{d}_2^{-1}\{\hat{d}_1^{-1}\hat{u}_2 - \ell\hat{u}_2\hat{d}_1^{-1}(1+\hat{u}_1)^{-1} + \ell\hat{u}_2^2\hat{d}_1^{-1}(1+\hat{u}_1)^{-1}\} \\ =&(\hat{d}_1\hat{d}_2)^{-1}\{(\hat{d}_2 + \hat{d}_1\hat{u}_2)\mu_1 + [\hat{u}_2 - \ell\hat{u}_2(1+\hat{u}_1)^{-1} + \ell\hat{u}_2^2(1+\hat{u}_1)^{-1}]\} > 0.\end{aligned} \tag{4.4}$$

For any $i \geq 1$, applying (4.4) we have

$$\begin{aligned}&\lim_{\hat{d}_3\to\infty} H(\mu_i; \hat{d}_1, \hat{d}_2, \hat{d}_3) \\ =&\mu_i^3 + (\hat{d}_1^{-1} + \hat{d}_2^{-1}\hat{u}_2)\mu_i^2 + (\hat{d}_1\hat{d}_2)^{-1}\{\hat{u}_2 - \ell\hat{u}_2(1+\hat{u}_1)^{-1} + \ell\hat{u}_2^2(1+\hat{u}_1)^{-1}\}\mu_i \\ >&\mu_1\{\mu_1^2 + (\hat{d}_1^{-1} + \hat{d}_2^{-1}\hat{u}_2)\mu_1 + (\hat{d}_1\hat{d}_2)^{-1}[\hat{u}_2 - \ell\hat{u}_2(1+\hat{u}_1)^{-1} + \ell\hat{u}_2^2(1+\hat{u}_1)^{-1}]\} > \mu_1^3.\end{aligned} \tag{4.5}$$

From (4.5) and Theorem 3.1 we see that there exists $\hat{d}_3 \gg 1$ such that

$$\hat{d}_3 > d_3\hat{d}_2/d_2, \quad H(\mu_i;\, \hat{d}_1,\ \hat{d}_2,\ \hat{d}_3) > 0, \qquad \forall\, i \geq 1, \tag{4.6}$$

and the problem (1.1) has no non-constant positive solution for $(d_1,\, d_2,\, d_3) = (\hat{d}_1,\, \hat{d}_2,\, \hat{d}_3)$. For $0 \leq t \leq 1$, we define

$$\alpha_i(t) = \hat{d}_i^{-1} + t(d_i^{-1} - \hat{d}_i^{-1}), \qquad \mathbf{F}(t, \mathbf{u}) = \begin{pmatrix} \alpha_1(t) f_1(u_1, u_2, u_3) \\ \alpha_2(t) f_2(u_1, u_2, u_3) \\ \alpha_3(t) f_3(u_1, u_2, u_3) \end{pmatrix},$$

and consider the boundary value problem

$$-\Delta \mathbf{u} = \mathbf{F}(t, \mathbf{u}) \quad \text{in} \ \ \Omega, \qquad \partial_n \mathbf{u} = 0 \quad \text{on} \ \ \partial\Omega. \tag{4.7}$$

Then $\mathbf{u}$ is a non-constant positive solution of (1.1) if and only if it is such a solution of (4.7) for $t = 1$. Set

$$\mathbf{X} = \{(u_1,\, u_2,\, u_3) \in [C^1(\overline{\Omega})]^3 \mid \partial_n u_i = 0 \ \text{ on } \partial\Omega,\ i = 1,\, 2,\, 3\,\}.$$

For any $0 \leq t \leq 1$, $\mathbf{u}$ is a non-constant positive solution of (4.7) if and only if it solves

$$\Phi(t;\, \mathbf{u}) := \mathbf{u} - (\mathbf{I} - \Delta)^{-1}\{\, \mathbf{F}(t,\, \mathbf{u}) + \mathbf{u}\,\} = 0 \qquad \text{on} \ \ \mathbf{X},$$

where $(\mathbf{I} - \Delta)^{-1}$ is the inverse of $\mathbf{I} - \Delta$ subject to the homogeneous Neumann boundary condition. The direct computation gives

$$D_{\mathbf{u}}\Phi(t; \hat{\mathbf{u}}) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}(\mathbf{F}_{\mathbf{u}}(t, \hat{\mathbf{u}}) + \mathbf{I}),$$

where

$$\mathbf{F}_{\mathbf{u}}(t, \hat{\mathbf{u}}) = \begin{pmatrix} -\alpha_1(t) & -\alpha_1(t)\ell & 0 \\ \alpha_2(t)\hat{u}_2(1 - \hat{u}_2)/(1 + \hat{u}_1) & -\alpha_2(t)\hat{u}_2 & -b\alpha_2(t)/a \\ \alpha_3(t)(\hat{u}_2 - 1)/b & \alpha_3(t)a(1 - \hat{u}_2)/b & 0 \end{pmatrix}.$$

It is easy to calculate that

$$\det[\mu_i \mathbf{I} - \mathbf{F}_{\mathbf{u}}(1, \hat{\mathbf{u}})] = H(\mu_i; d_1, d_2, d_3), \quad \det[\mu_i \mathbf{I} - \mathbf{F}_{\mathbf{u}}(0, \hat{\mathbf{u}})] = H(\mu_i; \hat{d}_1, \hat{d}_2, \hat{d}_3).$$

Let L be the total number of eigenvalues with negative real parts (counting multiplicity) of $D_{\mathbf{u}}\Phi(1;\, \hat{\mathbf{u}})$. In order to calculate L, we decompose $\mathbf{X} = \overset{\infty}{\underset{i=0}{\oplus}} \mathbf{X}_i$, where $\mathbf{X}_i$ is the eigenspace corresponding to μ_i, i.e. $\mathbf{X}_i = \overset{m(\mu_i)}{\underset{j=1}{\oplus}} \mathbf{X}_{ij} = \overset{m(\mu_i)}{\underset{j=1}{\oplus}} \mathrm{Span}\{\varphi_{i,j}\}$, and $\{\varphi_{i,1}, \cdots, \varphi_{i,m(\mu_i)}\}$ is the base of $\mathbf{X}_i$. Each $\mathbf{X}_{ij}$ is invariant for $D_{\mathbf{u}}\Phi(1;\, \hat{\mathbf{u}})$, and ξ is an eigenvalue of $D_{\mathbf{u}}\Phi(1;\, \hat{\mathbf{u}})$ on $\mathbf{X}_{ij}$ if and only if $\xi(1 + \mu_i)$ is an eigenvalue of the matrix $\mathbf{I} - (1 + \mu_i)^{-1}(\mathbf{F}_{\mathbf{u}}(1, \hat{\mathbf{u}}) + \mathbf{I})$. Hence, on each $\mathbf{X}_{ij}$, the total number of eigenvalues with negative real parts of $D_{\mathbf{u}}\Phi(1;\, \hat{\mathbf{u}})$ is the same as that of the matrix $\mathbf{I} - (1 + \mu_i)^{-1}(\mathbf{F}_{\mathbf{u}}(1, \hat{\mathbf{u}}) + \mathbf{I})$ or the matrix $\mu_i \mathbf{I} - \mathbf{F}_{\mathbf{u}}(1, \hat{\mathbf{u}})$. Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix $\mu_i \mathbf{I} - \mathbf{F}_{\mathbf{u}}(1, \hat{\mathbf{u}})$. Then

$$\lambda_1\lambda_2\lambda_3 = \det[\,\mu_i \mathbf{I} - \mathbf{F}_{\mathbf{u}}(1, \hat{\mathbf{u}})] = H(\mu_i; d_1, d_2, d_3).$$

Therefore, when $H(\mu_i; d_1, d_2, d_3) < 0$, the number of λ_i having negative real parts is 1 or 3, and when $H(\mu_i; d_1, d_2, d_3) > 0$, the number of λ_i having negative real parts is 0 or 2. Hence, modulo 2, the number of eigenvalues with negative real parts of $D_{\mathbf{u}}\Phi(1;\, \hat{\mathbf{u}})$ on $\mathbf{X}_{ij}$ is the same as

$$\frac{1}{2}(1 - \operatorname{sgn}\{\det[\,\mu_i \mathbf{I} - \mathbf{F}_{\mathbf{u}}(1, \hat{\mathbf{u}})]\}) = \frac{1}{2}(1 - \operatorname{sgn}\{\, H(\mu_i;\, d_1,\, d_2,\, d_3)\}),$$

provided that $\det[\,\mu_i \mathbf{I} - \mathbf{F}_{\mathbf{u}}(1, \hat{\mathbf{u}})] \neq 0$. Consequently, modulo 2, the total number of eigenvalues with negative real parts of $D_{\mathbf{u}}\Phi(1;\, \hat{\mathbf{u}})$ on $\mathbf{X}_i$ is 0 if $H(\mu_i;\, d_1,\, d_2,\, d_3) > 0$, and $m(\mu_i)$ if $H(\mu_i;\, d_1,\, d_2,\, d_3) < 0$. Therefore, modulo 2, $L = \sum\limits_{i \in \mathcal{A}} m(\mu_i)$

From the estimate of (4.6), similarly to the above arguments dealing with $D_{\mathbf{u}}\Phi(1;\, \hat{\mathbf{u}})$, we have that, modulo 2, the total number of eigenvalues with negative real parts of $D_{\mathbf{u}}\Phi(0;\, \hat{\mathbf{u}})$ is 0.

Applying the first inequality of (4.6), it is easy to check that

$$\alpha_2^{-1}(t)/\alpha_3^{-1}(t) = \alpha_3(t)/\alpha_2(t) \leq d_2/d_3 \qquad \text{for all } \ 0 \leq t \leq 1.$$

By Theorem 2.1, the positive solution of (4.7) satisfies

$$\max_{\overline{\Omega}} u_1 \leq L_0 + \ell, \quad \max_{\overline{\Omega}} u_2 \leq 1, \quad \max_{\overline{\Omega}} u_3 \leq (a/b)[1 + \alpha_2^{-1}(t)/\alpha_3^{-1}(t)] \leq (a/b)(1 + d_2/d_3)$$

for all $0 \leq t \leq 1$. Denote $\overline{C} = \max\{1,\, L_0 + \ell,\, (a/b)(1 + d_2/d_3)\}$. Let $d = \min\{d_1,\, d_2,\, d_3\} > 0$. By Theorem 2.2, there exists a positive constant $\underline{C} = \underline{C}(d) > 0$ such that the positive solution of (4.7) satisfies

$$\min_{\overline{\Omega}} u_i(x) \geq \underline{C}, \qquad i = 1,\, 2,\, 3, \quad \forall\, 0 \leq t \leq 1.$$

Set

$$\Sigma = \left\{ \mathbf{u} \in \mathbf{X} \mid \ \underline{C}/2 < u_1(x),\, u_2(x),\, u_3(x) < 2\overline{C} \ \text{ on } \ \overline{\Omega} \right\}.$$

Then $\Phi(t;\, \mathbf{u}) \neq 0$ for all $\mathbf{u} \in \partial\Sigma$ and $t \in [0, 1]$. By the homotopy invariance of the Leray-Schauder degree (see [3]),

$$\deg\,(\Phi(0\,;\,\cdot),\, \Sigma,\, 0) = \deg\,(\Phi(1\,;\,\cdot),\, \Sigma,\, 0). \tag{4.8}$$

Since both equations $\Phi(0; \mathbf{u}) = 0$ and $\Phi(1; \mathbf{u}) = 0$ have the unique positive solution $\hat{\mathbf{u}}$ in Σ, it follows that

$$\begin{cases} \deg\,(\Phi(0\,;\,\cdot),\, \Sigma,\, 0) = \operatorname{index}\,(\Phi(0\,;\,\cdot),\, \hat{\mathbf{u}}) = (-1)^0 = 1, \\ \deg\,(\Phi(1\,;\,\cdot),\, \Sigma,\, 0) = \operatorname{index}\,(\Phi(1\,;\,\cdot),\, \hat{\mathbf{u}}) = (-1)^L = -1. \end{cases}$$

This contradicts (4.8) and our proof is completed.

§ 5 . Bifurcation

In this section we discuss the bifurcation of non-constant positive solutions of (1.1). Let the parameters a, b, ℓ and L_0 be fixed, and consider d_1, d_2, $d_3 > 0$ as the bifurcation parameters. We shall only consider the bifurcation with respect to the parameter d_3 when d_1 and d_2 are kept fixed.

We say that $(\hat{d}_3;\, \hat{\mathbf{u}}) \in (0,\, \infty) \times \mathbf{X}$ is a bifurcation point of (1.1) if for any $\delta \in (0,\, \hat{d}_3)$, there exists $d_3 \in [\hat{d}_3 - \delta,\, \hat{d}_3 + \delta]$ such that (1.1) has a non-constant positive solution. Otherwise, we say that $(\hat{d}_3;\, \hat{\mathbf{u}})$ is a regular point.

In the sequel, we shall use S_p to denote the positive spectrum of $-\Delta$ on Ω with the homogeneous Neumann boundary condition, i.e., $S_p = \{\mu_1,\, \mu_2, \cdots\}$. We also introduce the notation

$$\mathcal{N}(d_3) = \{\, \mu > 0 \mid H(d_1,\, d_2,\, d_3;\, \mu) = 0\} \qquad \text{for} \;\; d_3 > 0,$$

where $H(d_1,\, d_2,\, d_3;\, \mu)$ is given by (4.1). Then $\mathcal{N}(d_3)$ contains at most two elements.

Theorem 5.1. (Local Bifurcation) *Let $\hat{d}_3 > 0$ and consider the point $(\hat{d}_3;\, \hat{\mathbf{u}})$.*

(i) *If $S_p \cap \mathcal{N}(\hat{d}_3) = \emptyset$, then $(\hat{d}_3;\, \hat{\mathbf{u}})$ is a regular point of* (1.1).

(ii) *Suppose $S_p \cap \mathcal{N}(\hat{d}_3) \neq \emptyset$. If the sum $\sum_{\mu_i \in \mathcal{N}(\hat{d}_3)} m(\mu_i)$ is odd, then $(\hat{d}_3;\, \hat{\mathbf{u}})$ is a bifurcation point of* (1.1).

Proof. Define

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, \quad \mathbf{F}(\mathbf{u}) = \begin{pmatrix} f_1(u_1, u_2, u_3) \\ f_2(u_1, u_2, u_3) \\ f_3(u_1, u_2, u_3) \end{pmatrix}, \quad M_i = \mu_i \mathbf{I} - D^{-1} D_{\mathbf{u}} \mathbf{F}(\hat{\mathbf{u}}),$$

and let $\Psi(x) = \mathbf{u}(x) - \hat{\mathbf{u}}$. Then the problem (1.1) is equivalent to

$$\begin{cases} -\Delta \Psi = D^{-1} \mathbf{F}(\hat{\mathbf{u}} + \Psi), & x \in \Omega, \\ \partial_n \Psi = 0, & x \in \partial\Omega, \end{cases}$$

which, in turn, is equivalent to

$$f(d_3;\, \Psi) \stackrel{\Delta}{=} \Psi - (\mathbf{I} - \Delta)^{-1} \{ D^{-1} \mathbf{F}(\hat{\mathbf{u}} + \Psi) + \Psi \} = 0 \qquad \text{on} \;\; \mathbf{X}. \tag{5.1}$$

By direct computation, we have

$$D_\Psi f(d_3;\, 0) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1} (D^{-1} D_{\mathbf{u}} \mathbf{F}(\hat{\mathbf{u}}) + \mathbf{I}\,),$$

and as in §4, for each i, ξ is an eigenvalue of $D_\Psi f(d_3;\, 0)$ on $\mathbf{X}_i$ if and only if $\xi(1 + \mu_i)$ is an eigenvalue of the matrix M_i. Moreover, $H(d_1,\, d_2,\, d_3;\, \mu_i) = \det M_i$.

(i) If $S_p \cap \mathcal{N}(\hat{d}_3) = \emptyset$, then $\det M_i \neq 0$ for all i, i.e., 0 is not the eigenvalue of $D_\Psi f(\hat{d}_3;\, 0)$. This implies that $D_\Psi f(\hat{d}_3;\, 0)$ is a homeomorphism from $\mathbf{X}$ to itself. The implicit function theorem shows that for all d_3 close to $\hat{d}_3$, $\Psi = 0$ is the only solution to $f(d_3;\, \Psi) = 0$ in a small neighborhood of the origin, i.e., $(\hat{d}_3;\, \hat{\mathbf{u}})$ is a regular point of (5.1).

(ii) If $S_p \cap \mathcal{N}(\hat{d}_3) \neq \emptyset$, it is easy to show that 0 is a simple eigenvalue of M_i for any i satisfying $\mu_i \in S_p \cap \mathcal{N}(\hat{d}_3)$. Now, suppose on the contrary that the assertion of the theorem is false. Then there exists a $\hat{d}_3 > 0$ such that the following are true:

(a) $S_p \cap \mathcal{N}(\hat{d}_3) \neq \emptyset$, and $\sum_{\mu_i \in \mathcal{N}(\hat{d}_3)} m(\mu_i)$ is odd.

(b) There exists $\delta \in (0,\, \hat{d}_3)$ such that for every $d_3 \in [\hat{d}_3 - \delta,\, \hat{d}_3 + \delta]$, $\Psi = 0$ is the only solution to $f(d_3;\, \Psi) = 0$ in a neighborhood B_δ of the origin.

Since $f(d_3;\,\cdot)$ is a compact perturbation of an identity function, in view of (b), the Leray-Schauder degree $\deg\left(f(d_3;\,\cdot),\,B_\delta,\,0\right)$ is well defined and does not depend on $d_3 \in [\hat{d}_3-\delta,\,\hat{d}_3+\delta]$. In addition, for those $d_3 \in [\hat{d}_3-\delta,\,\hat{d}_3+\delta]$ where $D_\Psi f(d_3;\,0)$ is invertible, $\deg\left(f(d_3;\,\cdot),\,B_\delta,\,0\right) = (-1)^{\nu(d_3)}$, where $\nu(d_3)$ is the total number of eigenvalues with negative real parts (counting multiplicities) of $D_\Psi f(d_3;\,0)$.

Let

$$\widetilde{H}(d_1,\,d_2,\,d_3;\,\mu) = d_1d_2d_3H(d_1,\,d_2,\,d_3;\,\mu).$$

For $\mu_i \in S_p \cap \mathcal{N}(\hat{d}_3)$, as $\widetilde{H}(d_1,\,d_2,\,\hat{d}_3;\,\mu_i) = 0$, the direct calculation yields

$$\frac{\partial}{\partial d_3}\widetilde{H}(d_1,\,d_2,\,\hat{d}_3;\,\mu_i) = -\hat{d}_3^{-1}[a^{-1}(a-1-\ell-L_0)+d_1(1-\hat{u}_2)\mu_i] < 0.$$

Since $S_p \cap \mathcal{N}(\hat{d}_3)$ contains at most two elements, there exists $\delta \ll 1$ such that

$$\frac{\partial}{\partial d_3}\widetilde{H}(d_1,\,d_2,\,d_3;\,\mu_i) < 0$$

for all $d_3 \in [\hat{d}_3-\delta,\,\hat{d}_3+\delta]$ and $\mu_i \in S_p \cap \mathcal{N}(\hat{d}_3)$. Therefore

$$\widetilde{H}(d_1,\,d_2,\,\hat{d}_3-\delta;\,\mu_i)\widetilde{H}(d_1,\,d_2,\,\hat{d}_3+\delta;\,\mu_i) < 0,$$

and in turn,

$$H(d_1,\,d_2,\,\hat{d}_3-\delta;\,\mu_i)H(d_1,\,d_2,\,\hat{d}_3+\delta;\,\mu_i) < 0, \qquad \forall\,\mu_i \in S_p \cap \mathcal{N}(\hat{d}_3). \tag{5.2}$$

Since S_p does not have any accumulation point, by taking δ sufficiently small, we may assume that $\mathcal{N}(d_3) \cap S_p = \emptyset$ for all $d_3 \in [\hat{d}_3-\delta,\,\hat{d}_3) \cup (\hat{d}_3,\,\hat{d}_3+\delta]$. Therefore, $D_\Psi f(d_3;\,0)$ is invertible for all $d_3 \in [\hat{d}_3-\delta,\,\hat{d}_3) \cup (\hat{d}_3,\,\hat{d}_3+\delta]$.

Now, for each i and $d_3 \in [\hat{d}_3-\delta,\,\hat{d}_3+\delta]$, $\mathbf{X}_i$ is invariant under $D_\Psi f(d_3;\,0)$, and the number of eigenvalues with negative real parts of $D_\Psi f(d_3;\,0)$ on $\mathbf{X}_i$ is the same as that of the matrix M_i. Hence, modulo 2, the number of eigenvalues with negative real parts of $D_\Psi f(d_3;\,0)$ on $\mathbf{X}_i$ is the same as

$$\frac{1}{2}(1-\operatorname{sgn}\{\det M_i\}) = \frac{1}{2}(1-\operatorname{sgn}\{H(d_1,\,d_2,\,d_3;\,\mu_i)\}),$$

provided that $H(d_1,\,d_2,\,d_3;\,\mu_i) \neq 0$.

On the other hand, if $\mu_i \notin \mathcal{N}(\hat{d}_3)$ then the number of eigenvalues with negative real parts of $D_\Psi f(d_3;\,0)$ on $\mathbf{X}_i$ is independent of $d_3 \in [\hat{d}_3-\delta,\,\hat{d}_3+\delta]$; whereas if $\mu_i \in \mathcal{N}(\hat{d}_3)$ then the difference between the number of eigenvalues with negative real parts of $D_\Psi f(d_3;\,0)$ on $\mathbf{X}_i$ for $d_3 = \hat{d}_3-\delta$ and $d_3 = \hat{d}_3+\delta$ is 1 by (5.2). Thus, modulo 2, $\nu(\hat{d}_3+\delta)-\nu(\hat{d}_3-\delta)$ is equal to the sum $\sum\limits_{\mu_i \in \mathcal{N}(\hat{d}_3)} m(\mu_i)$, which is odd. Therefore, $\deg\left(f(\hat{d}_3-\delta,\,\cdot),\,B_\delta,\,0\right) \neq \deg\left(f(\hat{d}_3+\delta,\,\cdot),\,B_\delta,\,0\right)$, and we have a contradiction. This shows that $(\hat{d}_3;\,\hat{\mathbf{u}})$ is a bifurcation point of (5.1).

Theorem 5.2. (Global Bifurcation) *Let $\hat{d}_3 > 0$ and suppose that $S_p \cap \mathcal{N}(\hat{d}_3) \neq \emptyset$. If the sum $\sum\limits_{\mu_i \in \mathcal{N}(\hat{d}_3)} m(\mu_i)$ is odd, then there exists an interval $(a,\,b) \subset \mathbf{R}^+$ such that for every $d_3 \in (a,\,b)$, the problem* (1.1) *admits a non-constant positive solution $\mathbf{u} = \mathbf{u}(d_3)$. Moreover, one of the following holds:*

(i) $\hat{d}_3 = a < b < \infty$ *and* $S_p \cap \mathcal{N}(b) \neq \emptyset$;

(ii) $0 < a < b = \hat{d}_3$ *and* $S_p \cap \mathcal{N}(a) \neq \emptyset$;

(iii) $\mathbf{u}(a) = \hat{\mathbf{u}}$ *or* $\mathbf{u}(b) = \hat{\mathbf{u}}$;

(iv) $(a,\, b) = (\hat{d}_3,\, \infty)$;

(v) $(a,\, b) = (0,\, \hat{d}_3)$.

Proof. Let

$$\begin{aligned} \Gamma =& \{\, d_3 > 0 \mid S_p \cap \mathcal{N}(d_3) \neq \emptyset \,\}, \\ S =& \text{closure}\{\, (d_3,\, \mathbf{u}) \in \mathbf{R}^+ \times \mathbf{X} \mid \mathbf{u} > 0, \quad \mathbf{u} \neq \hat{\mathbf{u}}, \quad \mathbf{u} \text{ solves } (1.1) \,\}. \end{aligned}$$

In view of the estimates (2.1) and (2.2), following the arguments of [6] or [8, pp.181–183] and incorporating the calculation of the degree $\deg(f(d_3;\, \cdot),\, B_\delta,\, 0)$ that we presented in the proof of Theorem 5.1, we can conclude that S contains a component (maximal connected subset) $\mathcal{C}$ which meets $(\hat{d}_3;\, \hat{\mathbf{u}})$ such that

(1) $\mathcal{C}$ meets $\Gamma \times \{\hat{\mathbf{u}}\}$ at a point $(d_3;\, \hat{\mathbf{u}})$ with $d_3 \neq \hat{d}_3$; or

(2) $\mathcal{C}$ meets $\{d_3 > 0\} \times \{\hat{\mathbf{u}}\}$ at a point $(d_3;\, \hat{\mathbf{u}})$ with $d_3 \neq \hat{d}_3$; or

(3) $\mathcal{C}$ is non-compact in $(0,\, \infty) \times \mathbf{X}$.

Now, in the case of (1), either the assertion (i) or the assertion (ii) of the theorem holds. If (2) happens, then (iii) holds. Finally, if (3) holds, then, applying the estimates (2.1) and (2.2), we see that either (iv) or (v) holds. This completes the proof.

§ 6. Asymptotic Behavior

In §4 we proved that for the suitable ranges of the parameters $\Lambda \triangleq \{a,\, \ell,\, L_0\}$ and d_1 and d_2, the problem (1.1) has at least one non-constant positive solution for all large d_3 (Corollary 4.1). In this section we shall discuss the asymptotic behavior of such solutions as $d_3 \to \infty$.

Theorem 6.1. *Let Λ, b, d_1 and d_2 be fixed, and let $(u_{1i},\, u_{2i},\, u_{3i})$ be non-constant positive solutions of* (1.1) *with $d_3 = d_{3i}$, where, $d_{3i} \to \infty$. By passing to a subsequence if necessary, we have*

$$\lim_{i \to \infty} (u_{1i},\, u_{2i},\, u_{3i}) = (\tilde{u}_1,\, \tilde{u}_2,\, \tau),$$

where τ is a positive constant, and $(\tilde{u}_1,\, \tilde{u}_2)$ is a positive solution to the problem

$$\begin{cases} -d_1 \Delta \tilde{u}_1 = \tilde{u}_1 \Big(1 - \dfrac{\tilde{u}_1}{L_0 + \ell \tilde{u}_2}\Big) & \text{in } \Omega, \qquad \partial_n \tilde{u}_1 = 0 \quad \text{on } \partial\Omega, \\ -d_2 \Delta \tilde{u}_2 = \tilde{u}_2 \Big(1 - \tilde{u}_2 - \dfrac{b\tau}{1 + \tilde{u}_1}\Big) & \text{in } \Omega, \qquad \partial_n \tilde{u}_2 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Proof. By applying Theorem 2.1 and the regularity for elliptic equations, it follows that, for any non-negative integer k, there exists a positive constant $C = C(k,\, \Lambda,\, b,\, n)$ such that, for all $d_{3i} \geq d_2$,

$$\|u_{1i}\|_{C^k(\overline{\Omega})}, \qquad \|u_{2i}\|_{C^k(\overline{\Omega})}, \qquad \|u_{3i}\|_{C^k(\overline{\Omega})} \leq C. \tag{6.2}$$

By passing to a subsequence if necessary, $(u_{1i}, u_{2i}, u_{3i}) \to (\tilde u_1, \tilde u_2, \tilde u_3)$ in $[C^2(\Omega)]^3$ for some nonnegative functions $\tilde u_1$, $\tilde u_2$ and $\tilde u_3$. Moreover, since $d_{3,i} \to \infty$, $(\tilde u_1, \tilde u_2, \tilde u_3)$ satisfies

$$\begin{cases} -d_1\Delta\tilde u_1 = \tilde u_1\Big(1 - \dfrac{\tilde u_1}{L_0 + \ell\tilde u_2}\Big) & \text{in } \Omega, \qquad \partial_n \tilde u_1 = 0 \quad \text{on } \partial\Omega, \\ -d_2\Delta\tilde u_2 = \tilde u_2\Big(1 - \tilde u_2 - \dfrac{b\tilde u_3}{1+\tilde u_1}\Big) & \text{in } \Omega, \qquad \partial_n \tilde u_2 = 0 \quad \text{on } \partial\Omega, \\ -\Delta\tilde u_3 = 0 & \text{in } \Omega, \qquad \partial_n \tilde u_3 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

Therefore, $\tilde u_3 \equiv$ constant $\triangleq \tau \ge 0$. Integrating the differential equation for u_{3i} we have

$$\int_\Omega u_{3i}\Big(-1 + \frac{a u_{2i}}{1+u_{1i}}\Big)dx = 0, \qquad u_{3i} > 0 \qquad \text{on } \overline{\Omega}, \quad \forall\, i. \tag{6.4}$$

If $\tau = 0$, the second equation of (6.3) yields $\tilde u_2 \equiv 0$ or $\tilde u_2 \equiv 1$. Since $u_{2i} \to \tilde u_2$ uniformly on $\overline{\Omega}$, we see from (6.4) that $\tilde u_2 \equiv 0$ is impossible. Hence $\tilde u_2 \equiv 1$, and consequently, the first equation of (6.3) yields $\tilde u_1 \equiv 0$ or $\tilde u_1 \equiv \ell + L_0$. Since $u_{1i} \to \tilde u_1$ uniformly on $\overline{\Omega}$ and $a > 1 + \ell + L_0$, we see from (6.4) that neither $\tilde u_1 \equiv 0$ nor $\tilde u_1 \equiv \ell + L_0$. This contradiction shows that $\tau > 0$.

As above, $\tilde u_1 \not\equiv 0$ and $\tilde u_2 \not\equiv 0$. The maximum principle asserts that $\tilde u_1 > 0$, $\tilde u_2 > 0$ on $\overline{\Omega}$. The proof is completed.

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