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GLOBAL EXPONENTIAL STABILITY IN HOPFIELD AND BIDIRECTIONAL ASSOCIATIVE MEMORY NEURAL NETWORKS WITH TIME DELAYS***

RONG Libin*   LU Wenlian*   CHEN Tianping**

Abstract

Without assuming the boundedness, strict monotonicity and differentiability of the activation functions, the authors utilize the Lyapunov functional method to analyze the global convergence of some delayed models. For the Hopfield neural network with time delays, a new sufficient condition ensuring the existence, uniqueness and global exponential stability of the equilibrium point is derived. This criterion concerning the signs of entries in the connection matrix imposes constraints on the feedback matrix independently of the delay parameters. From a new viewpoint, the bidirectional associative memory neural network with time delays is investigated and a new global exponential stability result is given.

Keywords Hopfield neural network, Bidirectional associative memory (BAM), Global exponential stability, Time delays, Lyapunov functional
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§ 1 . Introduction

In the past years, many applications of the dynamics of artificial neural networks in different areas have attracted the increasing interest of researchers. Among the most popular models in the literature of artificial neural networks is the following continuous time system described by the set of differential equations:

$$C_i \frac{du_i(t)}{dt} = -\frac{u_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j(u_j(t)) + I_i, \qquad i = 1, 2, \cdots, n, \tag{1.1}$$

where n is the number of neurons in the network. For neuron i, u_i is the neuron voltage, $C_i \geq 0$ is the neuron amplifier input capacitance, $R_i \geq 0$ is the resistance, and I_i is the constant input electric current from outside the system. The matrix T represents the connection strengths between neurons, and if the output from neuron j excites (resp., inhibits) neuron i, then $T_{ij} \geq 0$ (resp., ≤ 0). The matrix T is assumed to be irreducible, i.e., the network is strongly connected. The functions g_j are neuron activation functions. This model was

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*Laboratory of Nonlinear Science, Institute of Mathematics, Fudan University, Shanghai 200433, China.
Department of Mathematics, and Laboratory of Nonlinear Science, Institute of Mathematics, Fudan University, Shanghai 200433, China. **E-mail: tchen@fudan.edu.cn
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proposed by [1] with an electrical circuit implementation and is thereafter referred to in the literature as Hopfield neural network.

Time delays are inevitably present in electronic neural networks due to the finite switching speed of amplifiers. A single time delay $\tau \geq 0$ was first introduced into (1.1) by [2]. They considered the following system of differential equations with time delays:

$$C_i \frac{du_i(t)}{dt} = -\frac{u_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j(u_j(t-\tau)) + I_i, \qquad i = 1, 2, \cdots, n. \tag{1.2}$$

[3] recently considered a modification of (1.2) by incorporating different delays $\tau_{ij} \geq 0$ in different communication channels (from neuron j to neuron i), namely,

$$\frac{du_i(t)}{dt} = -d_i u_i(t) + \sum_{j=1}^{n} b_{ij} g_j(u_j(t-\tau_{ij})) + I_i, \qquad i = 1, 2, \cdots, n. \tag{1.3}$$

The initial conditions associated with (1.3) are of the form

$$u_i(s) = \phi_i(s) \qquad \text{for} \quad s \in [-\tau_0, 0], \quad \text{where} \quad \tau_0 = \max_{1 \leq i,j \leq n} \tau_{ij},$$

and it is usually assumed that $\phi_i \in C([-\tau_0, 0], R)$, $i = 1, 2, \cdots, n$.

Hopfield-type neural networks (1.1), (1.2) and their various generations have been deeply investigated due to their promising application either as associative memories (or pattern recognition) or as optimization solvers. In both applications, the stability analysis of the networks is prerequisite. Indeed, when they are applied as associative memories, the equilibrium points of networks represent the stored patterns, and the stability means that the stored patterns can be retrieved even in the presence of noise. While when applied as optimization solvers, the equilibrium points of networks characterize all possible optimal solutions of the optimization problem, and the stability then ensures the convergence to the optimal solutions. In particular, the global stability ensures the convergence to an optimal solution starting from any initial guess. On the other hand, the stability is fundamental for the network design. For these reasons, the stability analysis of the Hopfield-type networks has received extensive attention and many results have been obtained. For example, [4] has proved the following global asymptotic stability theorem for system (1.3).

Theorem A. *Suppose that the activation functions g_j is globally Lipschitz with Lipschitz constant G_j, i.e., $|g_j(u_j) - g_j(v_j)| \leq G_j |u_j - v_j|$ for all u_j, v_j. If $DG^{-1} - |B|$ is an M-matrix (A real $n \times n$ matrix $A = \{a_{ij}\}$ is said to be an M-matrix if $a_{ij} \leq 0$, $i, j = 1, 2, \cdots, n$, $i \neq j$, and all successive principle minors of A are positive.), then for each $I \in R^n$, system* (1.3) *has a unique equilibrium point, which is globally asymptotically stable, independent of the delays.*

For more results on these systems, see, for instance, [5, 6, 10–15]. It should be noted that most of the results deal with the asymptotic stability. However, in designing a neural circuit, one is not only interested in questions concerning the stability, but also in performance. Particularly, it is often desired that a neural network converges in an exponential rate to ensure fast response in the network. In this work, we give detailed global exponential stability analysis of some delayed models.

Conditions in Theorem A and some previous results in the literature are explicit and easily verified in practice. But they neglect the signs of entries in the connection matrix, and thus, the differences between excitatory and inhibitory effects might be ignored. In the case of cellular neural network with time delays, [16, 17] attempted to overcome this disadvantage with the sigmoidal activation functions.

In the present paper, without assuming the boundedness, strict monotonicity and differentiability of the activation functions, we consider the following dynamical system described by differential equations with time delays:

$$\frac{du_i(t)}{dt} = -d_i u_i(t) + \sum_{j=1}^{n} b_{ij} g_j(u_j(t-\tau_j)) + I_i, \qquad i = 1, 2, \cdots, n. \tag{1.4}$$

Assume that the nonlinear system (1.4) is supplemented with initial values of the type

$$u_i(t) = \phi_i(t), \qquad t \in [-\tau_0, 0], \qquad \tau_0 = \max_{1 \le i \le n} \tau_i$$

in which $\phi_i(t)$, $i = 1, 2, \cdots, n$, are continuous functions.

We also assume that the activation function g belongs to the class $G\{G_1, G_2, \cdots, G_n\}$ of functions, defined by the property that $g \in G\{G_1, G_2, \cdots, G_n\}$ if the function $g(x)$ satisfies

$$0 \le \frac{g_i(x) - g_i(y)}{x - y} \le G_i$$

for each $x, y \in R, x \ne y$ and for $i = 1, 2, \cdots, n$.

$$G = \operatorname{diag}(G_1, G_2, \cdots, G_n)$$

and G_i satisfies $0 < G_i < +\infty,\ i = 1, 2, \cdots, n$.

The paper is organized as follows. In Section 2, the result concerning the existence and uniqueness of the equilibrium point for (1.4) is given. By use of the Lyapunov functional method we obtain a new sufficient condition of the global exponential stability for the delayed Hopfield neural network. In Section 3, the delayed bidirectional associative memory neural network is simplified and similar stability result is presented. Some concluding remarks follow in Section 4.

§ 2 . Global Exponential Stability Result

In this section, we will prove the following main result.

Theorem 2.1. *Suppose that $g \in G\{G_1, G_2, \cdots, G_n\}$ and there exist positive diagonal matrices P and Q such that*

$$2PD - PBQ^{-1}(PB)^T - GQG > 0.$$

Then, for each $I \in R^n$, system (1.4) *has a unique equilibrium point which is globally exponentially stable, independent of the delays.*

In order to prove Theorem 2.1, we prove the following

Lemma 2.1. *Suppose that the conditions in Theorem* 2.1 *hold. Then, for each $I \in R^n$, system* (1.4) *has a unique equilibrium point.*

Proof. We define a map associated with system (1.4),

$$H(u) = -Du + Bg(u) + I,$$

then it is only needed to prove that H is a homeomorphism of R^n. According to [4], we prove it with two steps: firstly, we prove that H is injective. Suppose, for purposes of contradiction, that there exist $u, v \in R_n$ with $u \neq v$ such that $H(u) = H(v)$. Thus, we have $-D(u-v) + B(g(u) - g(v)) = 0$. Since $g \in G\{G_1, G_2, \cdots, G_n\}$, there exists a positive diagonal matrix K with $0 \leq K \leq G$ such that $(-D + BK)(u - v) = 0$. Now, we prove $\det(-D + BK) \neq 0$. For this consider the model

$$\frac{dx(t)}{dt} = (-D + BK)x(t).$$

Let $V(x(t)) = x^T Px$ and calculate the derivative of V along the above system. We get

$$\dot{V}(x) = x^T(-2PD + PBK + K(PB)^T)x. \tag{2.1}$$

From the inequality

$$(Q^{-\frac{1}{2}}(PB)^T - Q^{\frac{1}{2}}K)^T(Q^{-\frac{1}{2}}(PB)^T - Q^{\frac{1}{2}}K) \geq 0,$$

we obtain

$$PBQ^{-1}(PB)^T + KQK - PBK - K(PB)^T \geq 0.$$

Thus

$$PBK + K(PB)^T \leq PBQ^{-1}(PB)^T + GQG. \tag{2.2}$$

We get

$$\dot{V}(x) \leq x^T(-2PD + PBQ^{-1}(PB)^T + GQG)x. \tag{2.3}$$

Consequently, $\dot{V}(x)$ is negative positive. So the trivial solution of the system is asymptotically stable and $\det(-D + BK) \neq 0$. Therefore $u = v$, which is a contradiction. So H is injective.

In the following, we prove that when $\|u\| \to +\infty$, $\|H(u)\| \to +\infty$.

Let $\overline{H}(u) = -Du + B\bar{g}(u)$, $\bar{g}(u) = g(u) - g(0)$, then

$$\begin{aligned}
2u^T P\overline{H} &= u^T P\overline{H} + \overline{H}^T Pu \\
&= -2u^T PDu + u^T PB\bar{g}(u) + \bar{g}^T(u)(PB)^T u \\
&= -u^T[2PD - PBQ^{-1}(PB)^T - GQG]u - u^T PBQ^{-1}(PB)^T u \\
&\quad - u^T GQGu + u^T PB\bar{g}(u) + \bar{g}^T(u)(PB)^T u \\
&\leq -u^T[2PD - PBQ^{-1}(PB)^T - GQG]u - u^T PBQ^{-1}(PB)^T u \\
&\quad - \bar{g}^T(u)Q\bar{g}(u) + u^T PB\bar{g}(u) + \bar{g}^T(u)(PB)^T u \\
&= -u^T[2PD - PBQ^{-1}(PB)^T - GQG]u \\
&\quad - [Q^{-\frac{1}{2}}(PB)^T u - Q^{\frac{1}{2}}\bar{g}(u)]^T[Q^{-\frac{1}{2}}(PB)^T u - Q^{\frac{1}{2}}\bar{g}(u)] \\
&\leq -u^T[2PD - PBQ^{-1}(PB)^T - GQG]u.
\end{aligned}$$

If $2PD - PBQ^{-1}(PB)^T - GQG \geq \mu I_n > 0$, then $2u^T P\overline{H} \leq -\mu||u||^2$. Therefore

$$2\|u\|\|P\|\|\overline{H}\| \geq \mu\|u\|^2,$$

i.e., $\|\overline{H}\| \geq \frac{\mu\|u\|}{2\|P\|}$, so when $\|u\| \to +\infty$, $\|\overline{H}(u)\| \to +\infty$, i.e., $\|H(u)\| \to +\infty$. From this, we know that H is a homeomorphism of R^n, which implies that system (1.4) has a unique equilibrium point. Lemma 2.1 is proved.

Proof of Theorem 2.1. By Lemma 2.1, system (1.4) has a unique equilibrium point, namely, u^*.

Let

$$x_i(t) = u_i(t) - u_i^*.$$

Thus, (1.4) can be rewritten as

$$\frac{dx(t)}{dt} = -Dx(t) + B\varphi(t-\tau), \tag{2.4}$$

where

$$\begin{aligned} x(t) &= (x_1(t), \cdots, x_n(t))^T, \qquad \tau = (\tau_1, \tau_2, \cdots, \tau_n)^T, \\ \varphi(t-\tau) &= (\varphi_1(x_1(t-\tau_1)), \cdots, \varphi_n(x_n(t-\tau_n)))^T, \\ \varphi_i(x_i(\cdot)) &= g_i(x_i(\cdot) + u_i^*) - g_i(u_i^*), \qquad i = 1, 2, \cdots, n. \end{aligned}$$

Obviously, system (2.4) has a unique equilibrium at $x = 0$.

Clearly, u^* is globally exponentially stable for (1.4) if and only if the trivial solution of (2.4) is globally exponentially stable.

Define the functional

$$V(x(t), t) = x^T P x e^{\epsilon t} + \sum_{i=1}^{n} Q_i \int_{t-\tau_i}^{t} \varphi_i^2(x_i(s)) e^{\epsilon(s+\tau_i)} ds, \tag{2.5}$$

where ϵ is positive and will be chosen.

We have

$$\begin{aligned} \frac{dV(x(t),t)}{dt} &= \epsilon x^T P x e^{\epsilon t} + 2x^T[-PDx + PB\varphi(t-\tau)]e^{\epsilon t} \\ &\quad + \varphi^T(t) E^{\epsilon\tau} Q \varphi(t) e^{\epsilon t} - \varphi^T(t-\tau) Q \varphi(t-\tau) e^{\epsilon t} \\ &\leq e^{\epsilon t}\{-x^T(2PD - \epsilon P - GE^{\epsilon\tau}QG)x \\ &\quad + 2x^T PB\varphi(t-\tau) - \varphi^T(t-\tau)Q\varphi(t-\tau)\} \\ &= e^{\epsilon t}\{-x^T(2PD - \epsilon P - GE^{\epsilon\tau}QG - PBQ^{-1}(PB)^T)x \\ &\quad - [Q^{\frac{1}{2}}\varphi(t-\tau) - Q^{-\frac{1}{2}}B^T P x]^T [Q^{\frac{1}{2}}\varphi(t-\tau) - Q^{-\frac{1}{2}}B^T P x]\} \\ &\leq -e^{\epsilon t}\{x^T[2PD - \epsilon P - GE^{\epsilon\tau}QG - PBQ^{-1}(PB)^T]x\}, \end{aligned}$$

where $E^{\epsilon\tau}$ donates $\mathrm{diag}(e^{\epsilon\tau_1}, \cdots, e^{\epsilon\tau_n})$, and $Q = \mathrm{diag}(Q_1, \cdots Q_n)$.

From the assumption that $2PD - PBQ^{-1}(PB)^T - GQG > 0$, we can choose the sufficiently small ϵ such that

$$2PD - \epsilon P - PBQ^{-1}(PB)^T - GE^{\epsilon\tau}QG \geq 0. \tag{2.6}$$

Thus, $\dfrac{dV(x(t),t)}{dt} \leq 0$. So

$$V(x(t), t) \leq V(x(0), 0), \qquad \sum_{i=1}^{n} x_i^2 \leq \frac{V(x(0), 0)}{\min\limits_i P_i} e^{-\epsilon t}.$$

Therefore, $x(t)$ converges to 0 exponentially. That means the origin of system (2.4) is globally exponentially stable and so is u^* for system (1.4). The proof is finished.

Remark 2.1. The above theorems represent a generalization of the results in some existing literatures. In [9–11], only the asymptotic stability was investigated. In [8], global exponential stability of Hopfield neural networks without time delays was discussed. In [25], the absolute stability was considered for the cellular neural networks. Here, we give the global exponential stability analysis. One even can obtain the global convergence rate according to (2.6). On the other hand, the constraints imposed on the activation functions are less restrictive, for example, see [5–7, 10–13, 25].

Remark 2.2. If some entries in the connection matrices are negative, there are some examples [4] for which Theorem A in Section 1 fails to be true. However, Theorem 2.1 still holds. It illustrates that the hypothesis in Theorem A is conservative. This results from the ignoring differences between excitatory and inhibitory effects.

§ 3. Stability Analysis of BAM Networks

In the present section, we investigate the existence and exponential stability of unique equilibria of bidirectional associative memory (BAM) neural network described by the following delayed model

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(y_j(t-\sigma_j)) + I_i, \\ \frac{dy_i(t)}{dt} &= -c_i y_i(t) + \sum_{j=1}^{n} d_{ij} g_j(x_j(t-\tau_j)) + J_i \end{aligned} \tag{3.1}$$

for $i = 1, 2, \cdots, n$.

If we denote $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T$, then (3.1) changes to

$$\begin{aligned} \frac{dx(t)}{dt} &= -Ax(t) + Bf(y(t-\sigma)) + I, \\ \frac{dy(t)}{dt} &= -Cy(t) + Dg(x(t-\tau)) + J, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &\sigma = (\sigma_1, \cdots, \sigma_n)^T, && f(y(t-\sigma)) = (f_1(y_1(t-\sigma_1)), \cdots, f_n(y_n(t-\sigma_n)))^T, \\ &\tau = (\tau_1, \cdots, \tau_n)^T, && g(x(t-\tau)) = (g_1(x_1(t-\tau_1)), \cdots, g_n(x_n(t-\tau_n)))^T, \\ &A = \mathrm{diag}(a_1, \cdots, a_n), && C = \mathrm{diag}(c_1, \cdots, c_n). \end{aligned}$$

System (3.1) consists of two sets of n neurons arranged on two layers, namely, I-layer and J-layer. $x_i(\cdot)$ and $y_i(\cdot)$ denote membrane potentials of i-th neurons from the I-layer and J-layer, respectively; b_{ij}, d_{ij} correspond to synaptic connection matrices. I_i, J_i denote external inputs to the neurons introduced from outside the network; σ_j, τ_j are time delays.

Obviously, model (3.1) represents a generalization of those studied by [18–22]. System (3.1) was also considered by [23]. Almost all of the works dealt with the asymptotic stability and the conditions guaranteeing the convergence ignored the signs of entries in the connection matrices. In this section, we simplify the two-layer model and obtain the global exponential stability result with more general activation functions.

Theorem 3.1. *Suppose that $f \in G\{F_1, F_2, \cdots, F_n\}$, $g \in G\{G_1, G_2, \cdots, G_n\}$ and there exist positive diagonal matrices P_1, P_2, Q_1 and Q_2 such that*

$$2P_1A - P_1BQ_2^{-1}(P_1B)^T - GQ_1G > 0, \tag{3.3}$$

$$2P_2C - P_2DQ_1^{-1}(P_2D)^T - FQ_2F > 0. \tag{3.4}$$

Then, for each $I, J \in R^n$, system (3.1) *has a unique equilibrium point, say,* $\begin{pmatrix} x^* \\ y^* \end{pmatrix}$, *which is globally exponentially stable, independent of the delays.*

Proof. Firstly, we simplify the two-layer model and the bidirectional associative memory neural network can be regarded as a single-layer system, i.e., the delayed Hopfield neural network studied in the previous section.

We let $w(t) = (x_1(t), x_2(t), \cdots, x_n(t), y_1(t), y_2(t), \cdots, y_n(t))^T$. Then system (3.1) can be rewritten as

$$\frac{dw(t)}{dt} = -\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} w(t) + \begin{pmatrix} 0 & B \\ D & 0 \end{pmatrix} h(w(t-\eta)) + \begin{pmatrix} I \\ J \end{pmatrix}, \tag{3.5}$$

where

$$\eta = (\tau_1, \tau_2, \cdots, \tau_n, \sigma_1, \sigma_2, \cdots, \sigma_n)^T, \qquad h = (g_1, g_2, \cdots, g_n, f_1, f_2, \cdots, f_n)^T,$$
$$h(w(t-\eta)) = (g_1(x_1(t-\tau_1)), \cdots, g_n(x_n(t-\tau_n)), f_1(y_1(t-\sigma_1)), \cdots, f_n(y_n(t-\sigma_n)))^T.$$

In fact, it is a Hopfield-type neural network and the connection matrix is $\begin{pmatrix} 0 & B \\ D & 0 \end{pmatrix}$, the new activation function is h. η denotes the new time delays. Since $f \in G\{F_1, F_2, \cdots, F_n\}$, $g \in G\{G_1, G_2, \cdots, G_n\}$, we get $h \in G\{G_1, G_2, \cdots, G_n, F_1, F_2, \cdots, F_n\}$.

From the conditions (3.3) and (3.4), after a direct calculation, we can obtain

$$\begin{aligned} &2\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} - \begin{pmatrix} G & 0 \\ 0 & F \end{pmatrix}\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}\begin{pmatrix} G & 0 \\ 0 & F \end{pmatrix} \\ &-\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} 0 & B \\ D & 0 \end{pmatrix}\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}^{-1}\begin{pmatrix} 0 & B \\ D & 0 \end{pmatrix}^T\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} > 0. \end{aligned}$$

Therefore, by Theorem 2.1, system (3.5) has a unique equilibrium point, say, w^*, or equivalent to $\begin{pmatrix} x^* \\ y^* \end{pmatrix}$ for system (3.1), which is globally exponentially stable, independent of the delays. The proof is completed.

Remark 3.1. In the proof, we do a simplification and the two-layer neural network can be considered as a single-layer system. While the latter have been studied extensively and many interesting works on the global stability have been obtained. By this simplification, we can get many similar results. For example, see the results in [21–24].

§ 4 . Conclusion

In this article, a class of Hopfield-type neural networks with time delays have been investigated. We use a new Lyapunov function to analyze the global exponential stability of these models and obtain a new condition guaranteeing the convergence. This stability criterion concerns the differences between excitatory and inhibitory effects on units, thus, we extend some existing results in the literature. Furthermore, we get these results with more general activation functions.

We also consider the delayed bidirectional associative memory (BAM) neural networks. The BAM models are simplified and can be regarded as the popular Hopfield-type neural networks. The exponential stability result of these two-layer systems has been established. By use of the simplification method, some existing results can be easily obtained.

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