Chin. Ann. Math.
25B:2(2004),263–274.

A CENTRAL LIMIT THEOREM FOR STRONG NEAR-EPOCH DEPENDENT RANDOM VARIABLES***

LIN Zhengyan* QIU Jin**

Abstract

In this paper, a central limit theorem for strong near-epoch dependent sequences of random variables introduced in [9] is showed. Under the same moments condition, the authors essentially weaken the "size" requirement mentioned in other papers about near epoch dependence.

Keywords Mixingale, Mixing, Near-epoch dependent, Strong near-epoch dependent, Central limit theorem
2000 MR Subject Classification 60E15, 60F15, 60F05, 60G45

§ 1 . Introduction

Mixingales cover a general class of practical and theoretical stochastic models. The concept of mixingale generalized one-step-ahead unpredictability to asymptotic unpredictability. It was introduced by Mcleish [10] for the case $p = 2$, and was extended by Andrews (1988). Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. $\{\mathcal{F}_n, n \geq 1\}$ be a sequence of σ-subfields of $\mathcal{F}$, which are increasing in n. For $p > 0$, put $\|X\|_p = (\mathsf{E}|X|^p)^{1/p}$ and $\mathsf{E}_n X = \mathsf{E}(X|\mathcal{F}_n)$.

Definition 1.1. *Let $p \geq 1$. $\{X_n, \mathcal{F}_n, n \geq 1\}$ will be called an L_p-mixingale if there exist sequences $\{a_n\}$ and $\{\mu(m)\}$ of nonnegative constants, where $\mu(m) \to 0$ as $m \to \infty$, such that for all $n \geq 1$, and $m \geq 0$,*

$$\|\mathsf{E}_{n-m} X_n\|_p \leq \mu(m) a_n,$$
$$\|X_n - \mathsf{E}_{n+m} X_n\|_p \leq \mu(m+1) a_n.$$

A function $\mu(m)$ is said to be size $-\lambda$ if $\mu(m) = O(m^{-\lambda-\varepsilon})$ for some $\varepsilon > 0$. Furthermore, we will call $\{X_n, n \geq 1\}$ "a mixingale of size $-\lambda$" if $\mu(m)$ in Definition 1.1 is of size $-\lambda$. For mixingales, the earliest result was the moment inequality of the maximum of partial sums, which was obtained by Mcleish [10] for the case of $p = 2$. From this result, one

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Manuscript received September 4, 2002.
*Department of Mathematics, Zhejiang University (Xixi Campus), Hangzhou 310028, China.
E-mail: zlin@mail.hz.zj.cn
**Department of Mathematics, Zhejiang University (Xixi Campus), Hangzhou 310028, China.
Information School, Zhejiang University of Finance and Economics, Hangzhou 310012, China.
E-mail: zjqi@mail.hz.zj.cn
***Project supported by the National Natural Science Foundation of China (No.10131040) and the Zhejiang Provincial Natural Science Foundation of China (No.199016).

obtained some convergence theorems immediately. Also with the help of it, CLT and weak convergence were proved (cf. [4, 11, 12]). Mcleish's inequality is about 2-th moment and needs "size" $-1/2$, which are also required for CLT and weak convergence of a mixingale (see [4, Assumption 1]).

The relationship between mixingales and mixing processes is similar to that between martingale differences and independent processes. Just as martingale differences need not be independent, so mixingales need not be mixing. We know that a function of a mixing sequence, which depends on an infinite number of lags and/or leads of the sequence, is not generally mixing. The definition in the following captures the idea of asymptotically mixing, which goes back to Ibragimov (1962) and had been formalized in different ways by other researchers. Let $\{V_n, n \geq 1\}$ be a sequence of random variables, and put

$$\mathcal{F}_s^t = \sigma(V_s, \cdots, V_t), \quad \mathsf{E}_s^t X = \mathsf{E}(X|\mathcal{F}_s^t).$$

Definition 1.2. *Let $p > 0$. $\{X_n, n \geq 1\}$ will be called L_p-near epoch dependent (L_p-NED) on $\{V_n, n \geq 1\}$ if there exist sequences $\{d_n\}$ and $\{\nu(m)\}$ of nonnegative constants, where $\nu(m) \to 0$ as $m \to \infty$, such that for $n \geq 1$ and $m \geq 0$,*

$$\|X_n - \mathsf{E}_{n-m}^{n+m} X_n\|_p \leq \nu(m) d_n.$$

Both mixingales and NED sequences are widely used in some fields, such as econometrics, during the recent years. For NED sequences, the terminology size, which has been defined for mixingale, is also applicable. We will call $\{X_n, n \geq 1\}$ "an L_p-NED sequence of size $-\lambda$" if $\nu(m)$ in Definition 1.2 is of size $-\lambda$. Suppose that $\{X_n, n \geq 1\}$ is an L_p-NED on a mixing sequence $\{V_n, n \geq 1\}$, and further if the former has a size related to the mixing dependence size of the latter, $\{X_n, n \geq 1\}$ is then a mixingale (cf. [3, Theorem 17.5]). Therefore, the theory of mixingale can be applied to studying L_p-NED sequence. Transfer the conditions imposed on mixingale to an NED sequence, when the r-th moment, $r > 2$, of the sequence exists, "size" $-1/2$ of NED and "size" $-r/(r-2)$ (or $-r/(2(r-1))$) of α-mixing (or φ-mixing) are required (see [4, Assumption 2]). Based on them, one showed a CLT and weak convergence for an NED sequence (cf. [2, 4, 5]). For the CLT, Jong's result seems to be the most general one. So the following two questions are interesting: What about the Mcleish inequality in the p-th moment, $p > 2$, case? Can the sizes of both NED and mixing dependence be weakened?

Lin [9] considered these questions. In order to get an analogy of the Mcleish inequality in the case of $p > 2$ and weaken "size" condition, we make a little restriction on NED concept by introducing a new class of dependent random variables, which is a subclass of NED sequence, but also "approximately" mixing. We called such sequence a strong NED sequence. In that paper, we verified two strong NED examples which are given in [3] as two NED examples, and established a maximal inequality on p-th moment, $p > 2$, under weaker dependence sizes. Using this inequality, we will show a CLT for strong NED in this paper. The conditions we supposed here are general. The moments condition is the same as that required in the CLT for the original NED sequence, but the conditions imposed on "size" of both strong NED and mixing are weakened essentially.

The following definition of strong L_p-NED was introduced in [9]. Put

$$S_k(n) = \sum_{t=k+1}^{k+n} X_t.$$

Definition 1.3. *Let $p > 0$. $\{X_n, n \geq 1\}$ will be called a strong L_p-NED sequence on $\{V_n, n \geq 1\}$ if there exist sequences $\{d_n\}$ and $\{\nu(m)\}$ of nonnegative constants, where $\nu(m) \to 0$ as $m \to \infty$ such that for all $k > 0$, $n \geq 1$ and $m \geq 0$,*

$$\|S_k(n) - \mathsf{E}_{k+1-m}^{k+n+m} S_k(n)\|_p \leq \nu(m)\Big(\sum_{j=1}^{n} d_{k+j}^2\Big)^{1/2}.$$

§ 2. The Theorem

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables with $\mathsf{E}X_n = 0$ and $\mathsf{E}|X_n|^p \leq M < \infty$ for some $p > 2$, $n = 1, 2, \cdots$. Suppose that $\{X_n, n \geq 1\}$ is strong L_p-NED on a φ-mixing sequence $\{V_n, n \geq 1\}$ with*

$$\varphi(n) = O((\log n)^{-p(1+\delta/2)}), \tag{2.1}$$

$\{d_n\}$ and $\{\nu(m)\}$ satisfying

$$\limsup_{n\to\infty} \sup_{k\geq 0} \sum_{j=1}^{n} d_{k+j}^2 / n = B < \infty, \tag{2.2}$$

$$\nu(m) = O((\log m)^{-(1+\delta/2)}), \tag{2.3}$$

$$\sigma_n^2 := \mathsf{E}S_n^2 \to \infty \qquad \text{as} \quad n \to \infty, \tag{2.4}$$

where $S_n = \sum_{j=1}^{n} X_j$. Then we have

$$S_n/\sigma_n \to N(0,1) \qquad \text{in distribution.}$$

To illustrate the generality of our theorem, we give a simple example — a kind of linear process, which arises very frequently in econometric modelling applications.

Example 2.1. Let $\{V_n, -\infty < n < \infty\}$ be a zero-mean, L_p-bounded ($p > 2$) φ-mixing sequence with mixing coefficient $\varphi(n)$ satisfying (2.1), and define a linear sequence

$$X_n = \sum_{j=-\infty}^{\infty} \theta_j V_{n-j}.$$

If the coefficients of innovation satisfy

$$\theta_j = O\big((\log|j|)^{-2-\delta/2}/j\big),$$

we obtain

$$S_n/\sigma_n \to N(0,1) \qquad \text{in distribution.}$$

Proof. Note that $\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty$, and

$$\nu(m) = 2^{1/p} \sum_{j=m+1}^{\infty} (|\theta_j| + |\theta_{-j}|) = O\big((\log(m))^{-(1+\delta/2)}\big) \to 0 \qquad \text{as} \quad m \to \infty.$$

From [9], we know that $\{X_n, n \geq 1\}$ is strong L_p-NED on $\{V_n, -\infty < n < \infty\}$ with $d_n = \sup_n \|V_n\|_p$ and $\nu(m)$ is as above. It is obvious that (2.2) and (2.4) are satisfied. According to Theorem 2.1, we have

$$S_n/\sigma_n \to N(0,1) \qquad \text{in distribution.}$$

However, when applying Jong's Theorem 2 (cf. [4]), the "size" of mixing coefficient should be $-p/(2(p-1))$ and the "size" of NED dependence be $-1/2$, which implies that

$$\theta_j = O(|j|^{-3/2-\delta}).$$

So the conditions imposed on "size" are weakened essentially in our theorem.

To prove Theorem 2.1, we need some lemmas. Define $\{l_n\}$ and $\{b_n\}$ to be nondecreasing sequences of positive integers such that $b_n/n \to 0$, $l_n/b_n \to 0$, and $r_n := [n/b_n]$.

Lemma 2.1. (cf. [4]) *Let $\{X_{nt}, 1 \leq t \leq n, n \geq 1\}$ be an array of random variables, $\{\mathcal{F}_{nt}, 1 \leq t \leq n, n \geq 1\}$ be an array of σ-fields that is increasing in t for each n. Suppose that*

(a) $\displaystyle\sum_{t=r_nb_n+1}^{n} X_{nt} \xrightarrow{p} 0,$

(b) $\displaystyle\sum_{i=1}^{r_n} \sum_{t=(i-1)b_n+1}^{(i-1)b_n+l_n} X_{nt} \xrightarrow{p} 0,$

(c) $\displaystyle\sum_{i=1}^{r_n} \mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}) \xrightarrow{p} 0,$

(d) $\displaystyle\sum_{i=1}^{r_n} (Z_{ni} - \mathsf{E}(Z_{ni}|\mathcal{F}_{ni})) \xrightarrow{p} 0,$

(e) $\displaystyle\sum_{i=1}^{r_n} (\mathsf{E}(Z_{ni}|\mathcal{F}_{ni}) - \mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}))^2 \xrightarrow{p} 1,$

(f) $\displaystyle\sum_{i=1}^{r_n} \mathsf{E}((\mathsf{E}(Z_{ni}|\mathcal{F}_{ni}) - \mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}))^2 I(|\mathsf{E}(Z_{ni}|\mathcal{F}_{ni}) - \mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1})| > \varepsilon)) \to 0$

as $n \to \infty$ for all $\varepsilon > 0$, where $Z_{ni} = \sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_{nt}$. Then

$$\sum_{t=1}^{n} X_{nt} \to N(0,1) \qquad \textit{in distribution.}$$

Lemma 2.2. (cf. [9]) *Let $\{V_n, n \geq 1\}$ be a φ-mixing sequence with mixing coefficient $\varphi(n)$ satisfying (2.1), and let $\{X_n, n \geq 1\}$ be a means zero L_p-bounded and strong L_p-NED sequence on $\{V_n\}$, $p > 2$, with $\{d_n\}$ and $\{\nu(m)\}$ satisfying (2.2) and (2.3). Then there exists a finite constant C depending only on $\{\varphi(\cdot)\}$ and $\{\nu(\cdot)\}$ such that for all positive integers k and n,*

$$\mathsf{E}\Big(\max_{1 \leq i \leq n} |S_k(i)|^p\Big) \leq C(Dn)^{p/2}, \tag{2.5}$$

where $D = B \bigvee \sup_{n \geq 1} \|X_n\|_p^2$.

Remark 2.1. For the case of $p = 2$, under the conditions of Lemma 2.2, Lin [9] showed that

$$\mathsf{E}S_k(n)^2 \le CDn,$$

where $D = B \bigvee \sup_{n\ge 1} \|X_n\|_2^2$.

Remark 2.2. From the proof of Lemma 2.2 (cf. [9, Theorem 2.1]), we can see that for fixed $k \ge 0$, the B in Lemma 2.2 can be replaced by $\sum_{j=1}^n d_{k+j}^2/n$.

Lemma 2.3. (cf. [2]) *Let X be a $\mathcal{G}$-measurable, L_p-integrable random variable for $p \ge 1$, and $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{G}$. Then*

$$\|X - \mathsf{E}(X|\mathcal{G}_2)\|_p \le 2\|X - \mathsf{E}(X|\mathcal{G}_1)\|_p.$$

Lemma 2.4. (cf. [13]) *Let $s_0 < s_1 < s_2 < s_3$ and Y be an $\mathcal{F}_{s_2}^{s_3}$-measurable, L_p-integrable random variable for $p \ge 1$. Then*

$$\|\mathsf{E}(Y|\mathcal{F}_{s_0}^{s_1}) - \mathsf{E}Y\|_p \le 2\varphi(s_2 - s_1)^{1-1/p}\|Y\|_p.$$

§ 3. Proof of Theorem 2.1

From Lemma 2.1, Theorem 2.1 will be proved if we can verify the conditions (a)–(f) in Lemma 2.1 with

$$X_{nt} = X_t/\sigma_n, \quad Z_{ni} = \sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_{nt}$$

and

$$\mathcal{F}_{nt} = \mathcal{F}_{tb_n} := \mathcal{F}_1^{tb_n}.$$

Take

$$r_n = O\big(\nu(l_n)^{-2\eta} \bigwedge \varphi(l_n)^{-1/4}\big) \qquad \text{as} \quad n \to \infty,$$

where $0 < \eta < 1/2$ is a constant specified later on.

Verification of (a). By Remark 2.1, $\sigma_n^2 = O(n)$, and hence, $b_n = o(\sigma_n^2)$. Using Lemma 2.2, we have

$$\mathsf{P}\Big(\Big|\sum_{t=r_nb_n+1}^{n} X_{nt}\Big| \ge \varepsilon\Big) \le \frac{1}{\varepsilon^p}\mathsf{E}\Big|\frac{1}{\sigma_n}\sum_{t=r_nb_n+1}^{n} X_t\Big|^p \le \frac{1}{\varepsilon^p \cdot \sigma_n^p} C(D_1 \cdot b_n)^{p/2},$$

where

$$D_1 = B_1 \bigvee \sup_{n\ge 1} \|X_n\|_p^2, \quad B_1 = \limsup_{n\to\infty} \sum_{j=1}^{b_n} d_{r_nb_n+j}^2/b_n.$$

So the right hand side of the inequality above is

$$O(\sigma_n^{-2} \cdot b_n)^{p/2} = o(1).$$

Therefore by the Markov inequality, (a) is verified.

Verification of (b). Let T denote the set $\Big\{t : t \in \bigcup_{i=1}^{r_n}[(i-1)b_n+1,(i-1)b_n+l_n]\Big\}$. Then applying Lemma 2.2, we obtain

$$\mathsf{E}\Big|\sum_{i=1}^{r_n}\sum_{t=(i-1)b_n+1}^{(i-1)b_n+l_n} X_{nt}\Big|^p = \mathsf{E}\Big|\sum_{t\in T} X_{nt}\Big|^p \le (1/\sigma_n^p)C(D_2\cdot r_n l_n)^{p/2},$$

where

$$D_2 = B_2 \bigvee \sup_{n\ge 1}\|X_n\|_p^2, \quad B_2 = \limsup_{n\to\infty}\sum_{t\in T} d_t^2/r_n l_n.$$

So the right hand side of the inequality above is

$$O\Big(\sum_{i=1}^{r_n}\sum_{t=(i-1)b_n+1}^{(i-1)b_n+l_n}\frac{d_t^2}{\sigma_n^2}\Big)^{p/2} = O(r_n l_n \sigma_n^{-2})^{p/2} = o(1),$$

which implies (b) immediately.

Verification of (c). We will show that $\mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1})$ is a strong L_p-NED sequence with respect to

$$\mathcal{H}_{i-m}^{i+m} = \sigma(\{V_{(i-m)b_n+l_n+1},\cdots,V_{(i+m)b_n}\}).$$

By Lemma 2.3, we have

$$\begin{aligned}
&\Big\|\sum_{i=k+1}^{k+s}\mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}) - \mathsf{E}\Big(\sum_{i=k+1}^{k+s}\mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1})|\mathcal{H}_{k+1-m}^{k+s+m}\Big)\Big\|_p \\
&\le \sum_{i=k+1}^{k+s}\|\mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}) - \mathsf{E}\big(\mathsf{E}(Z_{ni}|\mathcal{H}_{k+1-m}^{k+s+m})|\mathcal{F}_{n,i-1}\big)\|_p \\
&\le \sum_{i=k+1}^{k+s}\Big\|\sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_{nt} - \mathsf{E}\Big(\sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_{nt}|\mathcal{H}_{k+1-m}^{k+s+m}\Big)\Big\|_p \\
&\le 2\sum_{i=k+1}^{k+s}\Big\|\sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_{nt} - \mathsf{E}\Big(\sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_{nt}|\mathcal{F}_{(i-1)b_n+l_n+1-ml_n}^{ib_n+ml_n}\Big)\Big\|_p \\
&\le 2\sum_{i=k+1}^{k+s}\nu(ml_n)\cdot\Big(\sum_{t=(i-1)b_n+l_n+1}^{ib_n} d_t^2/\sigma_n^2\Big)^{1/2} \\
&\le 2\min(\nu(m),\nu(l_n))\sum_{i=k+1}^{k+s} c_{ni} \\
&\le c\cdot\nu(m)^{1-\eta}\Big(\sum_{i=k+1}^{k+s} c_{ni}\nu(l_n)^{\eta}\Big)^{1/2},
\end{aligned}$$

where

$$c_{ni}^2 = \sum_{t=(i-1)b_n+l_n+1}^{ib_n} d_t^2/\sigma_n^2 = O(b_n\sigma_n^{-2}) = o(1).$$

Moreover

$$\sum_{i=1}^{r_n} c_{ni} = O(r_n b_n^{1/2} \sigma_n^{-1}) = O(r_n^{1/2}),$$

so

$$\nu(l_n)^\eta \sum_{i=k+1}^{k+s} c_{ni} \le c\Big(\sum_{i=k+1}^{k+s} c_{ni}\nu(l_n)^\eta \Big)^{1/2} \qquad \text{for some } \ c > 0.$$

The η in the last inequality is chosen such that we can define

$$(1-\eta)(1+\delta/2) = 1 + \delta^*/2,$$

where $0 < \delta^* < \delta$. So $\mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1})$ is a strong L_p-NED sequence with

$$\nu^*(m) = \nu(m)^{1-\eta} = O((\log n)^{-(1+\delta^*/2)}).$$

Furthermore, note that

$$\varphi(n) = O((\log n)^{-p(1+\delta/2)}) = o((\log n)^{-p(1+\delta^*/2)}).$$

Let

$$S_k^*(j) = \sum_{i=k+1}^{k+j} \mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}).$$

Then applying the method in the proof of Lemma 2.2 and Remark 2.2, we obtain

$$\mathsf{E}(|S_k^*(r_n)|^p) \le C(D^* r_n)^{p/2},$$

where

$$D^* = B^* \bigvee \sup_{1\le i\le r_n} \|\mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1})\|_p^2, \quad B^* = \sum_{i=1}^{r_n} c_{ni}\nu(l_n)^\eta / r_n.$$

Moreover, by Lemma 2.4 and the definition of strong NED, it follows that

$$\begin{aligned}
&\|\mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1})\|_p^2 \\
&\le c_p\big(\|\mathsf{E}(\mathsf{E}(Z_{ni}|\mathcal{F}_{(i-1)b_n+[l_n/2]}^{ib_n+[l_n/2]})|\mathcal{F}_{n,i-1})\|_p^2 \\
&\quad + \|Z_{ni} - \mathsf{E}(Z_{ni}|\mathcal{F}_{(i-1)b_n+[l_n/2]}^{ib_n+[l_n/2]})\|_p^2\big) \\
&\le c_p\big(4\varphi([l_n/2])^{2-2/p}\|Z_{ni}\|_p^2 + \nu([l_n/2])^2 c_{ni}^2\big) \\
&= O\big((\varphi([l_n/2])^{2-2/p} + \nu([l_n/2])^2) b_n \sigma_n^{-2}\big).
\end{aligned} \tag{3.1}$$

So we get

$$\mathsf{E}(|S_k^*(r_n)|^p) = O\big((\varphi([l_n/2])^{2-2/p} + \nu([l_n/2])^2) r_n b_n \sigma_n^{-2} + \nu(l_n)^\eta r_n b_n^{1/2}\sigma_n^{-1}\big)^{p/2} = o(1),$$

which implies (c).

Verification of (d). The proof is analogous to the verification of (c).

Verification of (e). From the definition of X_{nt}, we should only prove

$$\sum_{i=1}^{r_n} (\mathsf{E}(Z_{ni}|\mathcal{F}_{ni}) - \mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}))^2 - \mathsf{E}\Big(\sum_{t=1}^{n} X_{nt}\Big)^2 \xrightarrow{p} 0. \tag{3.2}$$

Note that

$$\begin{aligned}
&\mathsf{E}\Big(\sum_{t=1}^{n} X_{nt}\Big)^2 - \mathsf{E}\Big(\sum_{t=1}^{r_n b_n} X_{nt}\Big)^2 \\
&= \mathsf{E}\Big(\sum_{t=r_n b_n+1}^{n} X_{nt}\Big)\Big(\sum_{t=1}^{n} X_{nt} + \sum_{t=1}^{r_n b_n} X_{nt}\Big) \\
&\le \Big\|\sum_{t=r_n b_n+1}^{n} X_{nt}\Big\|_2 \cdot \Big\|\sum_{t=1}^{n} X_{nt} + \sum_{t=1}^{r_n b_n} X_{nt}\Big\|_2 = o(1)
\end{aligned}$$

by the argument of the verification of (a). Next, we note that

$$\Big\|\sum_{i=1}^{r_n}\sum_{t=(i-1)b_n+1}^{(i-1)b_n+l_n} X_{nt}\Big\|_2 = o(1)$$

by the argument of the verification of (b). Therefore, (3.2) will be verified if we can show that

$$\sum_{i=1}^{r_n}(\mathsf{E}(Z_{ni}|\mathcal{F}_{ni}) - \mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}))^2 - \mathsf{E}\Big(\sum_{i=1}^{r_n} Z_{ni}\Big)^2 \xrightarrow{p} 0. \tag{3.3}$$

We divide the left hand side of (3.3) into three parts, that is

$$\begin{aligned}
&\sum_{i=1}^{r_n}(\mathsf{E}(Z_{ni}|\mathcal{F}_{ni}) - \mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}))^2 - \mathsf{E}\Big(\sum_{i=1}^{r_n} Z_{ni}\Big)^2 \\
&= \Big\{\sum_{i=1}^{r_n}(\mathsf{E}(Z_{ni}|\mathcal{F}_{ni}) - \mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}))^2 - \sum_{i=1}^{r_n} Z_{ni}^2\Big\} \\
&\quad + \Big\{\sum_{i=1}^{r_n} Z_{ni}^2 - \sum_{i=1}^{r_n}\mathsf{E}Z_{ni}^2\Big\} + \Big\{\sum_{i=1}^{r_n}\mathsf{E}Z_{ni}^2 - \mathsf{E}\Big(\sum_{i=1}^{r_n} Z_{ni}\Big)^2\Big\} \\
&=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}$$

Next we show that $\mathrm{I}_1 \xrightarrow{p} 0$, $\mathrm{I}_2 \xrightarrow{p} 0$ and $\mathrm{I}_3 \to 0$ respectively.

Note that

$$\begin{aligned}
\mathsf{E}|\mathrm{I}_1| &\le \sum_{i=1}^{r_n}\mathsf{E}|(\mathsf{E}(Z_{ni}|\mathcal{F}_{ni}) - \mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}) - Z_{ni}) \\
&\qquad \cdot (\mathsf{E}(Z_{ni}|\mathcal{F}_{ni}) - \mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}) + Z_{ni})| \\
&\le \sum_{i=1}^{r_n}\big(\|Z_{ni} - \mathsf{E}(Z_{ni}|\mathcal{F}_{ni})\|_2 + \|\mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1})\|_2\big) \cdot 3\|Z_{ni}\|_2.
\end{aligned}$$

Similarly to (3.1), we have

$$\|\mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1})\|_2 = O\big((\varphi([l_n/2]))^{1/2} + \nu([l_n/2]))b_n^{1/2}\sigma_n^{-1}\big),$$

and similarly

$$\|Z_{ni} - \mathsf{E}(Z_{ni}|\mathcal{F}_{ni})\|_2 = O\big((\varphi([l_n/2]))^{1/2} + \nu([l_n/2]))b_n^{1/2}\sigma_n^{-1}\big).$$

Therefore

$$\mathsf{E}|\mathrm{I}_1| = O\big((\varphi([l_n/2]))^{1/2} + \nu([l_n/2]))r_n b_n \sigma_n^{-2}\big) = o(1),$$

which implies $\mathrm{I}_1 \xrightarrow{p} 0$.

To prove $\mathrm{I}_2 \xrightarrow{p} 0$, we denote

$$h_K(x) = xI(|x| \le K) + KI(x > K) - KI(x < -K).$$

It is obvious that

$$|h_K(x) - h_K(y)| \le |x - y|.$$

Let

$$Z_{ni}^* = h_{K_\delta c_{ni}}(Z_{ni}),$$

where K_δ will be chosen later on and c_{ni} is as in the verification of (c). Note that for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathsf{E}\big(Z_{ni}^2/c_{ni}^2 I(Z_{ni}^2/c_{ni}^2 > \varepsilon)\big) \\
&\le \varepsilon^{(2-p)/2}\mathsf{E}\Big|\sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_t\Big|^p \Big/ \Big(\sum_{t=(i-1)b_n+l_n+1}^{ib_n} d_t^2\Big)^{p/2} \\
&\le C_0 \varepsilon^{(2-p)/2}
\end{aligned}$$

for some absolute constant C_0. Hence, Z_{ni}^2/c_{ni}^2 is uniformly integrable. Then

$$\begin{aligned}
&\mathsf{E}\Big|\sum_{i=1}^{r_n}(Z_{ni}^2 - Z_{ni}^{*2})\Big| \\
&\le 2\mathsf{E}\sum_{i=1}^{r_n}(Z_{ni}^2 I(Z_{ni}^2 > K_\delta^2 c_{ni}^2)) \\
&\le 2\max_{n\ge 1}\max_{1\le i\le r_n} \mathsf{E}(Z_{ni}/c_{ni})^2 I(Z_{ni}^2/c_{ni}^2 > K_\delta^2)\sum_{i=1}^{r_n} c_{ni}^2 \le \delta
\end{aligned}$$

by the uniformly integrability of Z_{ni}^2/c_{ni}^2 and the fact that

$$\sum_{i=1}^{r_n} c_{ni}^2 = O(r_n b_n \sigma_n^{-2}) = O(1),$$

provided K_δ is large enough. So if we can show

$$\lim_{n\to\infty}\Big\|\sum_{i=1}^{r_n}(Z_{ni}^{*2} - \mathsf{E}Z_{ni}^{*2})\Big\|_p = 0 \qquad \text{for any} \quad \delta > 0, \tag{3.4}$$

$\mathrm{I}_2 \xrightarrow{p} 0$ is proved.

We will show that $Z_{ni}^{*2} - \mathsf{E}Z_{ni}^{*2}$ is strong L_p-NED with respect to $\mathcal{H}_{i-m}^{i+m}$ which is denoted as in the verification of (c).

$$\begin{aligned}
&\Big\|\sum_{i=k+1}^{k+s} Z_{ni}^{*2} - \mathsf{E}\Big(\sum_{i=k+1}^{k+s}(Z_{ni}^{*2}|\mathcal{H}_{k+1-m}^{k+s+m})\Big)\Big\|_p \\
\le& \sum_{i=k+1}^{k+s}\|h_{K_\delta c_{ni}}^2(Z_{ni}) - \mathsf{E}(h_{K_\delta c_{ni}}^2(Z_{ni})|\mathcal{H}_{k+1-m}^{k+s+m})\|_p \\
\le& \sum_{i=k+1}^{k+s}\|h_{K_\delta c_{ni}}^2(Z_{ni}) - h_{K_\delta c_{ni}}^2(\mathsf{E}(Z_{ni}|\mathcal{H}_{k+1-m}^{k+s+m}))\|_p \\
\le& \sum_{i=k+1}^{k+s}\|h_{K_\delta c_{ni}}(Z_{ni}) - h_{K_\delta c_{ni}}(\mathsf{E}(Z_{ni}|\mathcal{H}_{k+1-m}^{k+s+m}))\|_p\cdot 2K_\delta c_{ni} \\
\le& \sum_{i=k+1}^{k+s}\|Z_{ni} - \mathsf{E}(Z_{ni}|\mathcal{H}_{k+1-m}^{k+s+m})\|_p\cdot 2K_\delta c_{ni} \\
\le& 2\sum_{i=k+1}^{k+s}\Big\|\sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_{nt} \\
&- \mathsf{E}\Big(\sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_{nt}|\mathcal{F}_{(i-1)b_n+l_n+1-ml_n}^{ib_n+ml_n}\Big)\Big\|_p\cdot 2K_\delta c_{ni} \\
\le& 2\sum_{i=k+1}^{k+s}\nu(ml_n)\Big(\sum_{t=(i-1)b_n+l_n+1}^{ib_n} d_t^2/\sigma_n^2\Big)^{1/2}\cdot 2K_\delta c_{ni} \\
\le& 4K_\delta\cdot\nu(ml_n)\sum_{i=k+1}^{k+s}c_{ni}^2 \\
\le& c\cdot\nu(m)^{1-\eta}\Big(\sum_{i=k+1}^{k+s}c_{ni}^2\nu(l_n)^{2\eta}\Big)^{1/2},
\end{aligned}$$

where η is as in the verification of (c). Moreover we have

$$\|Z_{ni}^{*2} - \mathsf{E}Z_{ni}^{*2}\|_p \le 2K_\delta c_{ni}\|Z_{ni}\|_p = O(b_n\sigma_n^{-2}).$$

Therefore, applying the method in the proof of Lemma 2.2 and Remark 2.2, we obtain

$$\Big\|\sum_{i=1}^{r_n}(Z_{ni}^{*2} - \mathsf{E}Z_{ni}^{*2})\Big\|_p \le C(D^{**}r_n)^{1/2},$$

where

$$D^{**} = B^{**}\bigvee\sup_{1\le i\le r_n}\|Z_{ni}^{*2} - \mathsf{E}Z_{ni}^{*2}\|_p^2,\quad B^{**} = \sum_{i=1}^{r_n}c_{ni}^2\nu(l_n)^{2\eta}/r_n.$$

So

$$\Big\|\sum_{i=1}^{r_n}(Z_{ni}^{*2} - \mathsf{E}Z_{ni}^{*2})\Big\|_p = O\big(r_nb_n^2\sigma_n^{-4} + r_nb_n\sigma_n^{-2}\nu(l_n)^{2\eta}\big)^{1/2} = o(1).$$

Then $\mathrm{I}_2 \xrightarrow{p} 0$ is proved.

Consider I_3. Let $m=[l_n/3]$ and note that

$$\begin{aligned}
\frac{1}{2}|\mathrm{I}_3| &= \Big|\sum_{i=1}^{r_n}\sum_{j=i+1}^{r_n} \mathsf{E}Z_{ni}Z_{nj}\Big| \\
&\le \sum_{i=1}^{r_n}\sum_{j=i+1}^{r_n} |\mathsf{E}(Z_{ni}-\mathsf{E}(Z_{ni}|\mathcal{F}_{(i-1)b_n+l_n+1-m}^{ib_n+m}))(Z_{nj}-\mathsf{E}(Z_{nj}|\mathcal{F}_{(j-1)b_n+l_n+1-m}^{jb_n+m}))| \\
&\quad + \sum_{i=1}^{r_n}\sum_{j=i+1}^{r_n} (|\mathsf{E}((Z_{ni}-\mathsf{E}(Z_{ni}|\mathcal{F}_{(i-1)b_n+l_n+1-m}^{ib_n+m}))\mathsf{E}(Z_{nj}|\mathcal{F}_{(j-1)b_n+l_n+1-m}^{jb_n+m}))| \\
&\quad + |\mathsf{E}(\mathsf{E}(Z_{ni}|\mathcal{F}_{(i-1)b_n+l_n+1-m}^{ib_n+m})(Z_{nj}-\mathsf{E}(Z_{nj}|\mathcal{F}_{(j-1)b_n+l_n+1-m}^{jb_n+m})))|) \\
&\quad + \sum_{i=1}^{r_n}\sum_{j=i+1}^{r_n} |\mathsf{E}(\mathsf{E}(Z_{ni}|\mathcal{F}_{(i-1)b_n+l_n+1-m}^{ib_n+m})\mathsf{E}(Z_{nj}|\mathcal{F}_{(j-1)b_n+l_n+1-m}^{jb_n+m}))| \\
&=: \mathrm{I}_{31}+\mathrm{I}_{32}+\mathrm{I}_{33}.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and the definition of strong L_p-NED, we get

$$\begin{aligned}
\mathrm{I}_{31} &\le \sum_{i=1}^{r_n}\sum_{j=i+1}^{r_n} \|Z_{ni}-\mathsf{E}(Z_{ni}|\mathcal{F}_{(i-1)b_n+l_n+1-m}^{ib_n+m})\|_2 \\
&\qquad \cdot \|Z_{nj}-\mathsf{E}(Z_{nj}|\mathcal{F}_{(j-1)b_n+l_n+1-m}^{jb_n+m})\|_2 \\
&\le \sum_{i=1}^{r_n}\sum_{j=i+1}^{r_n} \nu(m)^2 b_n\sigma_n^{-2} \\
&\le r_n^2\nu(m)^2 b_n\sigma_n^{-2} = O(\nu(l_n)^{2-2\eta}) = o(1).
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathrm{I}_{32} &\le \sum_{i=1}^{r_n}\sum_{j=i+1}^{r_n} \|Z_{ni}-\mathsf{E}(Z_{ni}|\mathcal{F}_{(i-1)b_n+l_n+1-m}^{ib_n+m})\|_2\cdot\|Z_{nj}\|_2 \\
&\quad + \sum_{i=1}^{r_n}\sum_{j=i+1}^{r_n} \|Z_{nj}-\mathsf{E}(Z_{nj}|\mathcal{F}_{(j-1)b_n+l_n+1-m}^{jb_n+m})\|_2\cdot\|Z_{ni}\|_2 \\
&\le 2r_n^2\nu(m)b_n\sigma_n^{-2} = O(\nu(l_n)^{1-2\eta}) = o(1).
\end{aligned}$$

$\{\mathsf{E}(Z_{ni}|\mathcal{F}_{(i-1)b_n+l_n+1-m}^{ib_n+m})\}$ is a φ-mixing sequence since $\{V_n, n\ge 1\}$ is φ-mixing. Noting that $m=[l_n/3]$, we get

$$\begin{aligned}
\mathrm{I}_{33} &\le \sum_{i=1}^{r_n}\sum_{j=i+1}^{r_n} \varphi(m)^{1/2}\|\mathsf{E}(Z_{ni}|\mathcal{F}_{(i-1)b_n+l_n+1-m}^{ib_n+m})\|_2 \\
&\qquad \cdot \|\mathsf{E}(Z_{nj}|\mathcal{F}_{(j-1)b_n+l_n+1-m}^{jb_n+m})\|_2 \\
&\le \varphi(m)^{1/2}\sum_{i=1}^{r_n}\sum_{j=i+1}^{r_n}\|Z_{ni}\|_2\cdot\|Z_{nj}\|_2 \\
&\le \varphi(m)^{1/2}r_n^2 b_n\sigma_n^{-2} = O\big(\varphi(m)^{1/4}\big) = o(1).
\end{aligned}$$

Therefore, combining the facts above together, we obtain $\mathrm{I}_3\to 0$.

Verification of (f). Let

$$W_{ni} = \mathsf{E}(Z_{ni}|\mathcal{F}_{ni}) - \mathsf{E}(Z_{ni}|\mathcal{F}_{n,i-1}),$$

and c_{ni} be as in the verification of (c). Then

$$\begin{aligned}\sum_{i=1}^{r_n} \mathsf{E}W_{ni}^2 I(|W_{ni}| > \varepsilon) &\le \max_{1\le i\le r_n} \mathsf{E}(W_{ni}^2/c_{ni}^2) I(|W_{ni}/c_{ni}| > \varepsilon/c_{ni}) \sum_{i=1}^{r_n} c_{ni}^2 \\ &= O\Big(\max_{1\le i\le r_n} \mathsf{E}(W_{ni}^2/c_{ni}^2) I(|W_{ni}/c_{ni}| > \varepsilon/c_{ni})\Big).\end{aligned}$$

Applying Lemma 2.2, we get

$$\begin{aligned}\|W_{ni}/c_{ni}\|_p &\le \|\mathsf{E}(Z_{ni}/c_{ni}|\mathcal{F}_{ni})\|_p + \|\mathsf{E}(Z_{ni}/c_{ni}|\mathcal{F}_{n,i-1})\|_p \\ &\le 2\|Z_{ni}/c_{ni}\|_p = O\Big(\sum_{t=(i-1)b_n+l_n+1}^{ib_n} (d_t^2/\sigma_n^2)(1/c_{ni}^2)\Big)^{1/2} = O(1),\end{aligned}$$

which implies that W_{ni}^2/c_{ni}^2 is uniformly integrable. Therefore (f) is verified.

The proof of the theorem is complete.

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