Chin. Ann. Math.
25B:2(2004),275–286.

EXISTENCE, MULTIPLICITY AND STABILITY RESULTS FOR POSITIVE SOLUTIONS OF NONLINEAR p-LAPLACIAN EQUATIONS***

MA Li* SU Ning**

Abstract

This paper studies the existence of positive solutions of the Dirichlet problem for the nonlinear equation involving p-Laplacian operator: $-\Delta_p u = \lambda f(u)$ on a bounded smooth domain Ω in $\mathbb{R}^n$. The authors extend part of the Crandall-Rabinowitz bifurcation theory to this problem. Typical examples are checked in detail and multiplicity of the solutions are illustrated. Then the stability for the associated parabolic equation is considered and a Fujita-type result is presented.

Keywords p-Laplacian operator, Bifurcation, Multiplicity, Positive solutions
2000 MR Subject Classification 35J60, 35K55, 35B32, 35B40, 92D25

§ 1. Introduction

In this work we are mainly concerned with the positive solutions of the following boundary value problem of nonlinear p-Laplacian equations

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{E_λ}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $p > 1$, $\lambda > 0$, and $f(z) \in C(\mathbb{R})$ is nonnegative, nondecreasing. In certain physical settings $f(u) = e^u$, and in some others $f(u) = u^\alpha$ or $f(u) = (1+u)^\alpha$ with $\alpha \geq 0$. The more general case, e.g. (1.1) below, can be treated in the same way. Here we want to point out that in our study of (E_λ), we often assume $f(0) \neq 0$, except when it is expressed explicitly. So our non-negative solution of (E_λ) obtained is non-trivial. We also investigate the stability of solutions of the parabolic counterparts of (E_λ):

$$\begin{cases} v_t - \Delta_p v = \lambda f(v) & \text{in} \quad \Omega \times (0,T), \\ v = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ v = v_0(x) & \text{on} \quad \Omega \times \{0\}, \end{cases} \tag{P_λ}$$

and establish a new Fujita type result for (P_λ).

Manuscript received July 24, 2002. Revised January 7, 2003.
*Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.
E-mail: lma@math.tsinghua.edu.cn
**Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.
E-mail: nsu@math.tsinghua.edu.cn
***Project supported by the 973 Project of the Ministry of Science and Technology of China (No.G1999075107) and a Scientific Grant of Tsinghua University.

The motivation of this paper is to extend the famous Crandall-Rabinowitz bifurcation theory [7] for the Dirichlet problem of semi-linear elliptic partial differential equations to nonlinear elliptic equations involving the p-Laplacian operator. To be precise, we mention the following equation

$$-\Delta_p u = \lambda V(x)e^u \qquad \text{in} \quad \Omega, \tag{1.1}$$

where V is a given positive, bounded, smooth function on Ω. If $p = 2$ and $n = 2$, the problem above was treated by T. Suzuki [22], H. Brezis and L. Merle [2]. In [1], the authors studied (1.1) with $p = n$. There have been a lot of works on nonlinear equations with p-Laplacian operator, even in the one-dimensional cases [19]. One may see [8] for more references. When $p = 2$, the equation (1.1) is usually called Emden-Fowler equation [15], and its study attracts attention of many mathematicians and scientists. See [3] and [16] for the physical background. As mentioned in [4], such equations also appear as generalized models in population dynamics. Our equation is a generalized Emden-Fowler equation. The main difficulties in our case are due to the degeneracy and the nonlinearity of the p-Laplacian operator. Although we have not completely succeeded in establishing the Crandall-Rabinowitz theory for (1.1), we do obtain some good results, which should be very useful for future study. On the other hand, many people studied the uniqueness and multiplicity of solutions of nonlinear equations with the p-Laplacian operator. See the works of Pucci and Serrin [20, 21], Dang, Schmitt and Shivaji [9], De Coster [5], and others [6, 11, 13, 14]. In this directon we will check up typical examples, exhibit multiple solutions, and establish a stability result of Fujita type. It is worthwhile to note that our result is different from Fujita's result [12], and of some new feature even in the case $p = 2$. Our main results are Theorem 3.1, Theorem 4.1, and Theorem 5.1.

The rest of the paper is organized as follows. In Section 2, we introduce some basic concepts and lemmas. Then, in Section 3, we derive a priori L^∞-estimates for the approximation solutions, and then extend part of the Crandall-Rabinowitz theory to the problem (E_λ). In Section 4, we present some concrete examples to exhibit the multiple solutions. An elementary and direct calculation shows not only the multiplicity, but also the exact number of the solutions, which will be used in the next section. The final section is devoted to establishing a stability result of Fujita type for the parabolic counterparts of problem (E_λ).

§ 2. Preliminaries

First of all, we recall the notion of solutions.

Definition 2.1. *A function $u \in W^{1,p}(\Omega)$ is called a super-solution of the problem if $-\Delta_p u \geq \lambda f(u)$ in $D'(\Omega)$, and $u \geq 0$ on $\partial\Omega$ in the trace sense. For convenience, we also write it as*

$$\begin{cases} -\Delta_p u \geq \lambda f(u) & \text{in} \quad \Omega, \\ u \geq 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The sub-solution is defined in the similar way except the inequalities are reversed. u is called a solution if it is both a super- and sub-solution.

Throughout this work only positive solutions are concerned.

The following existence result is known even in much more general case, see e.g. [17]. For completeness, we sketch the argument, which will be used in the next section.

Lemma 2.1. *If the problem* (E_λ) *has a bounded, positive super-solution, say* $U \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, *then there exists at least a bounded, positive solution* $u \le U$.

Proof. We construct a sequence of approximate solutions as follows: Set $u_0 = U$; and by induction, for every positive integer k define $u = u_{k+1} \in W_0^{1,p}(\Omega)$ as the unique solution of the following problem

$$\begin{cases} -\Delta_p u = \lambda f(u_k) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

Then from the maximum principle we see that $\{u_k\}$ is a sequence of nonnegative functions, and non-increasing in k. Consequently, $u(x) := \lim_{k\to\infty} u_k(x)$ exists on $\overline{\Omega}$. It is not hard to verify (see the argument in the next section) that u is a solution of (E_λ), and $u \le U$.

Remark 2.1. The above conclusion also applies to the case $f = f(x,u)$, provided that $f(x,u)$ is continuous in (x,u), and nondecreasing in u.

Throughout this paper we denote

$$\lambda^* = \lambda^*(\Omega) := \sup\{\lambda > 0 : \text{The problem } (E_\lambda) \text{ admits solutions}\}.$$

Lemma 2.2. *If the problem* (E_λ) *has a bounded super-solution, then* $\lambda^* \ge \lambda$.

Lemma 2.3. *The problem* (E_λ) *admits solutions for all* $\lambda \in [0, \lambda^*)$ *if* $\lambda^* \ge 0$.

Proof. In fact, if $\lambda > \lambda'$, then a solution of the problem (E_λ) must be a super-solution of the problem $(\mathrm{E}_{\lambda'})$.

Lemma 2.4. *If both* Ω_1 *and* Ω_2 *are bounded smooth domains, and* $\Omega_1 \supset \Omega_2$, *then* $\lambda^*(\Omega_1) \le \lambda^*(\Omega_2)$.

Proof. It suffices to note that a solution of the problem with $\Omega = \Omega_1$ must be a super-solution of the problem on Ω_2.

§ 3. Existence of Solutions

In order to establish the existence of solutions, we utilize a similar iteration scheme as above, except now $u_0 \in L^\infty(\Omega)$ not necessarily to be a super-solution: For any positive integer k, suppose $u_k \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is known and let $u = u_{k+1} \in W_0^{1,p}(\Omega)$ be the solution of the problem (2.1). To guarantee the iteration process available, an L^∞-estimate is necessary. Define

$$X(M) := \{v \in W_0^{1,p}(\Omega) : 0 \le v \le M \text{ a.e. }\}$$

with $M > 0$. By induction we assume that $u_k \in X(M)$ with some $M > 0$, and we will prove that $u = u_{k+1} \in X(M)$ if $\lambda > 0$ is small enough.

Proposition 3.1. *For any* $M > 0$, *there exists* $\lambda(M) > 0$ *such that if* $\lambda \in [0, \lambda(M)]$ *and* $u_k \in X(M)$, *then* $u = u_{k+1} \in X(M)$, *i.e.* $0 \le u \le M$ *a.e.*

It should be noted that the sequence $\{u_k\}$, different from that in the proof of Lemma 2.1, is not necessarily monotone. In fact, this estimate does not rely on the monotonicity property of $f(z)$, and no growth restriction on it is imposed either.

Proof. For any $s \ge 0$, we have

$$\int_\Omega |\nabla(u-s)^+|^p dx = \lambda \int_\Omega f(v)(u-s)^+ dx \le \lambda f(M) \int_\Omega (u-s)^+ dx. \tag{3.1}$$

Now we consider three cases: $p < n$, $p = n$, or $p > n$, separately.

In the case $p < n$, by the Hölder inequality and the Sobolev imbedding theorem, we get from (3.1) that

$$\begin{aligned}\int_\Omega (u-s)^+ dx &\le A_s^{1-\frac{1}{p^*}} \Big(\int_\Omega |(u-s)^+|^{p^*} dx \Big)^{\frac{1}{p^*}} \\ &\le A_s^{1-\frac{1}{p^*}} \beta \Big(\int_\Omega |\nabla (u-s)^+|^p dx \Big)^{\frac{1}{p}} \\ &\le A_s^{1-\frac{1}{p^*}} \beta \Big(\lambda f(M) \int_\Omega (u-s)^+ dx \Big)^{\frac{1}{p}},\end{aligned}$$

where $p^* = np/(n-p)$, $A_s = |\Omega \cap \{u > s\}|$, β is the Sobolev's imbedding constant. This leads to

$$\int_\Omega (u-s)^+ dx \le A_s^{1+\delta} (\beta^p \lambda f(M))^{\frac{1}{p-1}}$$

with $\delta = p/[(p-1)n] > 0$. On the other hand, for any $r > s$, denoting again $A_r = |\Omega \cap \{u > r\}|$, we have

$$A_r (r-s) \le \int_\Omega (u-s)^+ dx,$$

and hence

$$A_r \le \frac{1}{r-s} (\beta^p \lambda f(M))^{\frac{1}{p-1}} A_s^{1+\delta}. \tag{3.2}$$

Now for arbitrary $\theta \in (0,1)$, we set $s_k = M(1-\theta^k),\ k = 0, 1, 2, \cdots$, and define

$$Y_k := A_{s_k} = |\Omega \cap \{u > s_k\}|.$$

Then by the induction inequality (3.2) we have

$$Y_{k+1} \le \frac{[\beta^p \lambda f(M)]^{\frac{1}{p-1}}}{M(1-\theta)\theta^k} Y_k^{1+\delta}.$$

By the fast geometric convergence lemma (Lemma 4.1 in [10]), $\lim\limits_{k\to\infty} Y_k = 0$ follows from

$$Y_0 \le \Big[\frac{M(1-\theta)}{(\beta^p \lambda f(M))^{1/(p-1)}} \Big]^{1/\delta} \theta^{1/\delta^2}.$$

Setting

$$\lambda(M) = \frac{(1-\theta)^{p-1} \theta^{(p-1)/\delta} M^{p-1}}{\beta^p |\Omega|^{p/n} f(M)},$$

we see that if $\lambda \in [0, \lambda(M)]$, then

$$Y_0 \le |\Omega| = \Big[\frac{M(1-\theta)}{(\beta^p \lambda(M) f(M))^{\frac{1}{p-1}}} \Big]^{1/\delta} \theta^{1/\delta^2} \le \Big[\frac{M(1-\theta)}{(\beta^p \lambda f(M))^{\frac{1}{p-1}}} \Big]^{1/\delta} \theta^{1/\delta^2},$$

and therefore, $Y_k \to 0$ as $k \to \infty$, that is, $u \le M$ a.e. in Ω.

If $p = n$, we replace p^* by any $q > n$ in the above argument, and obtain the same conclusion with

$$\lambda(M) = \frac{(1-\theta)^{n-1} \theta^{(n-1)/\delta} M^{n-1}}{\beta^n |\Omega| f(M)}, \qquad \delta = \frac{q-n}{(n-1)q},$$

where $\beta = \beta(q)$ is the Sobolev's constant of imbedding $W_0^{1,n}(\Omega) \to L^q(\Omega)$, namely,

$$\|u\|_{L^q(\Omega)} \leq \beta|\Omega|^{\frac{1}{q}}\|\nabla u\|_{L^n(\Omega)}.$$

In the case $p > n$, by the imbedding theorem, we get from (3.1) that

$$\|u\|_{L^\infty(\Omega)} \leq \beta|\Omega|^{\frac{1}{n}-\frac{1}{p}}\|\nabla u\|_{L^p(\Omega)} \leq \beta|\Omega|^{\frac{1}{n}-\frac{1}{p}}\left(\lambda f(M)|\Omega|\right)^{\frac{1}{p}}\|u\|_{L^\infty(\Omega)}^{\frac{1}{p}},$$

and thus

$$\|u\|_{L^\infty(\Omega)} \leq (\beta^p\lambda|\Omega|^{\frac{p}{n}}f(M))^{\frac{1}{p-1}}.$$

Hence, $\|u\|_{L^\infty(\Omega)} \leq M$ follows from $0 \leq \lambda \leq \lambda(M) = \beta^{-p}|\Omega|^{-p/n}f(M)^{-1}M^{p-1}$.

Using this proposition, we derive the following result.

Theorem 3.1. *There exists $\lambda^* \in (0,\infty]$ such that for any $\lambda \in [0,\lambda^*)$, the problem (E_λ) admits at least one solution $u \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.*

Proof. We use notation and the iteration process as before. Fix any $M > 0$. We let $\lambda \in (0,\lambda(M))$, and take $u_0 \in X(M)$ (arbitrarily). Then for every k and the solution u_k of the problem (2.1), we have $0 \leq u_k \leq M$, and further, $\|u_k\|_{W^{1,p}(\Omega)} \leq C(M)$ for all $k \geq 0$. Therefore there exist subsequence of $\{u_k\}$, denoted by $\{u_k\}$ again, $u \in X(M)$ and $w \in L^{p'}(\Omega)$ such that

$$\begin{aligned} u_k &\rightharpoonup u \quad &&\text{in} \quad W^{1,p}(\Omega),\\ u_k &\to u \quad &&\text{a.e. in} \quad \Omega,\\ |\nabla u_k|^{p-2}\nabla u_k &\rightharpoonup w \quad &&\text{in} \quad L^{p'}(\Omega), \end{aligned}$$

with $p' = p/(p-1)$. Finally, a monotone argument (Minty technique) can readily show that $w = |\nabla u|^{p-2}\nabla u$, which evidently implies that u is a solution of (E_λ), namely,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx = \lambda\int_\Omega f(u)\varphi dx$$

holds for any $\varphi \in W_0^{1,p}(\Omega)$. Furthermore, by the regularity theory of p-Laplace equations (cf. [18]) $u \in C^{1,\alpha}(\overline{\Omega})$ with $0 < \alpha < 1$. This completes the proof.

This result can be considered as a generalized version of the Crandall-Rabinowitz theory [7].

Remark 3.1. This theorem also applies to the general case where $f = f(x,u)$, which will be used in the following sections. From above L^∞-estimation process we may present a lower bound for λ^* as below:

$$\lambda^* \geq \sup_{M>0,0<\theta<1,p<q<p^*}\frac{(1-\theta)^{p-1}\theta^{(p-1)/\delta}M^{p-1}}{\beta^p|\Omega|^{p/n}\sup\limits_{\Omega\times[0,M]}f(x,z)},$$

where $f(x,z)$ is nonnegative, continuous on $\overline{\Omega}\times[0,\infty)$, $\delta = \delta(q) = (q-p)/[(p-1)q]$, and $\beta = \beta(q)$ is the imbedding constant of $W^{1,p}(\Omega) \to L^q(\Omega)$.

Corollary 3.1. *If $\liminf\limits_{z\to\infty} z^{p-1}/f(z) = \infty$, then $\lambda^* = \infty$, that is, the problem (E_λ) has solutions for all $\lambda \geq 0$.*

Remark 3.2. In the case that function $f(z)$ is nondecreasing, the iteration procedure above can be used to produce the minimal solution. In fact, if we denote $\underline{u}_\lambda := \lim_{k\to\infty} u_k$ with $u_0 \equiv 0$, and let u be a solution, then

$$u_0 \le u_1 \le \cdots \le u_{k-1} \le u_k \le \cdots \le u$$

follows from the induction argument and the comparison principle, and hence $\underline{u}_\lambda \le u$. Moreover, if $0 \le \lambda_1 < \lambda_2 < \lambda^*$, then $\underline{u}_{\lambda_1} \le \underline{u}_{\lambda_2}$.

§ 4. Multiplicity of Solutions: Typical Examples

This section contains some important examples. It should be pointed out that some of these results are not new and may be found or deduced from the results in, e.g. [5, 9, 13]. However, by elementary calculation we can show not only the multiplicity of positive solutions, but also the exact number of the solutions.

To begin with, let us check the one-dimensional case.

4.1. One-dimensional Case

Consider the solution of the following problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(u) & \text{in } (a,b), \\ u(a) = u(b) = 0. \end{cases} \tag{4.1}$$

It is not hard to see that if $u(x)$ is a nontrivial solution, then it reaches maximum at $x = (a+b)/2$, say $u((a+b)/2) = M > 0$, and satisfies

$$\Big(1 - \frac{1}{p}\Big)|u'(x)|^p + \lambda \int_M^{u(x)} f(s)ds = 0,$$

and thus

$$u'(x) = -\mathrm{sgn}\Big(x - \frac{a+b}{2}\Big)\Big[\Big(1-\frac{1}{p}\Big)^{-1}\lambda \int_{u(x)}^M f(s)ds\Big]^{1/p}.$$

Consequently, we arrive at

$$\int_0^u \Big[\int_s^M f(t)dt\Big]^{-1/p} ds = \Big[\Big(1-\frac{1}{p}\Big)^{-1}\lambda\Big]^{1/p}\Big(\frac{b-a}{2} - \Big|x - \frac{a+b}{2}\Big|\Big).$$

From this we derive a sufficient and necessary condition for the problem to have a non-trivial solution: There exists $M > 0$ satisfying

$$\lambda = \Big(1-\frac{1}{p}\Big)\Big(\frac{2}{b-a}\Big)^p\Big[\int_0^M \Big[\int_s^M f(t)dt\Big]^{-1/p} ds\Big]^p. \tag{4.2}$$

Moreover, there exists a one-to-one mapping between the solutions and the roots of equation $I(y) = \beta$, where

$$I(y) := \int_0^y \Big[\int_s^y f(t)dt\Big]^{-1/p} ds, \quad \beta = \Big(\frac{b-a}{2}\Big)\Big[\Big(1-\frac{1}{p}\Big)^{-1}\lambda\Big]^{1/p}. \tag{4.3}$$

In summary, we have

Theorem 4.1. *The problem* (4.1) *has a solution* u *satisfying* $\max u = M$ *if and only if the equation* $I(y) = \beta$ *possesses a root* $y = M$*.*

Example 4.1. Let $f(u) = u^\alpha$ with $\alpha \geq 0$. Of course, the problem has minimal solution $\underline{u} \equiv 0$ for any λ. Condition (4.2) reduces to

$$\begin{aligned}\lambda &= (\alpha+1)\Big(\frac{2}{b-a}\Big)^p\Big(1-\frac{1}{p}\Big)\Big[\int_0^M (M^{\alpha+1}-s^{\alpha+1})^{-1/p}ds\Big]^p \\ &= (\alpha+1)^{1-p}\Big(\frac{2}{b-a}\Big)^p\Big(1-\frac{1}{p}\Big)M^{p-(\alpha+1)}B^p\Big(\frac{1}{\alpha+1}, 1-\frac{1}{p}\Big),\end{aligned}$$

where $B(t,\tau) = \displaystyle\int_0^1 s^{t-1}(1-s)^{\tau-1}ds$ denotes the Beta function with $t, \tau > 0$. From this equality we conclude:

(1) If $\alpha \neq p-1$, then for every $\lambda > 0$, there exists a unique $M > 0$ so that (4.2) is fulfilled, and consequently the problem admits a unique nontrivial solution— the maximal solution;

(2) If $\alpha = p-1$, then condition (4.2) is satisfied only for a particular λ, namely, the problem has nontrivial solutions if and only if

$$\lambda = \overline{\lambda} := p^{1-p}\Big(\frac{2}{b-a}\Big)^p\Big(1-\frac{1}{p}\Big)B^p\Big(\frac{1}{p}, 1-\frac{1}{p}\Big). \tag{4.4}$$

In particular, if $p = 2$, then $\overline{\lambda} = [\pi/(b-a)]^2$, the first eigenvalue of Laplacian operator in $\Omega = (a,b)$.

Remark 4.1. The above calculation also shows:

(1) If $\alpha > p-1$, then $M = M(\lambda)$, the maximum value of the nontrivial solution, is decreasing in λ, and $M \to 0$ as $\lambda \to \infty$;

(2) If $\alpha < p-1$, then $M = M(\lambda)$ is increasing in λ, and $M \to \infty$ as $\lambda \to \infty$.

(3) In the case $\alpha = p-1, \lambda = \overline{\lambda}$, the problem admits a family of solutions. In fact, let $u_1(x)$ be a nontrivial solution, then $u_h(x) = hu_1(x)$ are also solutions for all $h \geq 0$. Moreover, it is not hard to verify that any solution must be of this form.

Example 4.2. Let $f(u) = (1+u)^\alpha$ with $\alpha \geq 0$. Then

$$\begin{aligned}I(M) &= (\alpha+1)^{1/p}\int_0^M [(M+1)^{\alpha+1}-(s+1)^{\alpha+1}]^{-1/p}ds \\ &= (\alpha+1)^{-1+1/p}G((M+1)^{-(\alpha+1)}),\end{aligned}$$

where we denote

$$G(z) = z^{1/p-1/(\alpha+1)}\int_z^1 (1-t)^{-1/p}t^{-1+1/(\alpha+1)}dt$$

with $z \in (0,1)$. Analyzing the behaviors of $G(z)$ as $z \to 0^+$ and as $z \to 1^{-0}$ we find that

(1) If $\alpha > p-1$, then the problem has exactly two solutions for $\lambda \in (0, \lambda^*)$, one solution for $\lambda = 0$ or λ^*, and no solution for $\lambda > \lambda^*$, where

$$\lambda^* = (\alpha+1)^{1-p}\Big(\frac{2}{b-a}\Big)^p\Big(1-\frac{1}{p}\Big)\max_{0<z<1} G^p(z) < \infty;$$

(2) In the case $\alpha = p-1$, the problem admits a unique solution for $\lambda \in [0, \overline{\lambda})$, and no solution for $\lambda \geq \overline{\lambda}$, where $\overline{\lambda}$ is given by (4.4);

(3) If $\alpha < p-1$, then the problem admits a unique solution for every $\lambda \geq 0$.

Example 4.3. Let $f(u)=e^u$. Then we have

$$I(M)=\int_0^M (e^M-e^s)^{-1/p}ds=e^{-M/p}\int_{e^{-M}}^1(1-s)^{-1/p}s^{-1}ds.$$

Let

$$E(z):=z\int_{z^p}^1(1-s)^{-1/p}s^{-1}ds,\qquad 0<z<1.$$

Noting that $E''(z)=-pz^{-1}(1-z^p)^{-1-1/p}<0$, $E(z)>0$, and $E(z)\to 0$ as $z\to 0^+$ or $z\to 1^{-0}$, we see that the problem admits solutions if and only if

$$0\le\lambda\le\lambda^*:=\Big(\frac{2}{b-a}\Big)^p\Big(1-\frac{1}{p}\Big)\max_{0<z<1}E^p(z).$$

Furthermore, for each $\lambda\in(0,\lambda^*)$, there are exactly two different solutions $\underline{u}_\lambda$ and $\overline{u}_\lambda$ with $\underline{u}_\lambda\le\overline{u}_\lambda$; and for $\lambda=0$ and $\lambda=\lambda^*$, the problem admits the solutions $u=u_0(\equiv 0)$ and $u=u_{\lambda^*}$, respectively.

Remark 4.2. In above examples, if $\lim\limits_{z\to\infty}z^{p-1}/f(z)=0$, then the problem has exactly two solutions $\underline{u}_\lambda$ and $\overline{u}_\lambda$ with $\underline{u}_\lambda\le\overline{u}_\lambda$ for $\lambda\in(0,\lambda^*)$, and it is easy to see that $\underline{u}_{\lambda_1}\le\underline{u}_{\lambda_2}$, but $\max\overline{u}_{\lambda_1}>\max\overline{u}_{\lambda_2}$ if $0\le\lambda_1<\lambda_2\le\lambda^*$.

It should be noted that the above anti-ordered property for the maximal solutions is important in the study on the stability of solutions of the corresponding parabolic problem in Section 5.

Example 4.4. More generally, we consider the case $\limsup\limits_{z\to\infty}z^{p-1}/f(z)=0$. This indicates that for any $K>0$ there exists $M_0>0$ such that

$$f(z)\ge Kz^{p-1}\qquad\text{for all}\quad z\ge M_0.$$

Then we have, for any $y\ge M_0$,

$$\begin{aligned}I(y)&\le\int_0^{M_0}\Big[\int_s^y f(t)dt\Big]^{-1/p}ds+K^{-1/p}p^{1/p}\int_{M_0}^y(y^p-s^p)^{-1/p}ds\\&\le\int_0^{M_0}\Big[\int_{M_0}^y f(t)dt\Big]^{-1/p}+K^{-1/p}p^{1/p-1}B\Big(\frac{1}{p},1-\frac{1}{p}\Big).\end{aligned}$$

Plainly, this inequality implies that $I(y)\to 0$ as $y\to\infty$, which means that there exists $\beta^*>0$ such that the equation $I(y)=\beta$ has at least two roots for $\beta\in(0,\beta^*)$, at least one for $\beta=\beta^*$, and none for $\beta>\beta^*$. Correspondingly, there must exist $\lambda^*>0$ such that the problem (4.1) possesses at least two solutions for every $\lambda\in(0,\lambda^*)$, at least one for $\lambda=\lambda^*$, and none for $\lambda>\lambda^*$.

Now we turn to multi-dimensional case.

4.2. Multi-dimensional Case

Let $\Omega\subset R^n$ be a symmetric domain, e.g. the annulus $R_1<|x|<R_2$, where $0<R_1<R_2<\infty$, and let consider radial solution $u=u(|x|)$. Then the problem (E_λ) reduces to

$$\begin{cases}-r^{-(n-1)}(r^{n-1}|u'|^{p-2}u')'=\lambda f(u),\qquad r\in(R_1,R_2),\\ u(R_1)=u(R_2)=0.\end{cases}\tag{4.5}$$

We first introduce a change of variables as follows: If $p \neq n$, we set

$$r = |m|^{-1}\rho^m, \quad v(\rho) = u(r),$$

where $m = (p-1)/(p-n)$, and the problem becomes

$$\begin{cases} -(|v'|^{p-2}v')' = \lambda\varphi(\rho)f(v), & \rho \in (a,b), \\ v(a) = v(b) = 0, \end{cases} \tag{4.6}$$

where

$$a = \min\{(|m|R_1)^{1/m}, (|m|R_2)^{1/m}\},$$
$$b = \max\{(|m|R_1)^{1/m}, (|m|R_2)^{1/m}\}, \qquad \varphi(\rho) = \rho^{p(m-1)}.$$

If $p = n$, we use the change of variables

$$r = e^\rho, \quad v(\rho) = u(r),$$

and arrive at the same problem but with $a = \log R_1, b = \log R_2, \varphi(\rho) = e^{n\rho}$.

Then applying Theorem 1 in [9] to the problem (4.6), the following result which is similar to Example 4.4 can be obtained:

Example 4.5. If $\limsup\limits_{z\to\infty} z^{p-1}/f(z) = 0$, then there exists $\lambda^* \in (0,\infty)$ such that the problem (4.5) has at least two solutions for $\lambda \in (0,\lambda^*)$, at least one for $\lambda = \lambda^*$, and none for $\lambda > \lambda^*$.

Remark 4.3. Similarly to the reasoning of (4.2) we can get that

$$\frac{1}{\overline{\varphi}}\Big(1-\frac{1}{p}\Big)\Big(\frac{2}{b-a}\Big)^p[I(M)]^p \le \lambda(M) \le \frac{1}{\underline{\varphi}}\Big(1-\frac{1}{p}\Big)\Big(\frac{2}{b-a}\Big)^p[I(M)]^p,$$

where $\overline{\varphi} = \max\limits_{a\le\rho\le b}\varphi(\rho)$, $\underline{\varphi} = \min\limits_{a\le\rho\le b}\varphi(\rho)$, $M = \max\limits_{a\le\rho\le b} v(\rho)$ and function $I(y)$ is given by (4.3). This clearly provides a priori estimates of upper bounds of the solutions, as well as the estimate of λ^*.

§ 5. Stability for the Parabolic Problem—A Fujita Type Result

In this section, we are concerned with the corresponding parabolic problem, and study the large time behavior of the solutions. Let v be a solution of the following problem

$$\begin{cases} v_t - \Delta_p v = \lambda f(v) & \text{in} \quad \Omega\times(0,T), \\ v = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ v = v_0(x) & \text{on} \quad \Omega\times\{0\}, \end{cases} \tag{P_λ}$$

where function $v_0(x)$ is nonnegative, and sufficient smooth. Since $C^{1,\alpha}$-estimates are established on the bases of L^∞-estimates for the solutions (cf. e.g. [10]), we have the following well-known conclusions:

Lemma 5.1. *Assume that there exists constant $C > 0$, independent of $T > 0$, such that $v(\cdot,t) \le C$ for all $t \in (0,T)$. Then there must exist a subsequence $\{t_k\}$ in $(0,\infty)$, such that $\lim\limits_{k\to\infty} v(x,t_k)$ exists and is a solution of the problem (E_λ). In particular, if $\lim\limits_{t\to\infty} v(x,t) = u(x)$ exists, then $u(x)$ must be a solutions of (E_λ).*

Our aim is to answer the following questions:

How is the behavior of $v(x,t)$ as $t \to \infty$? When does it converge to a steady solution, and when to infinity? Since the problem (E_λ) may have multiple solutions, which one is stable, and which is not?

Based on the knowledge on the solutions to the problem (E_λ), a partial answer to these questions is given as follows:

Theorem 5.1. *Let v be as above, a solution of (P_λ). Assume that for every $\lambda \in (0, \lambda^*)$, the problem (E_λ) possesses exactly two solutions, maximal and minimal ones, denoted by $\overline{u}_\lambda$ and $\underline{u}_\lambda$, respectively.*

(1) *If there exists $\lambda' > \lambda$, so that $v_0 \le \overline{u}_{\lambda'}$, and $\overline{u}_{\lambda'}(x) < \overline{u}_\lambda(x)$ at some point $x \in \Omega$, then*

$$\lim_{t\to\infty} v(x,t) = \underline{u}_\lambda(x);$$

(2) *If there exists $\lambda' < \lambda$, so that $v_0 \ge \overline{u}_{\lambda'}$, and $\overline{u}_{\lambda'}(x) > \overline{u}_\lambda(x)$ at some point $x \in \Omega$, then either there exists $T^* > 0$ such that*

$$\lim_{t\to T^*} \sup_{x\in\Omega} v(x,t) = \infty, \quad \text{or} \quad \lim_{t\to\infty} v(x,t) = \infty.$$

This conclusion is also true for $\lambda = \lambda^*$ if we denote $\overline{u}_{\lambda^*} = u_{\lambda^*}$.

Note that in this case $\underline{u}_\lambda$ is a stable solution of (P_λ) while $\overline{u}_\lambda$ is unstable.

Proof. First let $\lambda' > \lambda$, and consider a solution W to the following problem

$$\begin{cases} W_t - \Delta_p W = \lambda f(W) & \text{in} \quad \Omega \times (0,T), \\ W = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ W = \overline{u}_{\lambda'}(x) & \text{on} \quad \Omega \times \{0\}. \end{cases} \tag{5.1}$$

In order to avoid the non-uniqueness we choose the maximal solution as W. Let w be the (minimal) solution to the problem

$$\begin{cases} w_t - \Delta_p w = \lambda f(w) & \text{in} \quad \Omega \times (0,T), \\ w = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ w = 0 & \text{on} \quad \Omega \times \{0\}. \end{cases}$$

Then $w \le v \le W$ follows from the comparison principle. Next by setting $h = W_t$ and differentiating (5.1), we find that

$$\begin{cases} h_t - (p-1)\mathrm{div}(|\nabla u|^{p-2}\nabla h) = \lambda f'(W)h & \text{in} \quad \Omega \times (0,T), \\ h = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ h \le 0 & \text{on} \quad \Omega \times \{0\}. \end{cases}$$

Here we used in the last inequality the fact

$$W_t(x,0) = \Delta_p \overline{u}_{\lambda'} + \lambda f(\overline{u}_{\lambda'}) \le \Delta_p \overline{u}_{\lambda'} + \lambda' f(\overline{u}_{\lambda'}) = 0.$$

Thus by the maximum principle, we get $h \le 0$ in $\Omega \times (0,T)$. This means that W is decreasing in t, and hence we may set $T = \infty$. In view of Lemma 5.1, $W(x,t)$ must approach one of the solutions of (E_λ) as $t \to \infty$. Since

$$W(x,t) \le W(x,0) = \overline{u}_{\lambda'}(x) < \overline{u}_\lambda(x)$$

at some point $x \in \Omega$, we arrive at $\lim_{t\to\infty} W(x,t) = \underline{u}_\lambda(x)$. In a similar way we derive that $w(x,t)$ is increasing in t, and converges to $\underline{u}_\lambda(x)$ as $t \to \infty$. Therefore $\lim_{t\to\infty} v(x,t) = \underline{u}_\lambda(x)$.

In the case $v_0(x) \geq \overline{u}_{\lambda'}(x)$ with $\lambda' < \lambda$, we suppose that the conclusion is not true. In virtue of a priori $C^{1,\alpha}$-estimates (cf. e.g. [10]), we see that $\liminf_{t\to\infty} v(x,t)$ must exist for all $x \in \Omega$. On the other hand, by considering the minimal solution W of the problem (5.1) we find that

(a) $W \leq v$ in $\Omega \times (0,\infty)$;

(b) W is nondecreasing in t, since $W_t(x,0) = \Delta_p \overline{u}_{\lambda'} + \lambda f(\overline{u}_{\lambda'}) \geq \Delta_p \overline{u}_{\lambda'} + \lambda' f(\overline{u}_{\lambda'}) = 0$ in this case.

Therefore, $\lim_{t\to\infty} W(x,t)$ must exist and be a solution of (E_λ), and hence $\lim_{t\to\infty} W(x,t) \leq \overline{u}_\lambda(x)$, which contradicts the fact that $W(x,t) \geq W(x,0) = \overline{u}_{\lambda'}(x) > \overline{u}_\lambda(x)$ at some points $x \in \Omega$.

Example 5.1. $f(u) = u^\alpha$ with $\alpha > p-1$, and $n = 1$.

Example 5.2. $f(u) = (1+u)^\alpha$ with $\alpha > p-1$, and $n = 1$.

Example 5.3. $f(u) = e^u$, and $n = 1$.

It is clear from the statement of the theorem that our result is different from that of Fujita [12]. Nevertheless, we consider Theorem 5.1 as an extension of the Fujita theory to the problem (P_λ), and we believe that the Fujita theory keeps true in the present case.

Acknowledgement. The first author would like to thank Prof. M. Kiessling for helpful discussions on related topics when he was at Rutgers University, New Brunswick in 1996.

References

[1] Crespo, J. A. Aguilar & Alonso, I. Peral, Blow-up behavior for solutions of $\Delta_N u = V(x)e^u$ in bounded domains in R^N, *Nonl. Anal. TMA,* **29**(1997), 365–384.

[2] Brezis, H. & Merle, F., Uniform estimate and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, *Comm. PDE,* **16**(1991), 1223–1253.

[3] Caglioti, E., Lions, P. L., Marchioro, C. & Pulvirenti, M., A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description, Part 2, *Comm. Math. Phys.,* **174**(1995), 229–260.

[4] Canada, A., Drabek, P. & Gamez, J. L., Existence of positive solutions for some problems with nonlinear diffusion, *Trans. Amer. Math. Soc.,* **349**(1997), 4231–4249.

[5] Coster, C. D., Pairs of positive solutions for the one-dimensional p-Laplacian, *Nonl. Anal. TMA,* **23**(1994), 669–681.

[6] Citti, G., Positive solutions of quasilinear degenerate elliptic equations in R^n, *Rend. Circ. Mat. Palermo,* **35**(1986), 364–375.

[7] Crandall, M. & Rabinowitz, P., Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, *Arch. Rat. Mech. Anal.,* **58**(1975), 207–218.

[8] Damascelli, L., Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results, *Analyse Nonlineaire,* **15**(1998), 493–516.

[9] Dang, H., Schmitt, K. & Shivaji, R., On the number of solutions of boundary value problems involving the p-Laplacian, *Electron. J. Differential Equations,* **1996:**01(1996), 1–9.

[10] DiBenedetto, E., Degenerate Parabolic Equations, Springer-Verlag, New York, 1993.

[11] Erbe, L. & Tang, M., Uniqueness theorems for positive solutions of quasilinear elliptic equations in a ball, *J. Diff. Eqns.,* **138**(1997), 351–379.

[12] Fujita, H., On the nonlinear equations $\Delta u + e^u = 0$ and $v_t = \Delta v + e^v$, *Bull. Amer. Math. Soc.,* **75**(1969), 132–135.

[13] Hai, D. D., Schmitt, K. & Shivaji, R., Positive solutions of quasilinear boundary value problems, *J. Math. Anal. Appl.,* **217**(1998), 672–686.

[14] Huang, Y. X. & Metzen, G., The existence of solutions to a class of semilinear equations, *Diff. Int. Eqns.,* **8**(1995), 429–452.

[15] Kaper, H. G. & Kwong, M. K., Free boundary problems for Emdan-Fowler equations, *Diff. Int. Eqns.,* **3**(1990), 353–362.

[16] Kiessling, M., Statistical mechanics of classical particles with logarithmic interaction, *CPAM,* **46**(1983), 27–56.

[17] Le, V. K. & Schmitt, K., On boundary value problems for degenerate quasilinear elliptic equations and inequalities, *J. Diff. Eqns.,* **144**(1998), 170–218.

[18] Ladyzanskaya, O. A. & Ural'ceva, N. N., Linear and Quasi-linear Equations of Elliptic Type (in Russian), Nauk, Moscow, 1964.

[19] Manasevich, P. & Mawhin, J., Period solutions for nonlinear systems with p-Laplacian-like operators, *J. Diff. Eqns.,* **145**(1998), 367–393.

[20] Pucci, P. & Serrin, J., Uniqueness of ground states for quasilinear elliptic operators, *Indiana Univ. Math. J.,* **47**(1998), 501–528.

[21] Pucci, P. & Serrin, J., Uniqueness of ground states for quasilinear elliptic equations in the exponential case, *Indiana Univ. Math. J.,* **47**(1998), 529–539.

[22] Suzuki, T., Global analysis for two dimensional elliptic eigenvalue problems with exponential nonlinearities, *Ann. I. H. P. Analyse Nonlinearie,* **9**(1992), 367–398.