GLOBAL EXISTENCE OF WEAKLY DISCONTINUOUS SOLUTIONS TO THE CAUCHY PROBLEM WITH A KIND OF NON-SMOOTH INITIAL DATA FOR QUASILINEAR HYPERBOLIC SYSTEMS***

LI TATSien*      WANG Libin**

Abstract

The authors consider the Cauchy problem with a kind of non-smooth initial data for quasilinear hyperbolic systems and obtain a necessary and sufficient condition to guarantee the existence and uniqueness of global weakly discontinuous solution.

Keywords Quasilinear hyperbolic system, Cauchy problem, Global weakly discontinuous solution, Weakly linear degeneracy

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§ 1. Introduction and Main Result

Consider the following first order quasilinear hyperbolic system

\[ \frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = 0, \tag{1.1} \]

where \( u = (u_1, \cdots, u_n)^T \) is the unknown vector function of \((t, x)\) and \( A(u) \) is an \( n \times n \) matrix with suitably smooth elements \( a_{ij}(u) \) \((i, j = 1, \cdots, n)\).

By the definition of hyperbolicity, for any given \( u \) on the domain under consideration, \( A(u) \) has \( n \) real eigenvalues \( \lambda_1(u), \cdots, \lambda_n(u) \) and a complete set of left (resp. right) eigenvectors. For \( i = 1, \cdots, n \), let \( l_i(u) = (l_{i1}(u), \cdots, l_{in}(u)) \) (resp. \( r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T \)) be a left (resp. right) eigenvector corresponding to \( \lambda_i(u) \):

\[ l_i(u)A(u) = \lambda_i(u)l_i(u) \tag{1.2} \]

and

\[ A(u)r_i(u) = \lambda_i(u)r_i(u), \tag{1.3} \]

we have

\[ \det |l_{ij}(u)| \neq 0 \quad \text{(resp. } \det |r_{ij}(u)| \neq 0). \tag{1.4} \]

Without loss of generality, we assume that on the domain under consideration

\[ l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n), \tag{1.5} \]

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*Department of Mathematics, Fudan University, Shanghai 200433, China. E-mail: dqli@fudan.edu.cn

**Department of Mechanics, Fudan University, Shanghai 200433, China.

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where $\delta_{ij}$ stands for the Kronecker’s symbol.

In particular, if, for any given $u$ on the domain under consideration, $A(u)$ has $n$ distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u),$$

system (1.1) is called to be strictly hyperbolic.

For the Cauchy problem of system (1.1) with the initial data

$$t = 0 : \quad u = \phi(x) \quad (-\infty < x < \infty),$$

where $\phi(x)$ is a $C^1$ vector function with bounded $C^1$ norm, it was proved in [3–6] and [12, 13] that if system (1.1) is strictly hyperbolic, then, for any given initial data satisfying the following small and decaying property:

$$\theta \triangleq \sup_{x \in \mathbb{R}} \{(1 + |x|)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} \ll 1,$$

where $\mu > 0$ is a constant, Cauchy problem (1.1) and (1.7) admits a unique global $C^1$ solution $u = u(t, x)$ with small $C^1$ norm for all $t \in \mathbb{R}$, if and only if system (1.1) is weakly linearly degenerate, i.e., all the characteristics are weakly linearly degenerate (see also [9, 10] and [15–18] for some related results). Here, we call $\lambda_i(u)$ ($i \in \{1, \cdots, n\}$) a weakly linearly degenerate characteristic if, along the $i$-th characteristic trajectory $u = u_i(s)$ passing through $u = 0$, defined by

$$\begin{cases}
\frac{du}{ds} = r_i(u), \\
\quad s = 0 : \quad u = 0,
\end{cases}$$

we have

$$\nabla \lambda_i(u) r_i(u) = 0, \quad \forall \ |u| \ \text{small},$$

namely

$$\lambda_i(u_i(s)) \equiv \lambda_i(0), \quad \forall \ |s| \ \text{small}. \quad (1.11)$$

In the previous result, the initial data are supposed to be in the $C^1$ class. However, in some practical problems, we are required to deal with the Cauchy problem for system (1.1) with the following kind of non-smooth initial data

$$t = 0 : \quad u = \begin{cases}
\ u_l(x), & x \leq 0, \\
\ u_r(x), & x \geq 0,
\end{cases} \quad (1.12)$$

where $u_l(x)$ and $u_r(x)$ are $C^1$ vector functions on $x \leq 0$ and $x \geq 0$ respectively and satisfy the following small and decaying property

$$\theta \triangleq \sup_{x \leq 0} \{(1 + |x|)^{1+\mu}(|u_l(x)| + |u'_l(x)|)\} + \sup_{x \geq 0} \{(1 + x)^{1+\mu}(|u_r(x)| + |u'_r(x)|)\} < +\infty, \quad (1.13)$$

where $\mu > 0$ is a constant; moreover,

$$u_l(0) = u_r(0) \quad \text{and} \quad u'_l(0) \neq u'_r(0). \quad (1.14)$$

In this paper, we will generalize the previous result to Cauchy problem (1.1) and (1.12). In the meantime, the method used in [6] and [13] will be simplified and improved. In order to state the main result of this paper, we first give the following
Definition 1.1. A continuous and piecewise $C^1$ vector function
\[ u = u(t,x) = \begin{cases} u^-(t,x), & x \leq x_k(t), \\ u^+(t,x), & x \geq x_k(t) \end{cases} \]
(1.15)
is called a weakly discontinuous solution containing a $k$-th weak discontinuity $x = x_k(t)$ for system (1.1), if $u = u(t,x)$ satisfies system (1.1) in the classical sense on both sides of $x = x_k(t)$,
\[ u^-(t,x_k(t)) = u^+(t,x_k(t)) \]  
(1.16) and $x = x_k(t)$ is the corresponding $k$-th characteristic:
\[ \frac{dx_k(t)}{dt} = \lambda_k(u^-(t,x_k(t))) = \lambda_k(u^+(t,x_k(t))), \]
(1.17)
moreover, the first order derivatives of $u(t,x)$ have the first kind discontinuity on $x = x_k(t)$.

Our main result is the following

Theorem 1.1. Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and system (1.1) is strictly hyperbolic. Suppose furthermore that $u_l(x)$ and $u_r(x)$ are $C^1$ vector functions on $x \leq 0$ and $x \geq 0$ respectively. Then there exists $\theta_0 > 0$ so small that for any given initial data satisfying (1.13)–(1.14) with $\theta \in (0,\theta_0]$, Cauchy problem (1.1) and (1.12) admits a unique global weakly discontinuous solution $u = u(t,x)$ containing $n$ weak discontinuities $x = x_k(t)$ ($k = 1, \ldots, n$), where $x = x_k(t)$ with $x_k(0) = 0$ denotes a $k$-th weak discontinuity passing through the origin $(0,0)$, if and only if system (1.1) is weakly linearly degenerate. Precisely speaking, the solution $u = u(t,x)$ should have the following structure:
\[ u = u(t,x) = \begin{cases} u^{(0)}(t,x), & (t,x) \in R_0, \\ u^{(l)}(t,x), & (t,x) \in R_l \quad (l = 1, \ldots, n-1), \\ u^{(n)}(t,x), & (t,x) \in R_n, \end{cases} \]
(1.18)
in which $u^{(l)}(t,x) \in C^1$ satisfies system (1.1) in the classical sense on $R_l$ ($l = 0,1,\ldots,n$) with
\[ R_l = \begin{cases} \{(t,x) \mid t \geq 0, \; x \leq x_1(t)\} & (l = 0), \\ \{(t,x) \mid t \geq 0, \; x_l(t) \leq x \leq x_{l+1}(t)\} & (l = 1,\ldots,n-1), \\ \{(t,x) \mid t \geq 0, \; x \geq x_n(t)\} & (l = n). \end{cases} \]
(1.19)
Moreover, for $k = 1,\ldots,n$,
\[ u^{(k-1)}(t,x_k(t)) = u^{(k)}(t,x_k(t)), \]
(1.20)
\[ \frac{dx_k(t)}{dt} = \lambda_k(u^{(k-1)}(t,x_k(t))) = \lambda_k(u^{(k)}(t,x_k(t))). \]
(1.21)

Remark 1.1. In Theorem 1.1, some weak discontinuities may degenerate.

Remark 1.2. Suppose that (1.1) is a non-strictly hyperbolic system with characteristics with constant multiplicity, say,
\[ \lambda_1(u) < \cdots < \lambda_k(u) < \lambda_{k+1}(u) \equiv \cdots \equiv \lambda_k+\rho(u) < \lambda_{k+\rho+1}(u) < \cdots < \lambda_n(u) \quad (p > 1). \]
(1.22)
Then, if there exist normalized coordinates, similar conclusion holds as in Theorem 1.1 (some related results can be found in [7, 14]).

The paper is organized as follows. In Section 2 we give some preliminaries. Then, the main result is proved in Section 3. Finally, an application is given in Section 4.
By Lemma 2.5 in [12], when system (1.1) is strictly hyperbolic, there exists a suitably smooth invertible transformation $u = u(\tilde{u})$ ($u(0) = 0$) such that in the $\tilde{u}$-space, for each $i = 1, \cdots, n$, the $i$-th characteristic trajectory passing through $\tilde{u} = 0$ coincides with the $\tilde{u}_i$-axis at least for $|\tilde{u}_i|$ small, namely,

$$\tilde{r}_i(\tilde{u}_i e_i)/e_i, \quad \forall |\tilde{u}_i| \text{ small } (i = 1, \cdots, n),$$

(2.1)

where $\tilde{r}_i(\tilde{u})$ denotes the $i$-th right eigenvector corresponding to $r_i(u)$ and

$$e_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T.$$  

(2.2)

This transformation is called a normalized transformation, and the unknown variables $\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_n)^T$ are called normalized variables or normalized coordinates.

Let

$$w_i = l_i(u)u_x \quad (i = 1, \cdots, n).$$

(2.3)

By (1.5), it is easy to see that

$$u_x = \sum_{k=1}^{n} w_k r_k(u).$$

(2.4)

Let

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \lambda_i(u) \frac{\partial}{\partial x}$$

(2.5)

denote the directional derivative with respect to $t$ along the $i$-th characteristic. We have

$$\frac{du}{dt} = \sum_{k=1}^{n} (\lambda_i(u) - \lambda_k(u)) w_k r_k(u) \quad (i = 1, \cdots, n).$$

(2.6)

Then, in normalized coordinates, it is easy to see that

$$\frac{du_i}{dt} = \sum_{j,k=1}^{n} \rho_{ijk}(u) u_j w_k \quad (i = 1, \cdots, n),$$

(2.7)

where

$$\rho_{iji}(u) \equiv 0, \quad \forall i, j,$$

(2.8)

and

$$\rho_{ijk}(u) = (\lambda_i(u) - \lambda_k(u)) \int_0^1 \frac{\partial r_k}{\partial u_j} (\tau u_1, \cdots, \tau u_{k-1}, u_k, \tau u_{k+1}, \cdots, \tau u_n) d\tau, \quad \forall j \neq k.$$  

(2.9)

Obviously

$$\rho_{iji}(u) \equiv 0, \quad \forall i, j.$$  

(2.10)

Moreover, noting (2.4) and (2.7), we have

$$d[u_i(dx - \lambda_i(u)dt)] = \left[ \frac{du_i}{dt} + \sum_{k=1}^{n} \nabla \lambda_i(u) r_k(u) u_i w_k \right] dt \wedge dx$$

$$= \sum_{j,k=1}^{n} F_{ijk}(u) u_j w_k dt \wedge dx,$$

(2.11)
where
\[ F_{ijk}(u) = \rho_{ijk}(u) + \nabla \lambda_j(u) r_k(u) \delta_{ij}. \] (2.12)

Noting (2.8) and (2.10), it is easy to see that
\[ F_{ijj}(u) \equiv 0, \quad \forall j \neq i, \] (2.13)
\[ F_{iji}(u) \equiv 0, \quad \forall j \neq i, \] (2.14)
\[ F_{iii}(u) = \nabla \lambda_i(u) r_i(u), \quad \forall i. \] (2.15)

On the other hand, we have (see [1–3] or [12])
\[ \frac{d w_i}{d t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k \quad (i = 1, \cdots, n), \] (2.16)
where
\[ \gamma_{ijk}(u) = \frac{1}{2} \left\{ (\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_j(u) - \nabla \lambda_k(u) r_j(u) \delta_{ik} + (j|k) \right\}, \] (2.17)
in which \((j|k)\) stands for all terms obtained by changing \(j\) and \(k\) in the previous terms. Hence
\[ \gamma_{ijj}(u) \equiv 0, \quad \forall j \neq i, \] (2.18)
\[ \gamma_{iii}(u) = -\nabla \lambda_i(u) r_i(u), \quad \forall i. \] (2.19)

Noting (2.4), by (2.16) we have (see [1])
\[ d[w_i(dx - \lambda_i(u) dt)] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k dt \wedge dx, \] (2.20)
where
\[ \Gamma_{ijk}(u) = \frac{1}{2} (\lambda_j(u) - \lambda_k(u)) l_i(u) [\nabla r_k(u) r_j(u) - \nabla r_j(u) r_k(u)]. \] (2.21)
Hence
\[ \Gamma_{ijj}(u) \equiv 0, \quad \forall i, j. \] (2.22)

§ 3 . Proof of Theorem 1.1

In order to prove the sufficiency in Theorem 1.1, in what follows we always assume that \( \theta > 0 \) is suitably small.

By the existence and uniqueness of local weakly discontinuous solution to the Cauchy problem (see [11]), there exists \( T_0 > 0 \) so small that Cauchy problem (1.1) and (1.12) admits a unique weakly discontinuous solution \( u = u(t, x) \) containing at most \( n \) weak discontinuities \( x = x_k(t) (k = 1, \cdots, n) \) on the domain \( R(T_0) = \{(t, x) \mid 0 \leq t \leq T_0, -\infty < x < +\infty\} = \bigcup_{l=0}^{n} R_l(T_0) \):\n
\[ u = u(t, x) = \begin{cases} u^{(0)}(t, x), & (t, x) \in R_0(T_0), \\ u^{(l)}(t, x), & (t, x) \in R_l(T_0) \quad (l = 1, \cdots, n - 1), \\ u^{(n)}(t, x), & (t, x) \in R_n(T_0), \end{cases} \] (3.1)
we have to Cauchy problem (1.1) and (1.12), where

\[ R_0(T_0) = \begin{cases} \{(t, x) \mid 0 \leq t \leq T_0, \ x \leq x_1(t)\} & (l = 0), \\ \{(t, x) \mid 0 \leq t \leq T_0, \ x_i(t) \leq x \leq x_{i+1}(t)\} & (l = 1, \ldots, n - 1), \\ \{(t, x) \mid 0 \leq t \leq T_0, \ x \geq x_n(t)\} & (l = n). \end{cases} \] (3.2)

Without loss of generality, we may assume that

\[ C \text{ the piecewise } C^0 \text{ norm of } u \text{ and } \notag \]

In what follows, we establish a uniform a priori estimate on the $C^0$ norm of $u$ and the piecewise $C^0$ norm of $u_x$ on any given existence domain of the weakly discontinuous solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.12). Noting (2.3), we only need to establish a uniform a priori estimate on the $C^0$ norm of $u$ and the piecewise $C^0$ norm of $w = (w_1, \ldots, w_n)$ on any given existence domain of the weakly discontinuous solution $u = u(t,x)$.

Noting (1.6), we have

\[ \lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0). \] (3.3)

Then, there exist positive constants $\delta$ and $\delta_0$ so small that

\[ \lambda_{i+1}(u) - \lambda_i(u') \geq 2\delta_0, \quad \forall |u|, |u'| \leq \delta \quad (i = 1, \cdots, n - 1), \] (3.4)

\[ |\lambda_i(u) - \lambda_i(u')| \leq \frac{\delta_0}{2}, \quad \forall |u|, |u'| \leq \delta \quad (i = 1, \cdots, n). \] (3.5)

Without loss of generality, we may assume that

\[ \lambda_i(0) > \delta_0 \quad (i = 1, \cdots, n). \] (3.6)

For the time being we assume that on any given existence domain $R(T) = \{(t, x) \mid 0 \leq t \leq T, -\infty < x < +\infty\} = \bigcup_{t=0}^{n} R_t(T)$ of the weakly discontinuous solution

\[ u = u(t,x) = \begin{cases} u^{(0)}(t,x), & (t,x) \in R_0(T), \\ u^{(l)}(t,x), & (t,x) \in R_l(T) \quad (l = 1, \cdots, n - 1), \\ u^{(n)}(t,x), & (t,x) \in R_n(T) \end{cases} \] (3.7)

to Cauchy problem (1.1) and (1.12), where

\[ R_l(T) = \begin{cases} \{(t, x) \mid 0 \leq t \leq T, \ x \leq x_1(t)\} & (l = 0), \\ \{(t, x) \mid 0 \leq t \leq T, \ x_i(t) \leq x \leq x_{i+1}(t)\} & (l = 1, \cdots, n - 1), \\ \{(t, x) \mid 0 \leq t \leq T, \ x \geq x_n(t)\} & (l = n), \end{cases} \] (3.8)

we have

\[ |u(t,x)| \leq \delta, \quad \forall (t,x) \in R(T). \] (3.9)

At the end of the proof of Lemma 3.3, we will explain that this hypothesis is reasonable.

Let

\[ D^T_i = \begin{cases} \{(t, x) \mid 0 \leq t \leq T, \ x \leq (\lambda_i(0) + \delta_0)t\} & (i = 1), \\ \{(t, x) \mid 0 \leq t \leq T, \ (\lambda_i(0) - \delta_0)t \leq x \leq (\lambda_i(0) + \delta_0)t\} & (i = 2, \cdots, n - 1), \\ \{(t, x) \mid 0 \leq t \leq T, \ x \geq (\lambda_n(0) - \delta_0)t\} & (i = n). \end{cases} \] (3.10)
Obviously
\[ \bigcup_{i=1}^{n} D_{i}^{T} \subset R(T). \quad (3.11) \]

On any given existence domain \( R(T) = \bigcup_{i=0}^{n} R_{i}(T) \) of the weakly discontinuous solution \( u = u(t, x) \) to Cauchy problem (1.1) and (1.12), let
\[ w^{(l)} = (w_{1}^{(l)}, \ldots, w_{n}^{(l)}) \quad (l = 0, 1, \ldots, n) \quad (3.12) \]
with
\[ w_{i}^{(l)} = l_{i}(u^{(l)})u_{x}^{(l)} \quad (i = 1, \ldots, n), \quad (3.13) \]
\[ W_{\infty}(T) = \max_{i=1,\ldots,n} \max_{t=0,M_{i} \in R(T)} \sup_{k \in D_{i}^{T}} \left\{ (1 + |x - \lambda_{i}(0)t|^{1+\mu}|w_{i}^{(l)}(t,x)| \right\}, \quad (3.14) \]
\[ U_{\infty}(T) = \max_{i=1,\ldots,n} \max_{t=0,M_{i} \in R(T)} \sup_{k \in D_{i}^{T}} \left\{ (1 + |x - \lambda_{i}(0)t|^{1+\mu}|w_{i}^{(l)}(t,x)| \right\}, \quad (3.15) \]
\[ \tilde{W}_{1}(T) = \max_{i=1,\ldots,n} \max_{j \neq i} \left\{ \sup_{c_{j}} \int_{c_{j} \cap R_{i}(T)} |w_{i}^{(i-1)}(t,x)| \, dx + \sup_{c_{j}} \int_{c_{j} \cap R_{i}(T)} |w_{i}^{(i)}(t,x)| \, dx \right\}, \quad (3.16) \]
where \( c_{j} \) denotes any given \( j \)-th characteristic on \( D_{i}^{T} \),
\[ W_{1}(T) = \max_{i=1,\ldots,n} \sup_{0 \leq t \leq T} \left\{ \int_{a(t)}^{x(t)} |w_{i}^{(i-1)}(t,x)| \, dx + \int_{x(t)}^{b(t)} |w_{i}^{(i)}(t,x)| \, dx \right\}, \quad (3.17) \]
where
\[ a(t) = \begin{cases} -\infty, & \text{if } i = 1, \\ (\lambda_{i}(0) - \delta_{0})t, & \text{if } i = 2, \ldots, n, \end{cases} \quad (3.18) \]
\[ b(t) = \begin{cases} (\lambda_{i}(0) + \delta_{0})t, & \text{if } i = 1, \ldots, n-1, \\ +\infty, & \text{if } i = n \end{cases} \quad (3.19) \]
and
\[ U_{\infty}(T) = \|u(t,x)\|_{L^{\infty}(R(T))}, \quad (3.20) \]
\[ W_{\infty}(T) = \sum_{l=0}^{n} \|w^{(l)}(t,x)\|_{L^{\infty}(R(T))}. \quad (3.21) \]

According to the definition of the weak discontinuity, it is easy to get

**Lemma 3.1.** On the \( k \)-th weak discontinuity \( x = x_{k}(t) \), we have
\[ w_{i}^{(k-1)} = w_{i}^{(k)}, \quad \forall i \neq k. \quad (3.22) \]

**Lemma 3.2.** For each \( i = 1, \ldots, n \) and any given point \( (t,x) \notin D_{i}^{T} \), let \( c_{i} : \xi_{i} = \xi_{i}(\tau) \ (\tau \leq t) \) be the \( i \)-th characteristic passing through \((t,x)\) and intersecting the \( x \)-axis at
\( (0, x_{i0}) \). Then there exist positive constants \( d_k \) \((k = 1, 2, 3)\) independent of \((t, x)\) and \(i\), such that
\[
d_1|x| \leq |x - \lambda_i(0)t| \leq d_2|x_{i0}|
\] (3.23)
and, if \((\tau, \xi_i(\tau)) \in D^T_j\) for some \(j\), then
\[
|\xi_i(\tau) - \lambda_j(0)\tau| \geq d_3|x_{i0}|.
\] (3.24)

**Proof.** When \(i \in \{2, \cdots, n - 1\}\), for any given point \((t, x) \in D^T_i\), by the definition of \(D^T_i\), we have
\[
x \geq (\lambda_i(0) + \delta_0)t \quad \text{or} \quad x \leq (\lambda_i(0) - \delta_0)t.
\] (3.25)
In what follows, we prove (3.23)–(3.24) for the case \(x \geq (\lambda_i(0) + \delta_0)t\). When \(x \leq (\lambda_i(0) - \delta_0)t\), (3.23)–(3.24) can be similarly proved.

Noting (3.5), for \(\tau \leq t\), it is easy to get
\[
\xi_i(\tau) \geq (\lambda_i(0) + \delta_0)\tau,
\] (3.26)
\[
\left(\lambda_i(0) - \frac{\delta_0}{2}\right)\tau \leq \xi_i(\tau) - x_{i0} \leq \left(\lambda_i(0) + \frac{\delta_0}{2}\right)\tau.
\] (3.27)
Then, noting (3.6), we have
\[
\xi_i(\tau) \leq \frac{2(\lambda_i(0) + \delta_0)}{\delta_0}x_{i0},
\] (3.28)
in particular,
\[
x \leq \frac{2(\lambda_i(0) + \delta_0)}{\delta_0}x_{i0}.
\] (3.29)
Thus, noting \(x \geq (\lambda_i(0) + \delta_0)t\), we immediately get (3.23).

Since \((\tau, \xi_i(\tau)) \in D^T_i\), in order to prove (3.24), we first consider the case \(j = i\). By (3.26)–(3.27), it is easy to get
\[
|\xi_i(\tau) - \lambda_i(0)\tau| \geq \frac{\delta_0}{\lambda_i(0) + \delta_0}x_{i0}.
\] (3.30)
Now we consider the case that there exists \(j \neq i\) such that \((\tau, \xi_i(\tau)) \in D^T_j\). When \(j < i\), noting (3.3) and (3.30), we have
\[
|\xi_i(\tau) - \lambda_j(0)\tau| \geq |\xi_i(\tau) - \lambda_i(0)\tau| \geq \frac{\delta_0}{\lambda_i(0) + \delta_0}x_{i0}.
\] (3.31)
When \(j > i\), since \((\tau, \xi_i(\tau)) \in D^T_j\), we have
\[
\xi_i(\tau) \geq (\lambda_j(0) + \delta_0)\tau \quad \text{or} \quad \xi_i(\tau) \leq (\lambda_j(0) - \delta_0)\tau.
\]
If \(\xi_i(\tau) \geq (\lambda_j(0) + \delta_0)\tau\), similarly to (3.30) we get
\[
|\xi_i(\tau) - \lambda_j(0)\tau| \geq \frac{\delta_0}{\lambda_j(0) + \delta_0}x_{i0};
\] (3.32)
while, if \(\xi_i(\tau) \leq (\lambda_j(0) - \delta_0)\tau\), noting (3.27), it is easy to get
\[
|\xi_i(\tau) - \lambda_j(0)\tau| \geq \frac{\delta_0}{\lambda_j(0) - \delta_0}x_{i0}.
\] (3.33)
The combination of (3.30)–(3.33) proves (3.24).

When \( i = 1 \) or \( n \), noting the definition of \( D^T_1 \) and \( D^T_n \), similarly we can get (3.23)–(3.24).

**Lemma 3.3.** Suppose that in a neighbourhood of \( u = 0 \), \( A(u) \in C^2 \) and system \( (1.1) \) is strictly hyperbolic, i.e., \((1.6)\) holds. Suppose furthermore that the initial data satisfy \((1.13)\). Then there exists \( \theta_0 > 0 \) so small that for any fixed \( \theta \in (0, \theta_0] \), on any given existence domain \( R(T) \) of the weakly discontinuous solution \( u = u(t, x) \) (see (3.7)) to Cauchy problem \((1.1)\) and \((1.12)\), we have the following uniform a priori estimates

\[
W^c_\infty(T) \leq \kappa_1 \theta, \tag{3.34}
\]

\[
\tilde{W}_1(T), W_1(T) \leq \kappa_2 \theta, \tag{3.35}
\]

\[
U_\infty(T) \leq \kappa_3 \theta, \tag{3.36}
\]

here and henceforth \( \kappa_i (i = 1, 2, \cdots) \) are positive constants independent of \( \theta \) and \( T \).

**Proof.** We first estimate \( W^c_\infty(T) \).

For any given \( i \in \{1, \cdots, n\} \), passing through any fixed point \((t, x) \in R(T) \setminus D^T_i \), we draw the \( i \)-th characteristic \( c_i : \xi = \xi_i(\tau) \) \((\tau \leq t)\) which intersects the \( x \)-axis at a point \((0, x_{i0})\). When \((t, x) \in R_i(T) \setminus D^T_i \) for some \( l < i \), noting Lemma 3.1, integrating (2.16) along \( c_i \) from 0 to \( t \) yields

\[
w^{(l)}_i(t, x) = w^{(0)}_i(0, x_{i0}) + \sum_{j,m=1}^n \gamma_{ijm}(u^{(0)})w^{(0)}_j w^{(0)}_m \tag{3.37}(\tau, \xi_i(\tau))d\tau
\]

\[
+ \int^t_{t_{il+2}} \sum_{j,m=1}^n \gamma_{ijm}(u^{(k)})w^{(k)}_j w^{(k)}_m \tag{3.38}(\tau, \xi_i(\tau))d\tau;
\]

while, when \((t, x) \in R_i(T) \setminus D^T_i \) for some \( l \geq i \), similarly we have

\[
w^{(l)}_i(t, x) = w^{(n)}_i(0, x_{i0}) + \sum_{j,m=1}^n \gamma_{ijm}(u^{(n)})w^{(n)}_j w^{(n)}_m \tag{3.37}(\tau, \xi_i(\tau))d\tau
\]

\[
+ \sum_{k=1}^{n} \int^{t_{ik+2}}_{t_{ik}} \sum_{j,m=1}^n \gamma_{ijm}(u^{(k)})w^{(k)}_j w^{(k)}_m \tag{3.38}(\tau, \xi_i(\tau))d\tau;
\]

here and hereafter, \((t_{ik}, x_k(t_{ik}))\) stands for the intersection point of \( c_i \) with the \( k \)-th weak discontinuity \( x = x_k(t) \) \((k = 1, \cdots, n)\). Moreover, by the definition of \( D^T_1 \) and \( D^T_n \), when \( i = 1 \), (3.37) disappears, and, when \( i = n \), (3.38) disappears. Then, by using Lemma 3.2 and (2.18) and noting (3.9) and \(|\xi_i(\tau) - \lambda_j(0)\tau| \geq \delta_0 \tau \) when \((\tau, \xi_i(\tau)) \in D^T_j \), it is easy to see that

\[
(1 + |x - \lambda_i(0)t|^{1+\mu}|w^{(l)}_i(t, x)|)
\]
\[
\leq C(1 + |x_{i0}|)^{1+\mu}(|w^{(0)}_i(0, x_{i0})| + |w^{(n)}_i(0, x_{i0})|) + C\{W^c_\infty(T)\tilde{W}_1(T) + (W^c_\infty(T))^2\}, \tag{3.39}
\]
Moreover, for similarly we have the characteristic 
\[ c_i(\xi) \]
(\( t = 0 \)). Then, noting (2.22), (3.4) and (3.9) and by using Lemma 3.2, it is easy to get that

\[ (1.13) \]

Thus, we finally get
\[ \tilde{W}_1(T) \leq C(\theta + W^c_\infty(T)\tilde{W}_1(T) + (W^c_\infty(T))^2). \] (3.40)

We next estimate \( \tilde{W}_1(T) \) and \( W_1(T) \).

For \( i \in \{1, \ldots, n-1\} \), passing through any given point \( A(t, x) \in D_i^T \cap R_i(T) \), we draw the \( j \)-th characteristic \( c_j : \xi = \xi_j(\tau) \) \((\tau \leq t, j > i)\) which intersects the \( i \)-th weak discontinuity \( x = x_i(t) \) at a point \( B(t_B, x_B) \). In the meantime, the \( i \)-th characteristic \( c_i : \xi = \xi_i(\tau) \) \((\tau \leq t)\) passing through point \( A \) intersects the boundary \( x = (\lambda_i(0) + \delta_0)t \) of \( D_i^T \) at a point \( C \). By (2.20), using Stokes' formula on the domain \( ABOC \) we get

\[
\int_{t_B}^t |w^{(i)}_t(\lambda_j(u^{(i)}) - \lambda_i(u^{(i)}))|d\tau \\
\leq \int_{OC} |w^{(i)}_t(\lambda_i(0) + \delta_0 - \lambda_i(u^{(i)}))|d\tau \\
+ \int_{ABOC} \sum_{k,m=1}^n \Gamma_{ikm}(u^{(i)}w^{(i)}_k w^{(i)}_m(t, x))dtdx. \] (3.41)

Then, noting (2.22), (3.4) and (3.9) and by using Lemma 3.2, it is easy to get that

\[ \int_{c_j} |w^{(i)}_t|d\tau = \int_{t_B}^t |w^{(i)}_t(\tau, \xi_j(\tau))|d\tau \leq C\{W^c_\infty(T) + W^c_\infty(T)W_1(T) + (W^c_\infty(T))^2\}. \] (3.42)

When \( j < i \), the \( j \)-th characteristic \( c_j : \xi = \xi_j(\tau) \) \((\tau \leq t)\) intersects the boundary \( x = (\lambda_i(0) + \delta_0)t \) of \( D_i^T \) at a point \( B(t_B, x_B) \). Using Stokes' formula on the domain \( ACB \), similarly we still get (3.42).

For \( i = n \), passing through any given point \( A(t, x) \in D_i^T \cap R_n(T) \), both the \( j \)-th characteristic \( c_j : \xi = \xi_j(\tau) \) \((\tau \leq t)\) and the \( i \)-th characteristic \( c_i : \xi = \xi_i(\tau) \) \((\tau \leq t)\) intersect the \( x \)-axis at points \( B(0, x_B) \) and \( C(0, x_C) \) respectively. Using Stokes' formula on the domain \( ACB \), similarly we have

\[ \int_{c_j} |w^{(n)}_t|d\tau = \int_{t_B}^t |w^{(n)}_t(\tau, \xi_j(\tau))|d\tau \leq C\{\theta + W^c_\infty(T)W_1(T) + (W^c_\infty(T))^2\}. \] (3.43)

On the other hand, for \( i \in \{2, \ldots, n\} \) and any given point \( A(t, x) \in D_i^T \cap R_{i-1}(T) \), similarly we have

\[ \int_{c_j} |w^{(i-1)}_t|d\tau = \int_{t_B}^t |w^{(i-1)}_t(\tau, \xi_j(\tau))|d\tau \leq C\{W^c_\infty(T) + W^c_\infty(T)W_1(T) + (W^c_\infty(T))^2\}. \] (3.44)

Moreover, for \( i = 1 \), we have

\[ \int_{c_j} |w^{(0)}_t|d\tau = \int_{t_B}^t |w^{(0)}_t(\tau, \xi_j(\tau))|d\tau \leq C\{\theta + W^c_\infty(T)W_1(T) + (W^c_\infty(T))^2\}. \] (3.45)

Thus, we finally get

\[
\tilde{W}_1(T) \leq C(\theta + W^c_\infty(T) + W^c_\infty(T)W_1(T) + (W^c_\infty(T))^2). \] (3.46)
Similarly, we can obtain (cf. [9])
\[ W_1(T) \leq C\{ \theta + W_\infty^c(T) + W_\infty^c(T)W_1(T) + (W_\infty^c(T))^2 \}. \tag{3.47} \]

The combination of (3.40) and (3.46)–(3.47) gives (3.34)–(3.35) (cf. [13]).

Finally, we estimate \( U_\infty(T) \).

Passing through any given point \( (t, x) \in R(T) \), we draw the \( n \)-th characteristic \( c_n : \xi = \xi_n(\tau) \ (\tau \leq t) \) which intersects the \( x \)-axis at a point \( (0, x_0) \). When \( (t, x) \in R_l(T) \) for \( l \in \{0, 1, \cdots, n - 1\} \), integrating (2.6) (in which \( i = n \)) along \( c_n \) from 0 to \( t \) gives
\[ u^{(i)}(t, x) = u^{(0)}(0, x_0) + \int_0^t \sum_{m=1}^{n-1} (\lambda_n(u^{(0)})) w_m^{(0)} r_m(u^{(0)})(\tau, \xi_n(\tau))d\tau 
+ \sum_{k=1}^{l-1} \int_{t_{n,k}}^{t} \sum_{m=1}^{n-1} (\lambda_n(u^{(k)})) w_m^{(k)} r_m(u^{(k)})(\tau, \xi_n(\tau))d\tau 
+ \int_{t_{n,i}}^{t} \sum_{m=1}^{n-1} (\lambda_n(u^{(i)})) w_m^{(i)} r_m(u^{(i)})(\tau, \xi_n(\tau))d\tau; \tag{3.48} \]
while, when \( (t, x) \in R_n(T) \), similarly we have
\[ u^{(n)}(t, x) = u^{(n)}(0, x_0) + \int_0^t \sum_{m=1}^{n-1} (\lambda_n(u^{(n)})) w_m^{(n)} r_m(u^{(n)})(\tau, \xi_n(\tau))d\tau. \tag{3.49} \]

Then, noting (1.13) and by using (3.34)–(3.35), it is easy to see that
\[ |u(t, x)| \leq C\{ \theta + W_\infty^c(T) + W_1(T) \} \leq C\theta. \tag{3.50} \]
Thus, (3.36) follows immediately. At the same time, (3.50) also means that hypothesis (3.9) is reasonable.

**Lemma 3.4.** Under the assumptions of Lemma 3.3, suppose furthermore that system (1.1) is weakly linearly degenerate, then, in normalized coordinates there exists \( \theta_0 > 0 \) so small that for any fixed \( \theta \in (0, \theta_0) \), on any given existence domain \( R(T) \) of the weakly discontinuous solution \( u = u(t, x) \) to Cauchy problem (1.1) and (1.12), we have the following uniform a priori estimates
\[ U_\infty^c(T) \leq \kappa_4\theta, \tag{3.51} \]
\[ W_\infty(T) \leq \kappa_5\theta. \tag{3.52} \]

**Proof.** Similarly to (3.16)–(3.17), let
\[ \tilde{U}_1(T) = \max_{i=1, \cdots, n} \max_{j \neq i} \left\{ \sup_{c_j} \int_{c_j \cap R_{i-1}(T)} |u_{i}^{(i-1)}(t, x)|dt + \sup_{c_j} \int_{c_j \cap R_{i}(T)} |u_{i}^{(i)}(t, x)|dt \right\}, \tag{3.53} \]
\[ U_1(T) = \max_{i=1, \cdots, n} \sup_{0 \leq t \leq T} \left\{ \int_{a(t)}^{x(t)} |u_{i}^{(i-1)}(t, x)|dx + \int_{x(t)}^{b(t)} |u_{i}^{(i)}(t, x)|dx \right\}. \tag{3.54} \]

We now estimate \( U_\infty^c(T) \).
Similarly to (3.37)–(3.38), when \((t, x) \in R_l(T) \setminus D^+_l\) for some \(l < i\), integrating (2.7) along \(c_i\) from 0 to \(t\), we have
\[
u^{(l)}_{(t, x)} = u^{(l)}_{(0, x, 0)} + \int_0^{t_{l,1}} \sum_{j,m=1}^n \rho_{ijm}(u^{(n)}) u^{(n)}_j w^{(n)}_m (\tau, \xi_i(\tau)) d\tau + \int_0^{t_{l,1+k+1}} \sum_{j,m=1}^n \rho_{ijm}(u^{(k)}) u^{(k)}_j w^{(k)}_m (\tau, \xi_i(\tau)) d\tau + \int_0^t \sum_{j,m=1}^n \rho_{ijm}(u^{(l)}) u^{(l)}_j w^{(l)}_m (\tau, \xi_i(\tau)) d\tau; \tag{3.55}\]
while, when \((t, x) \in R_l(T) \setminus D^+_l\) for some \(l \geq i\), we have
\[
u^{(l)}_{(t, x)} = u^{(n)}_{(0, x, 0)} + \int_0^{t_{i,n}} \sum_{j,m=1}^n \rho_{ijm}(u^{(n)}) u^{(n)}_j w^{(n)}_m (\tau, \xi_i(\tau)) d\tau + \int_0^{t_{l+2,1+k-1}} \sum_{j,m=1}^n \rho_{ijm}(u^{(k-1)}) u^{(k-1)}_j w^{(k-1)}_m (\tau, \xi_i(\tau)) d\tau + \int_0^t \sum_{j,m=1}^n \rho_{ijm}(u^{(l)}) u^{(l)}_j w^{(l)}_m (\tau, \xi_i(\tau)) d\tau. \tag{3.56}\]
Then, noting (2.8) and using Lemma 3.2, similarly to (3.40) we get
\[U^\infty_\infty(T) \leq C \{ \theta + U^\infty_\infty(T) \bar{W}_1(T) + W^\infty_\infty(T) U^\infty_\infty(T) + \bar{U}_1(T) W^\infty_\infty(T) \}. \tag{3.57}\]
Hence, using Lemma 3.3, we get immediately
\[U^\infty_\infty(T) \leq C \theta \{ 1 + \bar{U}_1(T) \}. \tag{3.58}\]

We next estimate \(\bar{U}_1(T)\) and \(U_1(T)\).
For \(i \in \{ 1, \ldots, n-1 \}\), similarly to (3.41), by (2.11) we have
\[
\int_{t_B}^t |u^{(i)}_{(t)}(\lambda_ju^{(i)} - \lambda_iu^{(i)})(\tau, \xi_j(\tau))| d\tau \\
\leq \int_{OC} |u^{(i)}(\lambda_i(0) + \delta_0 - \lambda_i(u^{(i)})) (\tau, (\lambda_i(0) + \delta_0)\tau)| d\tau + \iint_{ABOC} \left| \sum_{k,m=1}^n F_{ikm}(u^{(i)}) u^{(i)}_k w^{(i)}_m(t, x) \right|dt dx. \tag{3.59}\]
Then, noting (2.13)–(2.14), the second term on the right hand side of (3.59) can be rewritten as
\[
\iint_{ABOC} \left| \sum_{k,m=1}^n F_{ikm}(u^{(i)}) u^{(i)}_k w^{(i)}_m(t, x) \right| dt dx = \iint_{ABOC} \left| \sum_{k \neq m} F_{ikm}(u^{(i)}) u^{(i)}_k w^{(i)}_m(t, x) + F_{iii}(u^{(i)}) u^{(i)}_i w^{(i)}_i(t, x) \right| dt dx. \tag{3.60}\]
Finally, (3.51) follows immediately from the combination of (3.58) and (3.67).

Thus we get

Similarly, we have

Hence, similarly to (3.42), using Lemma 3.3, from (3.59) we get

Then, using Hardmard’s formula, we have

Hence, we get

Moreover, similarly to (3.44)–(3.45), we can estimate

For \( i = n \), similarly to (3.43), we have

Moreover, similarly to (3.44)–(3.45), we can estimate

Hence, we get

Similarly, we have

Thus we get

Finally, (3.51) follows immediately from the combination of (3.58) and (3.67).

We finally estimate \( W_\infty(T) \).

For any given \( i \in \{1, \ldots, n\} \) and any given point \((t, x) \in D_T^1\), let \( c_i : \xi = \xi_i(\tau) \ (\tau \leq t) \) be the \( i \)-th characteristic passing through \((t, x)\), which intersects the \( x \)-axis at a point \((0, x_{i0})\).
When \( (t, x) \in R_{i-1}(T) \), integrating (2.16) along \( c_i \) from 0 to \( t \) gives

\[
\begin{align*}
\psi_i^{(i-1)}(t, x) &= \psi_i^{(0)}(0, x_0) + \int_0^t \sum_{j, m=1}^n \gamma_{ijm}(u^{(0)}) w_j^{(0)} w_m^{(0)}(\tau, \xi_i(\tau)) d\tau \\
&\quad + \sum_{k=1}^{i-2} \int_{t_{i,k}}^{t_{i,k+1}} \sum_{j, m=1}^n \gamma_{ijm}(u^{(k)}) w_j^{(k)} w_m^{(k)}(\tau, \xi_i(\tau)) d\tau \\
&\quad + \int_{t_{i,i-1}}^t \sum_{j, m=1}^n \gamma_{ijm}(u^{(i-1)}) w_j^{(i-1)} w_m^{(i-1)}(\tau, \xi_i(\tau)) d\tau;
\end{align*}
\]

while, when \( (t, x) \in R_i(T) \), similarly we have

\[
\begin{align*}
\psi_i^{(i)}(t, x) &= \psi_i^{(n)}(0, x_0) + \int_0^t \sum_{j, m=1}^n \gamma_{ijm}(u^{(n)}) w_j^{(n)} w_m^{(n)}(\tau, \xi_i(\tau)) d\tau \\
&\quad + \sum_{k=i+2}^{i+k-1} \sum_{j, m=1}^n \gamma_{ijm}(u^{(k-1)}) w_j^{(k-1)} w_m^{(k-1)}(\tau, \xi_i(\tau)) d\tau \\
&\quad + \int_{t_{i,i-1}}^t \sum_{j, m=1}^n \gamma_{ijm}(u^{(i)}) w_j^{(i)} w_m^{(i)}(\tau, \xi_i(\tau)) d\tau.
\end{align*}
\]

Since \( \lambda_i(u) \) is weakly linearly degenerate and \( u = (u_1, \cdots, u_n)^T \) are normalized coordinates, by (2.19) we have

\[
\gamma_{iii}(u_i e_i) \equiv 0, \quad \forall |u_i| \text{ small.} \tag{3.70}
\]

Then, noting (1.13) and (2.18), similarly to (3.63) and (3.64), it is easy to get

\[
|w_i^{(i-1)}(t, x)|, \quad |w_i^{(i)}(t, x)| \leq C\{\theta + W_\infty^c(T)\tilde{W}_1(T) + (W_\infty^c(T))^2 \\
+ W_\infty^c(T)W_\infty(T) + U_\infty^c(T)(W_\infty(T))^2\}. \tag{3.71}
\]

Thus, noting Lemma 3.3 and (3.51) we have

\[
W_\infty(T) \leq C\theta\{1 + W_\infty(T) + (W_\infty(T))^2\}, \tag{3.72}
\]

which implies (3.52).

From Lemmas 3.3 and 3.4, the sufficiency in Theorem 1.1 follows immediately.

We now prove the necessity in Theorem 1.1.

For the Cauchy problem of a scalar equation

\[
\begin{align*}
\begin{cases}
\frac{\partial v}{\partial t} + \lambda(v) \frac{\partial v}{\partial x} = 0, \\
t = 0: \quad v = \begin{cases}
\psi_l(x), & x \leq 0, \\
\psi_r(x), & x \geq 0
\end{cases}
\end{cases}
\end{align*}
\]

with

\[
\psi_l(0) = \psi_r(0) \quad \text{and} \quad \psi_l'(0) \neq \psi_r'(0), \tag{3.74}
\]

where \( \psi_l(x) \) and \( \psi_r(x) \in C^1 \) and satisfy (1.13), it is easy to get
Lemma 3.5. There exists \( \theta_0 > 0 \) so small that for any given \( \theta \in (0, \theta_0] \), Cauchy problem (3.73) admits a unique global weakly discontinuous solution if and only if \( \lambda(v) \) is a constant in a neighbourhood of \( v = 0 \).

Then, noting that in normalized coordinates the characteristic \( \lambda_i(u) \) is weakly linearly degenerate if and only if

\[
\lambda_i(u, c_i) \equiv \text{const.}, \quad \forall |u_i| \text{ small},
\]

we easily get the necessity in Theorem 1.1 (cf. [4, 5]).

Remark 3.1. Comparing with the method used in [6] and [13], the estimates on the domains \( D_{T^\pm} \) and \( D_{T^0} \) and the estimates for \( v_i = l_i(u)(i = 1, \cdots, n) \) are all omitted in the proof of Theorem 1.1.

§ 4. Application

Consider the following Cauchy problem for the system of the planar motion of an elastic string (cf. [8, 13])

\[
\begin{aligned}
&u_t - v_x = 0, \\
v_t - \left( \frac{T(r)}{r} u \right)_x = 0
\end{aligned}
\] (4.1)

with the initial condition

\[
t = 0: \quad (u, v) = \begin{cases}
(u_0 + u_l(x), v_0 + v_l(x)) & (x \leq 0), \\
(u_0 + u_r(x), v_0 + v_r(x)) & (x \geq 0),
\end{cases}
\] (4.2)

where

\[
(u_l(0), v_l(0)) = (u_r(0), v_r(0)) \quad \text{and} \quad (u_l'(0), v_l'(0)) \neq (u_r'(0), v_r'(0)),
\] (4.3)

\( u = (u_1, u_2)^T, \quad v = (v_1, v_2)^T, \quad r = |u| = \sqrt{u_1^2 + u_2^2}, \quad T(r) \) is a \( C^3 \) function of \( r > 1 \), such that

\[
T'(\tilde{r}_0) > \frac{T(\tilde{r}_0)}{\tilde{r}_0} > 0,
\] (4.4)

in which \( \tilde{u}_0 \) and \( \tilde{v}_0 \) are constant vectors and \( \tilde{r}_0 = |\tilde{u}_0| > 1, (u_l(x), v_l(x)) \) and \( (u_r(x), v_r(x)) \in C^1 \) and satisfy (1.13). Let

\[
U = \begin{pmatrix} u \\ v \end{pmatrix}.
\] (4.5)

By (4.4), in a neighbourhood of \( U_0 = \left( \frac{\tilde{u}_0}{\tilde{v}_0} \right) \), (4.1) is a strictly hyperbolic system with the following distinct real eigenvalues:

\[
\lambda_1(U) = -\sqrt{T'(r)} < \lambda_2(U) = -\sqrt{\frac{T(r)}{r}} < 0 < \lambda_3(U) = \sqrt{\frac{T(r)}{r}} < \lambda_4(U) = \sqrt{T'(r)}. \] (4.6)

\( \lambda_2(U) \) and \( \lambda_3(U) \) are linearly degenerate in the sense of P. D. Lax, then weakly linearly degenerate. Moreover, \( \lambda_1(U) \) and \( \lambda_4(U) \) are also linearly degenerate, then weakly linearly degenerate, provided that

\[
T''(r) \equiv 0, \quad \forall |r - \tilde{r}_0| \text{ small.} \] (4.7)
By Theorem 1.1 we get

**Theorem 4.1.** Suppose that (4.7) holds. There exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, Cauchy problem (4.1)–(4.2) admits a unique global weakly discontinuous solution $U = U(t, x)$ on $t \geq 0$, which possesses at most 4 weak discontinuities $x = x_k(t)$ ($k = 1, \ldots, 4$) passing through the origin.

**References**


