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GLOBAL EXISTENCE OF WEAKLY DISCONTINUOUS SOLUTIONS TO THE CAUCHY PROBLEM WITH A KIND OF NON-SMOOTH INITIAL DATA FOR QUASILINEAR HYPERBOLIC SYSTEMS***

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Abstract

The authors consider the Cauchy problem with a kind of non-smooth initial data for quasilinear hyperbolic systems and obtain a necessary and sufficient condition to guarantee the existence and uniqueness of global weakly discontinuous solution.

Keywords Quasilinear hyperbolic system, Cauchy problem, Global weakly discontinuous solution, Weakly linear degeneracy

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§ 1. Introduction and Main Result

Consider the following first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x) and $A(u)$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

By the definition of hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \cdots, \lambda_n(u)$ and a complete set of left (resp. right) eigenvectors. For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \tag{1.2}$$

and

$$A(u)r_i(u) = \lambda_i(u)r_i(u), \tag{1.3}$$

we have

$$\det |l_{ij}(u)| \neq 0 \qquad (\text{resp.}\ \ \det |r_{ij}(u)| \neq 0). \tag{1.4}$$

Without loss of generality, we assume that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \qquad (i, j = 1, \cdots, n), \tag{1.5}$$

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where δ_{ij} stands for the Kronecker's symbol.

In particular, if, for any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u), \tag{1.6}$$

system (1.1) is called to be strictly hyperbolic.

For the Cauchy problem of system (1.1) with the initial data

$$t = 0 : \quad u = \phi(x) \qquad (-\infty < x < \infty), \tag{1.7}$$

where $\phi(x)$ is a C^1 vector function with bounded C^1 norm, it was proved in [3–6] and [12, 13] that if system (1.1) is strictly hyperbolic, then, for any given initial data satisfying the following small and decaying property:

$$\theta \triangleq \sup_{x\in\mathbb{R}}\{(1+|x|)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} \ll 1, \tag{1.8}$$

where $\mu > 0$ is a constant, Cauchy problem (1.1) and (1.7) admits a unique global C^1 solution $u = u(t, x)$ with small C^1 norm for all $t \in \mathbb{R}$, if and only if system (1.1) is weakly linearly degenerate, i.e., all the characteristics are weakly linearly degenerate (see also [9, 10] and [15–18] for some related results). Here, we call $\lambda_i(u)$ $(i \in \{1, \cdots, n\})$ a weakly linearly degenerate characteristic if, along the i-th characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by

$$\begin{cases} \dfrac{du}{ds} = r_i(u), \\ s = 0 : \quad u = 0, \end{cases} \tag{1.9}$$

we have

$$\nabla\lambda_i(u)r_i(u) \equiv 0, \qquad \forall\, |u| \text{ small}, \tag{1.10}$$

namely

$$\lambda_i(u^{(i)}(s)) \equiv \lambda_i(0), \qquad \forall\, |s| \text{ small}. \tag{1.11}$$

In the previous result, the initial data are supposed to be in the C^1 class. However, in some practical problems, we are required to deal with the Cauchy problem for system (1.1) with the following kind of non-smooth initial data

$$t = 0 : \quad u = \begin{cases} u_l(x), & x \le 0, \\ u_r(x), & x \ge 0, \end{cases} \tag{1.12}$$

where $u_l(x)$ and $u_r(x)$ are C^1 vector functions on $x \le 0$ and $x \ge 0$ respectively and satisfy the following small and decaying property

$$\theta \triangleq \sup_{x\le 0}\{(1+|x|)^{1+\mu}(|u_l(x)| + |u_l'(x)|)\} + \sup_{x\ge 0}\{(1+x)^{1+\mu}(|u_r(x)| + |u_r'(x)|)\} < +\infty, \tag{1.13}$$

where $\mu > 0$ is a constant; moreover,

$$u_l(0) = u_r(0) \qquad \text{and} \qquad u_l'(0) \ne u_r'(0). \tag{1.14}$$

In this paper, we will generalize the previous result to Cauchy problem (1.1) and (1.12). In the meantime, the method used in [6] and [13] will be simplified and improved. In order to state the main result of this paper, we first give the following

Definition 1.1. *A continuous and piecewise C^1 vector function*

$$u = u(t,x) = \begin{cases} u_-(t,x), & x \le x_k(t), \\ u_+(t,x), & x \ge x_k(t) \end{cases} \tag{1.15}$$

is called a weakly discontinuous solution containing a k-th weak discontinuity $x = x_k(t)$ for system (1.1)*, if $u = u(t,x)$ satisfies system* (1.1) *in the classical sense on both sides of $x = x_k(t)$,*

$$u_-(t, x_k(t)) = u_+(t, x_k(t)) \tag{1.16}$$

and $x = x_k(t)$ is the corresponding k-th characteristic:

$$\frac{dx_k(t)}{dt} = \lambda_k(u_-(t, x_k(t))) = \lambda_k(u_+(t, x_k(t))), \tag{1.17}$$

moreover, the first order derivatives of $u(t,x)$ have the first kind discontinuity on $x = x_k(t)$.

Our main result is the following

Theorem 1.1. *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and system* (1.1) *is strictly hyperbolic. Suppose furthermore that $u_l(x)$ and $u_r(x)$ are C^1 vector functions on $x \le 0$ and $x \ge 0$ respectively. Then there exists $\theta_0 > 0$ so small that for any given initial data satisfying* (1.13)–(1.14) *with $\theta \in (0, \theta_0]$, Cauchy problem* (1.1) *and* (1.12) *admits a unique global weakly discontinuous solution $u = u(t,x)$ containing n weak discontinuities $x = x_k(t)$ $(k = 1, \cdots, n)$, where $x = x_k(t)$ with $x_k(0) = 0$ denotes a k-th weak discontinuity passing through the origin $(0,0)$, if and only if system* (1.1) *is weakly linearly degenerate. Precisely speaking, the solution $u = u(t,x)$ should have the following structure:*

$$u = u(t,x) = \begin{cases} u^{(0)}(t,x), & (t,x) \in R_0, \\ u^{(l)}(t,x), & (t,x) \in R_l \quad (l = 1, \cdots, n-1), \\ u^{(n)}(t,x), & (t,x) \in R_n, \end{cases} \tag{1.18}$$

in which $u^{(l)}(t,x) \in C^1$ satisfies system (1.1) *in the classical sense on R_l $(l = 0, 1, \cdots, n)$ with*

$$R_l = \begin{cases} \{(t,x) \mid t \ge 0,\ x \le x_1(t)\} & (l = 0), \\ \{(t,x) \mid t \ge 0,\ x_l(t) \le x \le x_{l+1}(t)\} & (l = 1, \cdots, n-1), \\ \{(t,x) \mid t \ge 0,\ x \ge x_n(t)\} & (l = n). \end{cases} \tag{1.19}$$

Moreover, for $k = 1, \cdots, n$,

$$u^{(k-1)}(t, x_k(t)) = u^{(k)}(t, x_k(t)), \tag{1.20}$$

$$\frac{dx_k(t)}{dt} = \lambda_k(u^{(k-1)}(t, x_k(t))) = \lambda_k(u^{(k)}(t, x_k(t))). \tag{1.21}$$

Remark 1.1. In Theorem 1.1, some weak discontinuities may degenerate.

Remark 1.2. Suppose that (1.1) is a non-strictly hyperbolic system with characteristics with constant multiplicity, say,

$$\lambda_1(u) < \cdots < \lambda_k(u) < \lambda_{k+1}(u) \equiv \cdots \equiv \lambda_{k+p}(u) < \lambda_{k+p+1}(u) < \cdots < \lambda_n(u) \qquad (p > 1). \tag{1.22}$$

Then, if there exist normalized coordinates, similar conclusion holds as in Theorem 1.1 (some related results can be found in [7, 14]).

The paper is organized as follows. In Section 2 we give some preliminaries. Then, the main result is proved in Section 3. Finally, an application is given in Section 4.

§ 2 . Preliminaries

By Lemma 2.5 in [12], when system (1.1) is strictly hyperbolic, there exists a suitably smooth invertible transformation $u = u(\widetilde{u})$ $(u(0) = 0)$ such that in the $\widetilde{u}$-space, for each $i = 1, \cdots, n$, the i-th characteristic trajectory passing through $\widetilde{u} = 0$ coincides with the $\widetilde{u}_i$-axis at least for $|\widetilde{u}_i|$ small, namely,

$$\widetilde{r}_i(\widetilde{u}_i e_i) // e_i, \qquad \forall\, |\widetilde{u}_i| \ \text{ small} \quad (i = 1, \cdots, n), \tag{2.1}$$

where $\widetilde{r}_i(\widetilde{u})$ denotes the i-th right eigenvector corresponding to $r_i(u)$ and

$$e_i = (0, \cdots, 0, \overset{(i)}{1}, 0, \cdots, 0)^T. \tag{2.2}$$

This transformation is called a normalized transformation, and the unknown variables $\widetilde{u} = (\widetilde{u}_1, \cdots, \widetilde{u}_n)^T$ are called normalized variables or normalized coordinates.

Let

$$w_i = l_i(u) u_x \qquad (i = 1, \cdots, n). \tag{2.3}$$

By (1.5), it is easy to see that

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.4}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u) \frac{\partial}{\partial x} \tag{2.5}$$

denote the directional derivative with respect to t along the i-th characteristic. We have

$$\frac{du}{d_i t} = \sum_{\substack{k=1 \\ k \neq i}}^{n} (\lambda_i(u) - \lambda_k(u)) w_k r_k(u) \qquad (i = 1, \cdots, n). \tag{2.6}$$

Then, in normalized coordinates, it is easy to see that

$$\frac{du_i}{d_i t} = \sum_{j,k=1}^{n} \rho_{ijk}(u) u_j w_k \qquad (i = 1, \cdots, n), \tag{2.7}$$

where

$$\rho_{ijj}(u) \equiv 0, \qquad \forall\, i, j \tag{2.8}$$

and

$$\rho_{ijk}(u) = (\lambda_i(u) - \lambda_k(u)) \int_0^1 \frac{\partial r_{ki}}{\partial u_j} (\tau u_1, \cdots, \tau u_{k-1}, u_k, \tau u_{k+1}, \cdots, \tau u_n) d\tau, \qquad \forall\, j \neq k. \tag{2.9}$$

Obviously

$$\rho_{iji}(u) \equiv 0, \qquad \forall\, i, j. \tag{2.10}$$

Moreover, noting (2.4) and (2.7), we have

$$\begin{aligned} d[u_i(dx - \lambda_i(u) dt)] &= \Big[\frac{du_i}{d_i t} + \sum_{k=1}^{n} \nabla \lambda_i(u) r_k(u) u_i w_k \Big] dt \wedge dx \\ &= \sum_{j,k=1}^{n} F_{ijk}(u) u_j w_k dt \wedge dx, \end{aligned} \tag{2.11}$$

where

$$F_{ijk}(u) = \rho_{ijk}(u) + \nabla\lambda_j(u)r_k(u)\delta_{ij}. \tag{2.12}$$

Noting (2.8) and (2.10), it is easy to see that

$$F_{ijj}(u) \equiv 0, \qquad \forall\, j \neq i, \tag{2.13}$$

$$F_{iji}(u) \equiv 0, \qquad \forall\, j \neq i, \tag{2.14}$$

$$F_{iii}(u) = \nabla\lambda_i(u)r_i(u), \qquad \forall\, i. \tag{2.15}$$

On the other hand, we have (see [1–3] or [12])

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u)w_j w_k \qquad (i = 1, \cdots, n), \tag{2.16}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u))l_i(u)\nabla r_k(u)r_j(u) - \nabla\lambda_k(u)r_j(u)\delta_{ik} + (j|k)\}, \tag{2.17}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms. Hence

$$\gamma_{ijj}(u) \equiv 0, \qquad \forall\, j \neq i, \tag{2.18}$$

$$\gamma_{iii}(u) = -\nabla\lambda_i(u)r_i(u), \qquad \forall\, i. \tag{2.19}$$

Noting (2.4), by (2.16) we have (see [1])

$$d[w_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u)w_j w_k dt \wedge dx, \tag{2.20}$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2}(\lambda_j(u) - \lambda_k(u))l_i(u)[\nabla r_k(u)r_j(u) - \nabla r_j(u)r_k(u)]. \tag{2.21}$$

Hence

$$\Gamma_{ijj}(u) \equiv 0, \qquad \forall\, i, j. \tag{2.22}$$

§ 3 . Proof of Theorem 1.1

In order to prove the sufficiency in Theorem 1.1, in what follows we always assume that $\theta > 0$ is suitably small.

By the existence and uniqueness of local weakly discontinuous solution to the Cauchy problem (see [11]), there exists $T_0 > 0$ so small that Cauchy problem (1.1) and (1.12) admits a unique weakly discontinuous solution $u = u(t, x)$ containing at most n weak discontinuities $x = x_k(t)$ $(k = 1, \cdots, n)$ on the domain $R(T_0) = \{(t, x) \mid 0 \le t \le T_0, -\infty < x < +\infty\} = \bigcup_{l=0}^{n} R_l(T_0)$:

$$u = u(t, x) = \begin{cases} u^{(0)}(t, x), & (t, x) \in R_0(T_0), \\ u^{(l)}(t, x), & (t, x) \in R_l(T_0) \quad (l = 1, \cdots, n-1), \\ u^{(n)}(t, x), & (t, x) \in R_n(T_0), \end{cases} \tag{3.1}$$

where

$$R_l(T_0)=\begin{cases}\{(t,x)\mid 0\le t\le T_0,\ x\le x_1(t)\} & (l=0),\\ \{(t,x)\mid 0\le t\le T_0,\ x_l(t)\le x\le x_{l+1}(t)\} & (l=1,\cdots,n-1),\\ \{(t,x)\mid 0\le t\le T_0,\ x\ge x_n(t)\} & (l=n).\end{cases}\tag{3.2}$$

In what follows, we establish a uniform a priori estimate on the C^0 norm of u and the piecewise C^0 norm of u_x on any given existence domain of the weakly discontinuous solution $u=u(t,x)$ to Cauchy problem (1.1) and (1.12). Noting (2.3), we only need to establish a uniform a priori estimate on the C^0 norm of u and the piecewise C^0 norm of $w=(w_1,\cdots,w_n)$ on any given existence domain of the weakly discontinuous solution $u=u(t,x)$.

Noting (1.6), we have

$$\lambda_1(0)<\lambda_2(0)<\cdots<\lambda_n(0).\tag{3.3}$$

Then, there exist positive constants δ and δ_0 so small that

$$\lambda_{i+1}(u)-\lambda_i(u')\ge 2\delta_0,\qquad \forall\,|u|,\,|u'|\le\delta\quad (i=1,\cdots,n-1),\tag{3.4}$$

$$|\lambda_i(u)-\lambda_i(u')|\le\frac{\delta_0}{2},\qquad \forall\,|u|,\,|u'|\le\delta\quad (i=1,\cdots,n).\tag{3.5}$$

Without loss of generality, we may assume that

$$\lambda_i(0)>\delta_0\qquad (i=1,\cdots,n).\tag{3.6}$$

For the time being we assume that on any given existence domain $R(T)=\{(t,x)\mid 0\le t\le T,-\infty<x<+\infty\}=\bigcup_{l=0}^{n}R_l(T)$ of the weakly discontinuous solution

$$u=u(t,x)=\begin{cases}u^{(0)}(t,x), & (t,x)\in R_0(T),\\ u^{(l)}(t,x), & (t,x)\in R_l(T)\quad (l=1,\cdots,n-1),\\ u^{(n)}(t,x), & (t,x)\in R_n(T)\end{cases}\tag{3.7}$$

to Cauchy problem (1.1) and (1.12), where

$$R_l(T)=\begin{cases}\{(t,x)\mid 0\le t\le T,\ x\le x_1(t)\} & (l=0),\\ \{(t,x)\mid 0\le t\le T,\ x_l(t)\le x\le x_{l+1}(t)\} & (l=1,\cdots,n-1),\\ \{(t,x)\mid 0\le t\le T,\ x\ge x_n(t)\} & (l=n),\end{cases}\tag{3.8}$$

we have

$$|u(t,x)|\le\delta,\qquad \forall\,(t,x)\in R(T).\tag{3.9}$$

At the end of the proof of Lemma 3.3, we will explain that this hypothesis is reasonable.

Let

$$D_i^T=\begin{cases}\{(t,x)\mid 0\le t\le T,\ x\le(\lambda_1(0)+\delta_0)t\} & (i=1),\\ \{(t,x)\mid 0\le t\le T,\ (\lambda_i(0)-\delta_0)t\le x\le(\lambda_i(0)+\delta_0)t\} & (i=2,\cdots,n-1),\\ \{(t,x)\mid 0\le t\le T,\ x\ge(\lambda_n(0)-\delta_0)t\} & (i=n).\end{cases}\tag{3.10}$$

Obviously

$$\bigcup_{i=1}^{n} D_i^T \subset R(T). \tag{3.11}$$

On any given existence domain $R(T) = \bigcup_{l=0}^{n} R_l(T)$ of the weakly discontinuous solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.12), let

$$w^{(l)} = (w_1^{(l)}, \cdots, w_n^{(l)}) \qquad (l = 0, 1, \cdots, n) \tag{3.12}$$

with

$$w_i^{(l)} = l_i(u^{(l)}) u_x^{(l)} \qquad (i = 1, \cdots, n), \tag{3.13}$$

$$W_\infty^c(T) = \max_{i=1,\cdots,n} \max_{l=0,1,\cdots,n} \sup_{(t,x)\in R_l(T)\setminus D_i^T} \{(1 + |x - \lambda_i(0)t|)^{1+\mu} |w_i^{(l)}(t,x)|\}, \tag{3.14}$$

$$U_\infty^c(T) = \max_{i=1,\cdots,n} \max_{l=0,1,\cdots,n} \sup_{(t,x)\in R_l(T)\setminus D_i^T} \{(1 + |x - \lambda_i(0)t|)^{1+\mu} |u_i^{(l)}(t,x)|\}, \tag{3.15}$$

$$\widetilde{W}_1(T) = \max_{i=1,\cdots,n} \max_{j\neq i} \Big\{ \sup_{c_j} \int_{c_j \cap R_{i-1}(T)} |w_i^{(i-1)}(t,x)| dt + \sup_{c_j} \int_{c_j \cap R_i(T)} |w_i^{(i)}(t,x)| dt \Big\}, \tag{3.16}$$

where c_j denotes any given j-th characteristic on D_i^T,

$$W_1(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T} \Big\{ \int_{a(t)}^{x_i(t)} |w_i^{(i-1)}(t,x)| dx + \int_{x_i(t)}^{b(t)} |w_i^{(i)}(t,x)| dx \Big\}, \tag{3.17}$$

where

$$a(t) = \begin{cases} -\infty, & \text{if } \ i = 1, \\ (\lambda_i(0) - \delta_0)t, & \text{if } \ i = 2, \cdots, n, \end{cases} \tag{3.18}$$

$$b(t) = \begin{cases} (\lambda_i(0) + \delta_0)t, & \text{if } \ i = 1, \cdots, n-1, \\ +\infty, & \text{if } \ i = n \end{cases} \tag{3.19}$$

and

$$U_\infty(T) = \|u(t,x)\|_{L^\infty(R(T))}, \tag{3.20}$$

$$W_\infty(T) = \sum_{l=0}^{n} \|w^{(l)}(t,x)\|_{L^\infty(R_l(T))}. \tag{3.21}$$

According to the definition of the weak discontinuity, it is easy to get

Lemma 3.1. *On the k-th weak discontinuity $x = x_k(t)$, we have*

$$w_i^{(k-1)} = w_i^{(k)}, \qquad \forall\, i \neq k. \tag{3.22}$$

Lemma 3.2. *For each $i = 1, \cdots, n$ and any given point $(t,x)\overline{\in} D_i^T$, let $c_i : \xi_i = \xi_i(\tau)$ $(\tau \le t)$ be the i-th characteristic passing through (t,x) and intersecting the x-axis at*

$(0, x_{i0})$. *Then there exist positive constants* d_k $(k = 1, 2, 3)$ *independent of* (t, x) *and* i*, such that*

$$d_1|x| \le |x - \lambda_i(0)t| \le d_2|x_{i0}| \tag{3.23}$$

and, if $(\tau, \xi_i(\tau))\overline{\in} D_j^T$ *for some* j*, then*

$$|\xi_i(\tau) - \lambda_j(0)\tau| \ge d_3|x_{i0}|. \tag{3.24}$$

Proof. When $i \in \{2, \cdots, n-1\}$, for any given point $(t, x)\overline{\in} D_i^T$, by the definition of D_i^T, we have

$$x \ge (\lambda_i(0) + \delta_0)t \qquad \text{or} \qquad x \le (\lambda_i(0) - \delta_0)t. \tag{3.25}$$

In what follows, we prove (3.23)–(3.24) for the case $x \ge (\lambda_i(0)+\delta_0)t$. When $x \le (\lambda_i(0)-\delta_0)t$, (3.23)–(3.24) can be similarly proved.

Noting (3.5), for $\tau \le t$, it is easy to get

$$\xi_i(\tau) \ge (\lambda_i(0) + \delta_0)\tau, \tag{3.26}$$

$$\Big(\lambda_i(0) - \frac{\delta_0}{2}\Big)\tau \le \xi_i(\tau) - x_{i0} \le \Big(\lambda_i(0) + \frac{\delta_0}{2}\Big)\tau. \tag{3.27}$$

Then, noting (3.6), we have

$$\xi_i(\tau) \le \frac{2(\lambda_i(0) + \delta_0)}{\delta_0} x_{i0}, \tag{3.28}$$

in particular,

$$x \le \frac{2(\lambda_i(0) + \delta_0)}{\delta_0} x_{i0}. \tag{3.29}$$

Thus, noting $x \ge (\lambda_i(0) + \delta_0)t$, we immediately get (3.23).

Since $(\tau, \xi_i(\tau))\overline{\in} D_i^T$, in order to prove (3.24), we first consider the case $j = i$. By (3.26)–(3.27), it is easy to get

$$|\xi_i(\tau) - \lambda_i(0)\tau| \ge \frac{\delta_0}{\lambda_i(0) + \delta_0} x_{i0}. \tag{3.30}$$

Now we consider the case that there exists $j \ne i$ such that $(\tau, \xi_i(\tau))\overline{\in} D_j^T$. When $j < i$, noting (3.3) and (3.30), we have

$$|\xi_i(\tau) - \lambda_j(0)\tau| \ge |\xi_i(\tau) - \lambda_i(0)\tau| \ge \frac{\delta_0}{\lambda_i(0) + \delta_0} x_{i0}. \tag{3.31}$$

When $j > i$, since $(\tau, \xi_i(\tau))\overline{\in} D_j^T$, we have

$$\xi_i(\tau) \ge (\lambda_j(0) + \delta_0)\tau \qquad \text{or} \qquad \xi_i(\tau) \le (\lambda_j(0) - \delta_0)\tau.$$

If $\xi_i(\tau) \ge (\lambda_j(0) + \delta_0)\tau$, similarly to (3.30) we get

$$|\xi_i(\tau) - \lambda_j(0)\tau| \ge \frac{\delta_0}{\lambda_j(0) + \delta_0} x_{i0}; \tag{3.32}$$

while, if $\xi_i(\tau) \le (\lambda_j(0) - \delta_0)\tau$, noting (3.27), it is easy to get

$$|\xi_i(\tau) - \lambda_j(0)\tau| \ge \frac{\delta_0}{\lambda_j(0) - \delta_0} x_{i0}. \tag{3.33}$$

The combination of (3.30)–(3.33) proves (3.24).

When $i = 1$ or n, noting the definition of D_1^T and D_n^T, similarly we can get (3.23)–(3.24).

Lemma 3.3. *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and system (1.1) is strictly hyperbolic, i.e., (1.6) holds. Suppose furthermore that the initial data satisfy (1.13). Then there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, on any given existence domain $R(T)$ of the weakly discontinuous solution $u = u(t,x)$ (see (3.7)) to Cauchy problem (1.1) and (1.12), we have the following uniform a priori estimates*

$$W_\infty^c(T) \le \kappa_1 \theta, \tag{3.34}$$

$$\widetilde{W}_1(T),\, W_1(T) \le \kappa_2 \theta, \tag{3.35}$$

$$U_\infty(T) \le \kappa_3 \theta, \tag{3.36}$$

here and henceforth κ_i $(i = 1, 2, \cdots)$ are positive constants independent of θ and T.

Proof. We first estimate $W_\infty^c(T)$.

For any given $i \in \{1, \cdots, n\}$, passing through any fixed point $(t,x) \in R(T)\backslash D_i^T$, we draw the i-th characteristic c_i: $\xi = \xi_i(\tau)$ $(\tau \le t)$ which intersects the x-axis at a point $(0, x_{i0})$. When $(t,x) \in R_l(T)\backslash D_i^T$ for some $l < i$, noting Lemma 3.1, integrating (2.16) along c_i from 0 to t yields

$$\begin{aligned} w_i^{(l)}(t,x) = w_i^{(0)}(0, x_{i0}) &+ \int_0^{t_{i1}} \sum_{j,m=1}^n \gamma_{ijm}(u^{(0)}) w_j^{(0)} w_m^{(0)}(\tau, \xi_i(\tau)) d\tau \\ &+ \sum_{k=1}^{l-1} \int_{t_{ik}}^{t_{i,k+1}} \sum_{j,m=1}^n \gamma_{ijm}(u^{(k)}) w_j^{(k)} w_m^{(k)}(\tau, \xi_i(\tau)) d\tau \\ &+ \int_{t_{il}}^t \sum_{j,m=1}^n \gamma_{ijm}(u^{(l)}) w_j^{(l)} w_m^{(l)}(\tau, \xi_i(\tau)) d\tau; \end{aligned} \tag{3.37}$$

while, when $(t,x) \in R_l(T)\backslash D_i^T$ for some $l \ge i$, similarly we have

$$\begin{aligned} w_i^{(l)}(t,x) = w_i^{(n)}(0, x_{i0}) &+ \int_0^{t_{in}} \sum_{j,m=1}^n \gamma_{ijm}(u^{(n)}) w_j^{(n)} w_m^{(n)}(\tau, \xi_i(\tau)) d\tau \\ &+ \sum_{k=l+2}^{n} \int_{t_{ik}}^{t_{i,k-1}} \sum_{j,m=1}^n \gamma_{ijm}(u^{(k-1)}) w_j^{(k-1)} w_m^{(k-1)}(\tau, \xi_i(\tau)) d\tau \\ &+ \int_{t_{i,l+1}}^t \sum_{j,m=1}^n \gamma_{ijm}(u^{(l)}) w_j^{(l)} w_m^{(l)}(\tau, \xi_i(\tau)) d\tau, \end{aligned} \tag{3.38}$$

here and hereafter, $(t_{ik}, x_k(t_{ik}))$ stands for the intersection point of c_i with the k-th weak discontinuity $x = x_k(t)$ $(k = 1, \cdots, n)$. Moreover, by the definition of D_1^T and D_n^T, when $i = 1$, (3.37) disappears, and, when $i = n$, (3.38) disappears. Then, by using Lemma 3.2 and (2.18) and noting (3.9) and $|\xi_i(\tau) - \lambda_j(0)\tau| \ge \delta_0 \tau$ when $(\tau, \xi_i(\tau)) \overline{\in} D_j^T$, it is easy to see that

$$\begin{aligned} &(1 + |x - \lambda_i(0)t|)^{1+\mu} |w_i^{(l)}(t,x)| \\ \le\ & C(1 + |x_{i0}|)^{1+\mu} (|w_i^{(0)}(0, x_{i0})| + |w_i^{(n)}(0, x_{i0})|) + C\{W_\infty^c(T) \widetilde{W}_1(T) + (W_\infty^c(T))^2\}, \end{aligned} \tag{3.39}$$

here and henceforth, C denotes different positive constants independent of θ and T. Noting (1.13), it turns out that

$$W^c_\infty(T) \le C\{\theta + W^c_\infty(T)\widetilde{W}_1(T) + (W^c_\infty(T))^2\}. \tag{3.40}$$

We next estimate $\widetilde{W}_1(T)$ and $W_1(T)$.

For $i \in \{1, \cdots, n-1\}$, passing through any given point $A(t, x) \in D^T_i \cap R_i(T)$, we draw the j-th characteristic $c_j : \xi = \xi_j(\tau)$ $(\tau \le t,\ j > i)$ which intersects the i-th weak discontinuity $x = x_i(t)$ at a point $B(t_B, x_B)$. In the meantime, the i-th characteristic $c_i : \xi = \xi_i(\tau)$ $(\tau \le t)$ passing through point A intersects the boundary $x = (\lambda_i(0) + \delta_0)t$ of D^T_i at a point C. By (2.20), using Stokes' formula on the domain $ABOC$ we get

$$\begin{aligned}
&\int_{t_B}^{t} |w_i^{(i)}(\lambda_j(u^{(i)}) - \lambda_i(u^{(i)}))(\tau, \xi_j(\tau))| d\tau \\
\le &\int_{OC} |w_i^{(i)}(\lambda_i(0) + \delta_0 - \lambda_i(u^{(i)}))(\tau, (\lambda_i(0) + \delta_0)\tau)| d\tau \\
&+ \iint_{ABOC} \Big| \sum_{k,m=1}^{n} \Gamma_{ikm}(u^{(i)}) w_k^{(i)} w_m^{(i)}(t, x) \Big| dt dx.
\end{aligned} \tag{3.41}$$

Then, noting (2.22), (3.4) and (3.9) and by using Lemma 3.2, it is easy to get that

$$\int_{c_j} |w_i^{(i)}| d\tau = \int_{t_B}^{t} |w_i^{(i)}(\tau, \xi_j(\tau))| d\tau \le C\{W^c_\infty(T) + W^c_\infty(T)W_1(T) + (W^c_\infty(T))^2\}. \tag{3.42}$$

When $j < i$, the j-th characteristic c_j: $\xi = \xi_j(\tau)$ $(\tau \le t)$ intersects the boundary $x = (\lambda_i(0) + \delta_0)t$ of D^T_i at a point $B(t_B, x_B)$. Using Stokes' formula on the domain ACB, similarly we still get (3.42).

For $i = n$, passing through any given point $A(t, x) \in D^T_n \cap R_n(T)$, both the j-th characteristic c_j: $\xi = \xi_j(\tau)$ $(\tau \le t)$ and the i-th characteristic c_i: $\xi = \xi_i(\tau)$ $(\tau \le t)$ intersect the x-axis at points $B(0, x_B)$ and $C(0, x_C)$ respectively. Using Stokes' formula on the domain ACB, similarly we have

$$\int_{c_j} |w_n^{(n)}| d\tau = \int_0^t |w_n^{(n)}(\tau, \xi_j(\tau))| d\tau \le C\{\theta + W^c_\infty(T)W_1(T) + (W^c_\infty(T))^2\}. \tag{3.43}$$

On the other hand, for $i \in \{2, \cdots, n\}$ and any given point $A(t, x) \in D^T_i \cap R_{i-1}(T)$, similarly we have

$$\int_{c_j} |w_i^{(i-1)}| d\tau = \int_{t_B}^{t} |w_i^{(i-1)}(\tau, \xi_j(\tau))| d\tau \le C\{W^c_\infty(T) + W^c_\infty(T)W_1(T) + (W^c_\infty(T))^2\}. \tag{3.44}$$

Moreover, for $i = 1$, we have

$$\int_{c_j} |w_1^{(0)}| d\tau = \int_0^t |w_1^{(0)}(\tau, \xi_j(\tau))| d\tau \le C\{\theta + W^c_\infty(T)W_1(T) + (W^c_\infty(T))^2\}. \tag{3.45}$$

Thus, we finally get

$$\widetilde{W}_1(T) \le C\{\theta + W^c_\infty(T) + W^c_\infty(T)W_1(T) + (W^c_\infty(T))^2\}. \tag{3.46}$$

Similarly, we can obtain (cf. [9])

$$W_1(T) \le C\{\theta + W_\infty^c(T) + W_\infty^c(T) W_1(T) + (W_\infty^c(T))^2\}. \tag{3.47}$$

The combination of (3.40) and (3.46)–(3.47) gives (3.34)–(3.35) (cf. [13]).

Finally, we estimate $U_\infty(T)$.

Passing through any given point $(t,x) \in R(T)$, we draw the n-th characteristic c_n : $\xi = \xi_n(\tau)$ $(\tau \le t)$ which intersects the x-axis at a point $(0, x_0)$. When $(t,x) \in R_l(T)$ for $l \in \{0, 1, \cdots, n-1\}$, integrating (2.6) (in which $i = n$) along c_n from 0 to t gives

$$\begin{aligned} u^{(l)}(t,x) = u^{(0)}(0,x_0) &+ \int_0^{t_{n1}} \sum_{m=1}^{n-1} (\lambda_n(u^{(0)}) - \lambda_m(u^{(0)})) w_m^{(0)} r_m(u^{(0)})(\tau, \xi_n(\tau)) d\tau \\ &+ \sum_{k=1}^{l-1} \int_{t_{nk}}^{t_{n,k+1}} \sum_{m=1}^{n-1} (\lambda_n(u^{(k)}) - \lambda_m(u^{(k)})) w_m^{(k)} r_m(u^{(k)})(\tau, \xi_n(\tau)) d\tau \\ &+ \int_{t_{nl}}^{t} \sum_{m=1}^{n-1} (\lambda_n(u^{(l)}) - \lambda_m(u^{(l)})) w_m^{(l)} r_m(u^{(l)})(\tau, \xi_n(\tau)) d\tau; \end{aligned} \tag{3.48}$$

while, when $(t,x) \in R_n(T)$, similarly we have

$$u^{(n)}(t,x) = u^{(n)}(0,x_0) + \int_0^t \sum_{m=1}^{n-1} (\lambda_n(u^{(n)}) - \lambda_m(u^{(n)})) w_m^{(n)} r_m(u^{(n)})(\tau, \xi_n(\tau)) d\tau. \tag{3.49}$$

Then, noting (1.13) and by using (3.34)–(3.35), it is easy to see that

$$|u(t,x)| \le C\{\theta + W_\infty^c(T) + \widetilde{W}_1(T)\} \le C\theta. \tag{3.50}$$

Thus, (3.36) follows immediately. At the same time, (3.50) also means that hypothesis (3.9) is reasonable.

Lemma 3.4. *Under the assumptions of Lemma* 3.3*, suppose furthermore that system* (1.1) *is weakly linearly degenerate, then, in normalized coordinates there exists* $\theta_0 > 0$ *so small that for any fixed* $\theta \in (0, \theta_0]$*, on any given existence domain* $R(T)$ *of the weakly discontinuous solution* $u = u(t,x)$ *to Cauchy problem* (1.1) *and* (1.12)*, we have the following uniform a priori estimates*

$$U_\infty^c(T) \le \kappa_4 \theta, \tag{3.51}$$

$$W_\infty(T) \le \kappa_5 \theta. \tag{3.52}$$

Proof. Similarly to (3.16)–(3.17), let

$$\widetilde{U}_1(T) = \max_{i=1,\cdots,n} \max_{j \ne i} \Big\{ \sup_{c_j} \int_{c_j \cap R_{i-1}(T)} |u_i^{(i-1)}(t,x)| dt + \sup_{c_j} \int_{c_j \cap R_i(T)} |u_i^{(i)}(t,x)| dt \Big\}, \tag{3.53}$$

$$U_1(T) = \max_{i=1,\cdots,n} \sup_{0 \le t \le T} \Big\{ \int_{a(t)}^{x_i(t)} |u_i^{(i-1)}(t,x)| dx + \int_{x_i(t)}^{b(t)} |u_i^{(i)}(t,x)| dx \Big\}. \tag{3.54}$$

We now estimate $U_\infty^c(T)$.

Similarly to (3.37)–(3.38), when $(t,x)\in R_l(T)\backslash D_i^T$ for some $l<i$, integrating (2.7) along c_i from 0 to t, we have

$$\begin{aligned} u_i^{(l)}(t,x) = u_i^{(0)}(0,x_{i0}) &+ \int_0^{t_{i1}} \sum_{j,m=1}^n \rho_{ijm}(u^{(0)})u_j^{(0)}w_m^{(0)}(\tau,\xi_i(\tau))d\tau \\ &+ \sum_{k=1}^{l-1}\int_{t_{ik}}^{t_{i,k+1}} \sum_{j,m=1}^n \rho_{ijm}(u^{(k)})u_j^{(k)}w_m^{(k)}(\tau,\xi_i(\tau))d\tau \\ &+ \int_{t_{il}}^{t} \sum_{j,m=1}^n \rho_{ijm}(u^{(l)})u_j^{(l)}w_m^{(l)}(\tau,\xi_i(\tau))d\tau; \end{aligned} \tag{3.55}$$

while, when $(t,x)\in R_l(T)\backslash D_i^T$ for some $l\ge i$, we have

$$\begin{aligned} u_i^{(l)}(t,x) = u_i^{(n)}(0,x_{i0}) &+ \int_0^{t_{in}} \sum_{j,m=1}^n \rho_{ijm}(u^{(n)})u_j^{(n)}w_m^{(n)}(\tau,\xi_i(\tau))d\tau \\ &+ \sum_{k=l+2}^{n}\int_{t_{ik}}^{t_{i,k-1}} \sum_{j,m=1}^n \rho_{ijm}(u^{(k-1)})u_j^{(k-1)}w_m^{(k-1)}(\tau,\xi_i(\tau))d\tau \\ &+ \int_{t_{i,l+1}}^{t} \sum_{j,m=1}^n \rho_{ijm}(u^{(l)})u_j^{(l)}w_m^{(l)}(\tau,\xi_i(\tau))d\tau. \end{aligned} \tag{3.56}$$

Then, noting (2.8) and using Lemma 3.2, similarly to (3.40) we get

$$U_\infty^c(T) \le C\{\theta + U_\infty^c(T)\widetilde{W}_1(T) + W_\infty^c(T)U_\infty^c(T) + \widetilde{U}_1(T)W_\infty^c(T)\}. \tag{3.57}$$

Hence, using Lemma 3.3, we get immediately

$$U_\infty^c(T) \le C\theta\{1+\widetilde{U}_1(T)\}. \tag{3.58}$$

We next estimate $\widetilde{U}_1(T)$ and $U_1(T)$.

For $i\in\{1,\cdots,n-1\}$, similarly to (3.41), by (2.11) we have

$$\begin{aligned} &\int_{t_B}^t |u_i^{(i)}(\lambda_j(u^{(i)})-\lambda_i(u^{(i)}))(\tau,\xi_j(\tau))|d\tau \\ \le &\int_{OC} |u_i^{(i)}(\lambda_i(0)+\delta_0-\lambda_i(u^{(i)}))(\tau,(\lambda_i(0)+\delta_0)\tau)|d\tau \\ &+ \iint_{ABOC}\Big|\sum_{k,m=1}^n F_{ikm}(u^{(i)})u_k^{(i)}w_m^{(i)}(t,x)\Big|dtdx. \end{aligned} \tag{3.59}$$

Then, noting (2.13)–(2.14), the second term on the right hand side of (3.59) can be rewritten as

$$\begin{aligned} &\iint_{ABOC}\Big|\sum_{k,m=1}^n F_{ikm}(u^{(i)})u_k^{(i)}w_m^{(i)}(t,x)\Big|dtdx \\ =&\iint_{ABOC}\Big|\sum_{k\ne m} F_{ikm}(u^{(i)})u_k^{(i)}w_m^{(i)}(t,x) + F_{iii}(u^{(i)})u_i^{(i)}w_i^{(i)}(t,x)\Big|dtdx. \end{aligned} \tag{3.60}$$

Since $\lambda_i(u)$ is weakly linearly degenerate and $u = (u_1, \cdots, u_n)^T$ are normalized coordinates, by (2.15) we have

$$F_{iii}(u_i e_i) \equiv 0, \qquad \forall\, |u_i| \;\text{ small.} \tag{3.61}$$

Then, using Hardmard's formula, we have

$$\begin{aligned} F_{iii}(u^{(i)}) &= F_{iii}(u^{(i)}) - F_{iii}(u_i^{(i)} e_i) \\ &= \int_0^1 \sum_{l \neq i} \frac{\partial F_{iii}}{\partial u_l}(\tau u_1^{(i)}, \cdots, \tau u_{i-1}^{(i)}, u_i^{(i)}, \tau u_{i+1}^{(i)}, \cdots, \tau u_n^{(i)}) u_l^{(i)} d\tau. \end{aligned} \tag{3.62}$$

Hence, similarly to (3.42), using Lemma 3.3, from (3.59) we get

$$\begin{aligned} \int_{c_j} |u_i^{(i)}| d\tau = \int_{t_B}^t &|u_i^{(i)}(\tau, \xi_j(\tau))| d\tau \\ &\le C\{U_\infty^c(T) + U_1(T) W_\infty^c(T) + U_\infty^c(T) W_1(T) \\ &\quad + U_\infty^c(T) W_\infty^c(T) + U_\infty(T) U_\infty^c(T) W_1(T)\} \\ &\le C\{U_\infty^c(T) + \theta U_1(T)\}. \end{aligned} \tag{3.63}$$

For $i = n$, similarly to (3.43), we have

$$\begin{aligned} \int_{c_j} |u_n^{(n)}| d\tau = \int_0^t &|u_n^{(n)}(\tau, \xi_j(\tau))| d\tau \\ &\le C\{\theta + U_1(T) W_\infty^c(T) + U_\infty^c(T) W_1(T) \\ &\quad + U_\infty^c(T) W_\infty^c(T) + U_\infty(T) U_\infty^c(T) W_1(T)\} \\ &\le C\theta\{1 + U_\infty^c(T) + U_1(T)\}. \end{aligned} \tag{3.64}$$

Moreover, similarly to (3.44)–(3.45), we can estimate

$$\int_{c_j} |u_i^{(i-1)}| d\tau \qquad \text{for } \; i = 1, \cdots, n.$$

Hence, we get

$$\widetilde{U}_1(T) \le C\{U_\infty^c(T) + \theta(1 + U_1(T))\}. \tag{3.65}$$

Similarly, we have

$$U_1(T) \le C\{U_\infty^c(T) + \theta(1 + U_1(T))\}. \tag{3.66}$$

Thus we get

$$\widetilde{U}_1(T), U_1(T) \le C\{\theta + U_\infty^c(T)\}. \tag{3.67}$$

Finally, (3.51) follows immediately from the combination of (3.58) and (3.67).

We finally estimate $W_\infty(T)$.

For any given $i \in \{1, \cdots, n\}$ and any given point $(t, x) \in D_i^T$, let $c_i : \xi = \xi_i(\tau)$ $(\tau \le t)$ be the i-th characteristic passing through (t, x), which intersects the x-axis at a point $(0, x_{i0})$.

When $(t,x)\in R_{i-1}(T)$, integrating (2.16) along c_i from 0 to t gives

$$\begin{aligned} w_i^{(i-1)}(t,x) = w_i^{(0)}(0,x_{i0}) &+ \int_0^{t_{i1}} \sum_{j,m=1}^n \gamma_{ijm}(u^{(0)}) w_j^{(0)} w_m^{(0)}(\tau,\xi_i(\tau))d\tau \\ &+ \sum_{k=1}^{i-2} \int_{t_{ik}}^{t_{i,k+1}} \sum_{j,m=1}^n \gamma_{ijm}(u^{(k)}) w_j^{(k)} w_m^{(k)}(\tau,\xi_i(\tau))d\tau \\ &+ \int_{t_{i,i-1}}^{t} \sum_{j,m=1}^n \gamma_{ijm}(u^{(i-1)}) w_j^{(i-1)} w_m^{(i-1)}(\tau,\xi_i(\tau))d\tau; \end{aligned} \tag{3.68}$$

while, when $(t,x)\in R_i(T)$, similarly we have

$$\begin{aligned} w_i^{(i)}(t,x) = w_i^{(n)}(0,x_{i0}) &+ \int_0^{t_{in}} \sum_{j,m=1}^n \gamma_{ijm}(u^{(n)}) w_j^{(n)} w_m^{(n)}(\tau,\xi_i(\tau))d\tau \\ &+ \sum_{k=i+2}^{n} \int_{t_{ik}}^{t_{i,k-1}} \sum_{j,m=1}^n \gamma_{ijm}(u^{(k-1)}) w_j^{(k-1)} w_m^{(k-1)}(\tau,\xi_i(\tau))d\tau \\ &+ \int_{t_{i,i+1}}^{t} \sum_{j,m=1}^n \gamma_{ijm}(u^{(i)}) w_j^{(i)} w_m^{(i)}(\tau,\xi_i(\tau))d\tau. \end{aligned} \tag{3.69}$$

Since $\lambda_i(u)$ is weakly linearly degenerate and $u=(u_1,\cdots,u_n)^T$ are normalized coordinates, by (2.19) we have

$$\gamma_{iii}(u_i e_i)\equiv 0, \qquad \forall\, |u_i| \ \text{ small.} \tag{3.70}$$

Then, noting (1.13) and (2.18), similarly to (3.63) and (3.64), it is easy to get

$$\begin{aligned} |w_i^{(i-1)}(t,x)|,\ |w_i^{(i)}(t,x)| \le C\{&\theta + W_\infty^c(T)\widetilde{W}_1(T) + (W_\infty^c(T))^2 \\ &+ W_\infty^c(T)W_\infty(T) + U_\infty^c(T)(W_\infty(T))^2\}. \end{aligned} \tag{3.71}$$

Thus, noting Lemma 3.3 and (3.51) we have

$$W_\infty(T) \le C\theta\{1 + W_\infty(T) + (W_\infty(T))^2\}, \tag{3.72}$$

which implies (3.52).

From Lemmas 3.3 and 3.4, the sufficiency in Theorem 1.1 follows immediately.

We now prove the necessity in Theorem 1.1.

For the Cauchy problem of a scalar equation

$$\begin{cases} \dfrac{\partial v}{\partial t} + \lambda(v)\dfrac{\partial v}{\partial x} = 0, \\ t=0: \quad v = \begin{cases} \psi_l(x), & x\le 0, \\ \psi_r(x), & x\ge 0 \end{cases} \end{cases} \tag{3.73}$$

with

$$\psi_l(0)=\psi_r(0) \qquad \text{and} \qquad \psi_l'(0)\ne\psi_r'(0), \tag{3.74}$$

where $\psi_l(x)$ and $\psi_r(x)\in C^1$ and satisfy (1.13), it is easy to get

Lemma 3.5. *There exists $\theta_0 > 0$ so small that for any given $\theta \in (0, \theta_0]$, Cauchy problem* (3.73) *admits a unique global weakly discontinuous solution if and only if $\lambda(v)$ is a constant in a neighbourhood of $v = 0$.*

Then, noting that in normalized coordinates the characteristic $\lambda_i(u)$ is weakly linearly degenerate if and only if

$$\lambda_i(u_i e_i) \equiv \text{const.}, \qquad \forall\, |u_i| \ \text{ small}, \tag{3.75}$$

we easily get the necessity in Theorem 1.1 (cf. [4, 5]).

Remark 3.1. Comparing with the method used in [6] and [13], the estimates on the domains $D_{\pm}^T$ and D_0^T and the estimates for $v_i = l_i(u)u$ $(i = 1, \cdots, n)$ are all omitted in the proof of Theorem 1.1.

§ 4 . Application

Consider the following Cauchy problem for the system of the planar motion of an elastic string (cf. [8, 13])

$$\begin{cases} u_t - v_x = 0, \\ v_t - \Big(\dfrac{T(r)}{r}u\Big)_x = 0 \end{cases} \tag{4.1}$$

with the initial condition

$$t = 0: \quad (u, v) = \begin{cases} (\widetilde{u}_0 + u_l\,(x), \widetilde{v}_0 + v_l\,(x)) & (x \le 0), \\ (\widetilde{u}_0 + u_r(x), \widetilde{v}_0 + v_r(x)) & (x \ge 0), \end{cases} \tag{4.2}$$

where

$$(u_l(0), v_l(0)) = (u_r(0), v_r(0)) \qquad \text{and} \qquad (u_l'(0), v_l'(0)) \neq (u_r'(0), v_r'(0)), \tag{4.3}$$

$u = (u_1, u_2)^T$, $v = (v_1, v_2)^T$, $r = |u| = \sqrt{u_1^2 + u_2^2}$, $T(r)$ is a C^3 function of $r > 1$, such that

$$T'(\widetilde{r}_0) > \frac{T(\widetilde{r}_0)}{\widetilde{r}_0} > 0, \tag{4.4}$$

in which $\widetilde{u}_0$ and $\widetilde{v}_0$ are constant vectors and $\widetilde{r}_0 = |\widetilde{u}_0| > 1$, $(u_l(x), v_l(x))$ and $(u_r(x), v_r(x)) \in C^1$ and satisfy (1.13). Let

$$U = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.5}$$

By (4.4), in a neighbourhood of $U_0 = \begin{pmatrix} \widetilde{u}_0 \\ \widetilde{v}_0 \end{pmatrix}$, (4.1) is a strictly hyperbolic system with the following distinct real eigenvalues:

$$\lambda_1(U) = -\sqrt{T'(r)} < \lambda_2(U) = -\sqrt{\frac{T(r)}{r}} < 0 < \lambda_3(U) = \sqrt{\frac{T(r)}{r}} < \lambda_4(U) = \sqrt{T'(r)}. \tag{4.6}$$

$\lambda_2(U)$ and $\lambda_3(U)$ are linearly degenerate in the sense of P. D. Lax, then weakly linearly degenerate. Moreover, $\lambda_1(U)$ and $\lambda_4(U)$ are also linearly degenerate, then weakly linearly degenerate, provided that

$$T''(r) \equiv 0, \qquad \forall\, |r - \tilde{r}_0| \ \text{ small}. \tag{4.7}$$

By Theorem 1.1 we get

Theorem 4.1. *Suppose that* (4.7) *holds. There exists* $\theta_0 > 0$ *so small that for any fixed* $\theta \in (0, \theta_0]$*, Cauchy problem* (4.1)–(4.2) *admits a unique global weakly discontinuous solution* $U = U(t, x)$ *on* $t \geq 0$*, which possesses at most* 4 *weak discontinuities* $x = x_k(t)$ $(k = 1, \cdots, 4)$ *passing through the origin.*

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