Chin. Ann. Math.
25B:3(2004),335–346.

UNCONDITIONAL CAUCHY SERIES AND UNIFORM CONVERGENCE ON MATRICES

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Abstract

The authors obtain new characterizations of unconditional Cauchy series in terms of separation properties of subfamilies of $\mathcal{P}(\mathbb{N})$, and a generalization of the Orlicz-Pettis Theorem is also obtained. New results on the uniform convergence on matrices and a new version of the Hahn-Schur summation theorem are proved. For matrices whose rows define unconditional Cauchy series, a better sufficient condition for the basic Matrix Theorem of Antosik and Swartz, new necessary conditions and a new proof of that theorem are given.

Keywords Unconditional Cauchy series, Orlicz-Pettis theorem, Summation, Hahn-Schur theorem, Basic matrix theorem
2000 MR Subject Classification 46B15, 46B25

§ 1. Introduction

In the literature, many applications of the Basic Matrix Theorem of [3] to problems of measure theory and Banach spaces theory have been found (see [10, 8], etc.) In this paper, we generalize several results that appear in [10, 5–7] and obtain new characterizations of unconditional Cauchy series (uca) in terms of separation properties of subfamilies of $\mathcal{P}(\mathbb{N})$ and a generalization of the Orlicz-Pettis Theorem. For other interesting generalizations of this theorem, we refer the reader to [11, 12]. In this paper, Wu Junde and Lu Shijie obtain a generalization which considers a dual pair $[X, Y]$ and replaces the subseries-$\sigma(X, Y)$ convergence (or, equivalently, the m_0-multiplier-$\sigma(X, Y)$ convergence, where m_0 denotes the scalar-valued sequence space which satisfies that for each $(t_i)_i \in m_0$ $\{t_i : i \in \mathbb{N}\}$ is a finite set) by the λ-multiplier-$\sigma(X, Y)$ convergence, where λ is a scalar-valued sequence space which has the S-WGHP and contains c_{00}. Wu Junde and Lu Shijie [12] prove that these assumptions imply the λ-$\tau(X, Y)$-multiplier convergence of a series in X, where $\tau(X, Y)$ denotes the Mackey topology. The paper shows the λ-multiplier convergence of series depends completely upon the AK-property of λ.

The mentioned separation properties let us prove new version of interesting results on the uniform convergence on matrices. The main result of [7] follows at once from the new

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version of the Hahn-Schur summation theorem that we prove in this paper. For matrices whose rows are unconditional Cauchy series, the case more frequently found in applications, we improve some sufficient conditions for the Basic Matrix Theorem. We also prove some new necessary conditions for that theorem. As a consequence, we give a proof of this theorem and of the converse result.

Although this paper is developed in the framework of normed space theory, most of the results can be extended, with some warnings, to Abelian topological groups.

Our Theorem 2.1, Lemma 2.1 and Theorem 2.2 are contained in [2], although they are proved here with a different technic. We include it for the sake of completeness.

§ 2. Unconditional Cauchy Series

Let us denote by $\phi_0(\mathbb{N})$ the family of finite subsets of $\mathbb{N}$. Let us suppose that $\mathcal{F}$ is a Boolean algebra such that $\phi_0(\mathbb{N}) \subset \mathcal{F} \subset P(\mathbb{N})$ and $\mathcal{F}$ has the Vitali-Hahn-Saks property (see [9]). Let X be a Banach space and let $\sum_{i\in\mathbb{N}} x_i$ be a series in X such that $\sum_{i\in A} x_i$ is convergent, for $A \in \mathcal{F}$. It can be proved (see [1]), that $\sum_{i\in\mathbb{N}} x_i$ is unconditionally convergent (uco).

If $\sum_{i\in\mathbb{N}} x_i$ is a series in X such that for any disjoint sequence $(A_i)_i$ in $\phi_0(\mathbb{N})$, there exists an infinite set $M \subseteq \mathbb{N}$ such that $\sum_{i\in A} x_i$ is convergent, for $A = \bigcup_{i\in M} A_i$, then $\sum x_i$ is uco (see [3, 7]).

In what follows, we obtain some new characterizations of unconditional Cauchy series. Some characterizations of uco series can be found in ([10, 12]).

Theorem 2.1. *Let $\sum_i x_i$ be a series in a normed space X. The following properties are equivalent:*

(i) *The series $\sum_i x_i$ is uca.*

(ii) *For any pair $[(A_i)_i, (B_i)_i]$ of disjoint sequences of mutually disjoint elements of $\phi_0(\mathbb{N})$, there exists $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ such that $\sum_{i\in B} x_i$ is of Cauchy, $A_i \subseteq B$ and $B_i \subseteq B^c$ for $i \in M$.*

Proof. Since any uca series is a Cauchy subseries, it is clear that (i)⇒(ii). Let us suppose that (ii) is true and that (i) is false. There exist $\varepsilon > 0$ and a sequence $(F_n)_n$ in $\phi_0(\mathbb{N})$ with $\sup F_n < \inf F_{n+1}$ and $\left\| \sum_{i\in F_n} x_i \right\| > \varepsilon$ for $n \in \mathbb{N}$. Let $A_k = \{\inf F_k,\ \inf F_k + 1,\ \cdots,\ \sup F_k\}$ and $B_k = A_k \backslash F_k$ for $k \in \mathbb{N}$. By (ii), there exists $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ such that $\sum_{i\in B} x_i$ is a Cauchy series, $F_i \subseteq B$ and $B_i \subseteq B^c$ for $i \in M$. It is obvious that $\left\| \sum_{i\in B\cap A_k} x_i \right\| = \left\| \sum_{i\in F_k} x_i \right\| > \varepsilon$, $k \in M$. This contradicts (ii).

Corollary 2.1. *Let $\sum_i x_i$ be a series in a normed space X. The following properties are equivalent:*

(i) *The series $\sum_i x_i$ is uco.*

(ii) *The series $\sum_i x_i$ is convergent and verifies property* (ii) *in Theorem* 2.1.

The following lemma will let us prove a result that weakens the usual hypothesis of Orlicz-Pettis Theorem.

Lemma 2.1. *Let $(a_{ij})_{i,j\in\mathbb{N}}$ be a real matrix (of infinite dimension) with the following properties:*

(i) *For $j \in \mathbb{N}$, $(a_{ij})_i$ is convergent.*

(ii) *For any pair $[(A_i)_i, (B_i)_i]$ of disjoint sequences of mutually disjoint elements of $\phi_0(\mathbb{N})$, there exists $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ such that $\Big(\sum_{j\in B} a_{ij}\Big)_i$ is a convergent sequence, $A_i \subseteq B$ and $B_i \subseteq B^c$ for $i \in M$.*

Then, for every $P \subseteq \mathbb{N}$, the sequence $\Big(\sum_{j\in P} a_{ij}\Big)_i$ is convergent and $\Big(\sum_{j\in A_n} a_{ij}\Big)_i$ is uniformly convergent, for every disjoint sequence $(A_n)_n$ of $P(\mathbb{N})$.

Proof. By Theorem 2.1, it is clear that $\sum_j a_{ij}$ is unconditionally convergent for every $i \in \mathbb{N}$.

Let us suppose that there exists $P \subseteq \mathbb{N}$ such that the sequence $\Big(\sum_{j\in p} a_{ij}\Big)_i$ is not a Cauchy sequence; let $\varepsilon > 0$ be such that for every $k \in \mathbb{N}$, there exists $n > k$ such that $\Big|\sum_{j\in P} a_{kj} - a_{nj}\Big| > \varepsilon$.

For $k_1 = 1$, let $n_1 > k_1$ and $m_1 \in \mathbb{N}$ be such that $\Big|\sum_{j\in P\cap\{1,2,\cdots,m_1\}} (a_{k_1 j} - a_{n_1 j})\Big| > \varepsilon$. Since $(a_{ij})_i$ is a Cauchy sequence, there exist $l_1 > n_1$ such that $\Big|\sum_{j\in C} (a_{lj} - a_{rj})\Big| < \frac{\varepsilon}{8}$ for $l, r \geq l_1$ and $C \subseteq \{1, 2, \cdots, m_1\}$. Let $k_2 > l_1$ and $n_2 > k_2$ be such that $\Big|\sum_{j\in P} (a_{k_2 j} - a_{n_2 j})\Big| > \varepsilon$. It is clear that for some $m_2 \in \mathbb{N}$, $m_2 > m_1$ we have $\Big|\sum_{j\in B} (a_{k_2 j} - a_{n_2 j})\Big| < \frac{\varepsilon}{8}$ for $B \subseteq \{m_2 + 1, m_2 + 2, \cdots\}$. Then

$$\begin{aligned}
&\Big|\sum_{j\in P\cap\{m_1+1,\cdots,m_2\}} (a_{k_{2j}} - a_{n_{2j}})\Big| \\
&\geq \Big|\sum_{j\in P} (a_{k_{2j}} - a_{n_{2j}})\Big| - \Big|\sum_{j\in P, j\leq m_1} (a_{k_{2j}} - a_{n_{2j}})\Big| - \Big|\sum_{j\in P, j>m_2} (a_{k_{2j}} - a_{n_{2j}})\Big| \\
&\geq \varepsilon - \frac{\varepsilon}{8} - \frac{\varepsilon}{8} = \frac{3\varepsilon}{4}
\end{aligned}$$

We can obtain, inductively, three increasing sequences $(k_i)_i$, $(n_i)_i$ and $(m_i)_i$ of natural numbers with the following properties:

(a) $k_1 < n_1 < k_2 < n_2 < \cdots$.

(b) $\Big|\sum_{j\in C} a_{k_i j} - a_{n_i j}\Big| < \frac{\varepsilon}{8}$ for $C \subseteq \{1, 2, \cdots, m_{i-1}\}$ if $i \in \mathbb{N}$.

(c) If $F_i = P \cap \{m_{i-1} + 1, \cdots, m_i\}$ for $i \in \mathbb{N}$ and $i > 1$, then $\Big|\sum_{j\in F_i} a_{k_i j} - a_{n_i j}\Big| > \frac{3\varepsilon}{4}$.

(d) $\Big|\sum_{j\in B} a_{k_i j} - a_{n_i j}\Big| < \frac{\varepsilon}{8}$, if $B \subseteq \{m_i + 1, m_i + 2, \cdots\}$ for $i > 1$.

Let $B_i = \{m_{i-1} + 1, \cdots, m_i\} \setminus F_i$ for $i > 1$. By applying the hypothesis (ii) to the pair $[(F_i)_i, (B_i)_i]$, we can find $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ such that $\Big(\sum_{j\in B} a_{ij}\Big)_i$ is convergent, $F_i \subseteq B$ and $B_i \subseteq B^c$ for $i \in M$. However, for $i \in M$,

$$\Big| \sum_{j\in B} a_{k_i j} - a_{n_i j} \Big|$$
$$\geq \Big| \sum_{j\in F_i} (a_{k_i j} - a_{n_i j}) \Big| - \Big| \sum_{j\in B,\, j\leq m_{i-1}} (a_{k_i j} - a_{n_i j}) \Big| - \Big| \sum_{j\in B,\, j>m_i} (a_{k_i j} - a_{n_i j}) \Big| > \frac{\varepsilon}{2},$$

which contradicts the Cauchy condition for $\Big(\sum_{j\in B} a_{ij} \Big)_i$.

For $i \in \mathbb{N}$, let us consider the measure $\mu_i : P(\mathbb{N}) \to \mathbb{R}$ defined by $\mu_i(A) = \sum_{j\in A} a_{ij}$ for $A \in P(\mathbb{N})$. It follows, from the Vitali-Hahn-Saks theorem (see [4, 9]), that the sequence $(\mu_i)_i$ is uniformly strongly additive; i.e., $(\mu_i(A_j))_j$ converges to zero uniformly in $i \in \mathbb{N}$, where $(A_j)_j$ is a disjoint sequence of $P(\mathbb{N})$. Now it is easy to check that $(\mu_i(A_j))_i$ is uniformly convergent, in $j \in \mathbb{N}$.

Remark 2.1. The final step of former theorem can be done, without using the Vitali-Hahn-Saks theorem, through the following result:

Let $(a_{ij})_{i,j}$ be a real matrix (of infinite dimension) such that, for every $P \subseteq \mathbb{N}$, the sequence $\Big(\sum_{j\in P} a_{ij} \Big)_i$ is convergent. Then, for every disjoint sequence $(A_n)_n$ in $P(\mathbb{N})$, $\Big(\sum_{j\in A_n} a_{ij} \Big)_i$ is uniformly convergent in n.

A sketch of the proof of this result is as follows:

First, we observe that if $\sum_j \alpha_j$ is an absolutely convergent series in $\mathbb{R}$, then for every disjoint sequence $(A_n)_n$ in $P(\mathbb{N})$, we have $\sum_{i\in\mathbb{N}} \Big| \sum_{j\in A_i} \alpha_j \Big| < +\infty$.

If the conclusion of the result is false, we can inductively construct the sequences of natural numbers $(k_r)_r$, $(i_r)_r$ with $k_1 < i_1 < k_2 < i_2 < \cdots$ and $(n_r)_r$, $(m_r)_r$ with $n_1 < m_1 < n_2 < m_2 < \cdots$ such that for every natural number $r > 1$,

(a) $\sum_{m\in C} \Big| \sum_{j\in A_m} (a_{pj} - a_{qj}) \Big| < \frac{\varepsilon}{4}$ if $p, q \geq k_r$ and $C \subseteq \{1, 2, \cdots, m_{r-1}\}$.

(b) $\Big| \sum_{j\in A_{n_r}} (a_{k_r j} - a_{i_r j}) \Big| > \varepsilon$.

(c) $\sum_{m\in B} \Big| \sum_{j\in A_m} (a_{k_r j} - a_{i_r j}) \Big| < \frac{\varepsilon}{4}$ if B is a subset of $\{m_r + 1, m_r + 2, \cdots\}$.

Let us define $A = \bigcup_{r\in\mathbb{N}} A_{n_r}$. It is easy to check that the sequence $\Big(\sum_{j\in A} x_{ij} \Big)_i$ does not verify the Cauchy condition, which contradicts our hypothesis.

Theorem 2.2. *Let $\sum_i x_i$ be a series in the normed space X. The following properties are equivalent if X is complete:*

(i) *The series $\sum_i x_i$ is uca.*

(ii) *For every pair $[(A_i)_i, (B_i)_i]$ of disjoint sequences of mutually disjoint elements of $\phi_0(\mathbb{N})$, there exists $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ such that $\sum_{i\in B} x_i$ is weakly convergent, $A_i \subseteq B$ and $B_i \subseteq B^c$ if $i \in M$.*

Without the hypothesis of completeness, we have (ii)⇒(i).

Proof. It is clear that (i)⇒(ii). Let us assume (ii). If (i) is false, let $\varepsilon > 0$ and let us suppose that the sequence $(F_n)_n \subseteq \phi_0(\mathbb{N})$ is such that $\sup F_n < \inf F_{n+1}$ and $\left\| \sum_{i \in F_n} x_i \right\| > \varepsilon$ for $n \in \mathbb{N}$. Let $f_n \in S_{X^*}$ be such that $f_n\Big(\sum_{i \in F_n} x_i \Big) > \varepsilon$ for $n \in \mathbb{N}$. We can assume that X is separable; therefore, a subsequence of $(f_n)_n$, which will be also denoted by $(f_n)_n$, is *-w-convergent to some $f_0 \in B_{X^*}$.

The matrix $(f_i(x_j))_{i,j}$ verifies the hypothesis of Lemma 2.1, which contradicts the inequality $f_i\Big(\sum_{j \in F_i} x_j \Big) > \varepsilon$.

Remark 2.2. Let $\mathcal{L}$ be a family of elements of $P(\mathbb{N})$ such that $\phi_0(\mathbb{N}) \subset \mathcal{L}$ and with the following property:

(a) for every pair $[(A_i)_i, (B_i)_i]$ of disjoint sequence of mutually disjoint elements of $\phi_0(\mathbb{N})$, there exists an infinite set $M \subset \mathbb{N}$ and a $B \in \mathcal{L}$ such that $A_i \subset B$ and $B_i \subset B^c$ for $i \in M$.

Let $\sum_i x_i$ be a series in the normed space X. If $\sum_{i \in A} x_i$ is weakly convergent for every $A \in \mathcal{L}$ then it follows, from Theorem 2.2, that $\sum_i x_i$ is uca. It is easy to check that if $\sum_{i \in A} x_i$ is a weakly Cauchy series for every $A \in \mathcal{L}$, then $\sum_i x_i$ is a weakly unconditional Cauchy series. Let $\mathcal{L}$ be a family in $P(\mathbb{N})$ such that $\phi_0(\mathbb{N}) \subset \mathcal{L}$ and with the following property:

(b) for every disjoint sequence $(A_i)_i$ in $\phi_0(\mathbb{N})$, there exists an infinite set $M \subseteq \mathbb{N}$ such that $\bigcup_{i \in M} A_i \in \mathcal{L}$.

It is well known that if $\sum x_i$ is a series such that $\sum_{i \in A} x_i$ is weakly convergent for every $A \in \mathcal{L}$, then $\sum_{i \in A} x_i$ is uca.

We now give an example (see [2]) of a family $\mathcal{L}$ in $P(\mathbb{N})$ with the property (a) that lacks the property (b).

Let B_1 the family of subsets $A \subset \mathbb{N}$ such that A and A^c have infinite even numbers and infinite odd numbers. Let $\mathcal{L} = B_1 \cup \phi_0(\mathbb{N})$. It is easy to check that for any given pair $[(A_n)_n, (B_n)_n]$ of disjoint sequences of mutually disjoint elements of $\phi_0(\mathbb{N})$, there exists $B \in \mathcal{L}$ and an infinite set $M \subseteq \mathbb{N}$ such that $A_i \subseteq B$ and $B_i \subseteq B^c$ for $i \in M$. However, in general, it is not true that for any disjoint sequence $(A_n)_n$ in $\phi_0(\mathbb{N})$, there exists an infinite set $M \subseteq \mathbb{N}$ such that $\bigcup_{i \in M} A_i \in \mathcal{L}$. Therefore, the union of the members of any subsequence of $(\{2n\})_{n \in \mathbb{N}}$ does not belong to $\mathcal{L}$.

§ 3. Uniform Convergence in Matrices

In this section we will study, for sequences and series, the uniform convergence on matrices.

The following result is a version of the Hahn-Schur summation theorem (see [7]) for normed spaces that improves Theorem 1 in [10].

Theorem 3.1. *Let $(x_{ij})_{i,j}$ be a matrix in the normed space X, where $(x_{ij})_i$ is a Cauchy sequence for every $j \in \mathbb{N}$ and, for $i \in \mathbb{N}$, either $\sum_j x_{ij}$ is uca or $\sum_j x_{ij}$ verifies condition* (ii) *in Theorem 2.2. Then, the following properties are equivalent:*

(1) *If $[(A_n)_n, (B_n)_n]$ is a pair of disjoint sequences of mutually disjoint elements of $\phi_0(\mathbb{N})$, then there exists $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ such that* (i) *$A_i \subseteq B$ and $B_i \subseteq B^c$ for $i \in M$;* (ii) *For every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that if $p, q \geq k$ there exists l_0 with $\left\| \sum_{j \in B \cap [1,l]} (x_{pj} - x_{qj}) \right\| < \varepsilon$ for $l \geq l_0$.*

(2) *$\left(\sum_{j \in A_n} x_{ij} \right)_i$ is a Cauchy sequence, uniformly in $n \in \mathbb{N}$, for any disjoint sequence $(A_n)_n$ in $\phi_0(\mathbb{N})$.*

(3) *The series $\sum_{j \in A} x_{ij}$ is uca uniformly in $i \in \mathbb{N}$, for $A \subseteq \mathbb{N}$.*

(4) *The series $\sum_{j \in A} x_{ij}$ is uca uniformly in $i \in \mathbb{N}$ and $A \in P(\mathbb{N})$.*

(5) *Let CX be a completion of X and, for $j \in \mathbb{N}$, let $x_{0j} \in CX$ be the limit of $(x_{ij})_i$. Then, the series $\sum_{j \in \mathbb{N}} x_{0j}$ is ico in CX and $\lim_i \sum_{j \in A} x_{ij} = \sum_{j \in A} x_{0j}$ uniformly in $A \in P(\mathbb{N})$.*

(6) *Let $A \in P(\mathbb{N})$. For every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for $p, q \geq k$ there exists l_0 with $\left\| \sum_{j \in A \cap [1,l]} (x_{pj} - x_{qj}) \right\| < \varepsilon$ for $l \geq l_0$.*

Proof. We first prove that (1)⇒(2). Let us suppose that there exists a disjoint sequence $(A_n)_n$ in $\phi_0(\mathbb{N})$ and an $\varepsilon > 0$ such that for any $k \in \mathbb{N}$ there exist $i > k$ and n_k with $\left\| \sum_{j \in A_{n_k}} (x_{kj} - x_{ij}) \right\| > \varepsilon$.

For $k_1 = 1$, there exist $i_1 > k_1$ and n_1 such that $\left\| \sum_{j \in A_{n_1}} (x_{k_1 j} - x_{i_1 j}) \right\| > \varepsilon$. Let $m_1 > \sup A_{n_1}$ be such that $\left\| \sum_{j \in B} (x_{k_1 j} - x_{i_1 j}) \right\| < \frac{\varepsilon}{8}$ for a finite set $B \subseteq \{m_1 + 1, \cdots\}$ and let i_{01} be such that $\left\| \sum_{j \in C} (x_{lj} - x_{rj}) \right\| < \frac{\varepsilon}{8}$ for $l, r \geq i_{01}$, $C \subseteq \{1, 2, \cdots, m_1\}$.

Let $k_2 > i_{01}$. Let $i_2 > k_2$ and n_2 be such that $\left\| \sum_{j \in A_{n_2}} (x_{k_2 j} - x_{i_2 j}) \right\| > \varepsilon$. If we define $B_2 = A_{n_2} \setminus \{1, 2, \cdots, m_1\}$, then $\left\| \sum_{j \in B_2} (x_{k_2 j} - x_{i_2 j}) \right\| > \frac{7\varepsilon}{8}$.

Let $m_2 > \sup A_{n_2}$ be such that $\left\| \sum_{j \in B} (x_{k_2 j} - x_{i_2 j}) \right\| < \frac{\varepsilon}{8}$ for any finite set $B \subseteq \{m_2 + 1, \cdots\}$ and let i_{02} be such that $\left\| \sum_{j \in C} (x_{lj} - x_{nj}) \right\| < \frac{\varepsilon}{8}$ for $l, r \geq i_{02}$, $C \subseteq \{1, 2, \cdots, m_2\}$.

We can obtain, inductively, three sequences of natural numbers $(k_n)_n$, $(i_n)_n$ and $(m_n)_n$ such that $k_1 < i_1 < k_2 < i_2 < \cdots$ and $m_1 < m_2 < \cdots$, and a disjoint sequence $(B_i)_i$ in $\phi_0(\mathbb{N})$ (where $B_1 = A_{n_1}$) such that for every $r \in \mathbb{N}$, $r > 1$,

(a) $B_r \subseteq \{m_{r-1} + 1, \cdots, m_r\}$ and $\left\| \sum_{j \in B_r} (x_{k_r j} - x_{i_r j}) \right\| > \frac{7\varepsilon}{8}$.

(b) $\left\| \sum_{j \in C} (x_{k_r j} - x_{i_r j}) \right\| < \frac{\varepsilon}{8}$ for $C \subseteq \{1, 2, \cdots, m_{r-1}\}$.

(c) $\left\| \sum_{j\in B}(x_{k_r j} - x_{i_r j})\right\| < \dfrac{\varepsilon}{8}$ for any finite set $B \subseteq \{m_r+1, \cdots\}$.

Let us define $F_r = \{m_{r-1}+1, \cdots, m_r\} \setminus B_r$ for $r \in \mathbb{N}$. By hypothesis, there exists $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ such that $B_i \subseteq B$ and $F_i \subseteq B^c$ for $i \in M$, and verify the property (ii) that appears in (1).

It is clear that for $r \in M$, $r > 1$, and $l \geq m_r$,

$$\left\| \sum_{j\in B\cap[1,l]} x_{k_r j} - x_{i_r j}\right\| \geq \left\| \sum_{j\in B_r}(x_{k_r j} - x_{i_r j})\right\| - \left\| \sum_{j\in B,\, j\leq m_{r-1},\, j\in[1,l]}(x_{k_r j} - x_{i_r j})\right\| - \left\| \sum_{j\in B,\, j> m_r,\, j\in[1,l]}(x_{k_r j} - x_{i_r j})\right\| \geq \frac{5\varepsilon}{8},$$

which contradicts (ii) of (1).

We now prove that (2)⇒(3). Let us suppose that for some $A \in P(\mathbb{N})$ and $\varepsilon > 0$, we have that for any $k \in \mathbb{N}$, there exists $F \in \phi_0(\mathbb{N})$, $\inf F > k$, and i such that $\left\| \sum_{j\in A\cap F} x_{ij}\right\| > \varepsilon$.

For $k_1 = 1$, let $F_1 \in \phi_0(\mathbb{N})$, $\inf F_1 > k_1$, and i_1 such that $\left\| \sum_{j\in A\cap F_1} x_{i_1 j}\right\| > \varepsilon$. Let $m_1 \in \mathbb{N}$ be such that $\left\| \sum_{j\in B} x_{ij}\right\| < \dfrac{\varepsilon}{8}$ for any finite set $B \subseteq \{m_1+1, \cdots\}$ and $i \in \{1, 2, \cdots, i_1\}$.

For $k_2 > \max\{m_1, \sup F_1\}$, let $F_2 \in \phi_0(\mathbb{N})$, $\inf F_2 > k_2$, and i_2 such that $\left\| \sum_{j\in A\cap F_2} x_{i_2 j}\right\| > \varepsilon$ (let us observe that $i_2 > i_1$). Then, it is clear that

$$\left\| \sum_{j\in A\cap F_2}(x_{i_2 j} - x_{i_1 j})\right\| > \frac{7\varepsilon}{8}.$$

We can inductively obtain an increasing sequence $(i_r)_r$ of natural numbers and a disjoint sequence $(F_r)_r$ in $\phi_0(\mathbb{N})$ with $\inf F_r > \sup F_{r-1}$ for $r \in \mathbb{N}$, such that

$$\left\| \sum_{j\in A\cap F_r}(x_{i_r j} - x_{i_{r-1} j})\right\| > \frac{7\varepsilon}{8},$$

which contradicts our hypothesis.

It is obvious that (3)⇒(4). Now, we prove (4)⇒(5). Let us observe that $\left(\sum_{j\in A} x_{ij}\right)_i$ is a Cauchy sequence. We fix $A \in P(\mathbb{N})$ and let $l_0 \in \mathbb{N}$ be such that $\left\| \sum_{j\in B} x_{ij}\right\| < \dfrac{\varepsilon}{4}$ for $B \subseteq \{l_0+1, \cdots\}$ and $i \in \mathbb{N}$. Let us consider $q > p > l_0$ and let $i \in \mathbb{N}$ be such that $\left\| \sum_{j\in C}(x_{ij} - x_{0j})\right\| < \dfrac{\varepsilon}{4}$ for $C \subseteq \{p, \cdots, q\}$. Then, it is clear that

$$\left\| \sum_{j\in A\cap[p,q]} x_{0j}\right\| \leq \left\| \sum_{j\in A\cap[p,q]}(x_{0j} - x_{ij})\right\| + \left\| \sum_{j\in A\cap[p,q]} x_{ij}\right\| \leq \varepsilon.$$

This proves that $\sum_{j\in\mathbb{N}} x_{0j}$ is ico in CX. By a similar procedure, it can be obtained that $\left(\sum_{j\in A} x_{ij}\right)_i$ is a sequence that converges to $\sum_{j\in A} x_{0j}$ uniformly in $A \in P(\mathbb{N})$.

The remaining parts of the proof are obvious.

Corollary 3.1. *Let $(x_{ij})_{i,j}$ be a matrix in a normed space X, where $(x_{ij})_i$ is a Cauchy sequence for $j \in \mathbb{N}$ and, for $i \in \mathbb{N}$, the series $\sum_j x_{ij}$ is subseries convergent. Then the following conditions are equivalent:*

(1) *If $[(A_n)_n, (B_n)_n]$ is a pair of disjoint sequences of mutually disjoint elements of $\phi_0(\mathbb{N})$, then there exists $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ such that* (i) *$A_i \subseteq B$ and $B_i \subseteq B^c$ for $i \in M$,* (ii) *$\left(\sum_{j\in B} x_{ij}\right)_i$ is a Cauchy sequence.*

(2) *The sequence $\left(\sum_{j\in A_n} x_{ij}\right)_i$ is of Cauchy uniformly in $n \in \mathbb{N}$, for every disjoint sequence $(A_n)_n$ in $\phi_0(\mathbb{N})$.*

(3) *For every $A \subseteq \mathbb{N}$, the series $\sum_{j\in A} x_{ij}$ is uco uniformly in $i \in \mathbb{N}$.*

(4) *The series $\sum_{j\in A} x_{ij}$ is uco uniformly in $i \in \mathbb{N}$ and $A \in P(\mathbb{N})$.*

(5) *Let CX be a completion of X. For every $j \in \mathbb{N}$, let $x_{0j} \in CX$ be the limit of $(x_{ij})_i$. Then, the series $\sum_{j\in\mathbb{N}} x_{0j}$ is uco in CX and $\lim_i \sum_{j\in A} x_{ij} = \sum_{j\in A} x_{0j}$ uniformly in $A \in P(\mathbb{N})$.*

(6) *The sequence $\left(\sum_{j\in A} x_{ij}\right)_i$ is of Cauchy, for $A \in P(\mathbb{N})$.*

Lemma 3.1. *Let X be a normed space and let $(x_{ij})_{i,j}$ be a matrix in X such that, for $i \in \mathbb{N}$, $(x_{ij})_j$ is a Cauchy sequence and, for $j \in \mathbb{N}$, $(x_{ij})_i$ is a Cauchy sequence.*

(a) *The following conditions are equivalent:*

(1) *$(x_{ij})_i$ is uniformly of Cauchy in $j \in \mathbb{N}$.*

(2) *$(x_{ij})_j$ is uniformly of Cauchy in $i \in \mathbb{N}$.*

(3) *For every $\varepsilon > 0$, there exist $k, l \in \mathbb{N}$ such that $\|x_{kl} - x_{ij}\| < \varepsilon$ for $i \geq k$ and $j \geq l$.*

Under any of these conditions, $(x_{ii})_i$ is a Cauchy sequence.

(b) *Let CX be a completion of X, let x_{0j} be the limit in CX of the sequence $(x_{ij})_i$, for $j \in \mathbb{N}$ and, similarly, let x_{i0} be the limit of $(x_{ij})_j$, for $i \in \mathbb{N}$. Then, if for every $\varepsilon > 0$, there exist $k, l \in \mathbb{N}$ such that $\|x_{kl} - x_{ij}\| < \varepsilon$ for $i \geq k$ and $j \geq l$, there exists x_{00} such that*

$$\lim_j x_{0j} = \lim_i x_{i0} = \lim_i x_{ii} = \lim_{i,j} x_{ij} = x_{00}.$$

Proof. The proof of (a) is straightforward. Let us prove (b). For any given $\varepsilon > 0$, we consider $k, l \in \mathbb{N}$ such that $\|x_{kl} - x_{ij}\| < \frac{\varepsilon}{6}$ for $i \geq k$ and $j \geq l$. For $j \geq l$, let $i_0 \geq k$ be such that $\|x_{ij} - x_{0j}\| < \frac{\varepsilon}{3}$ and $\|x_{il} - x_{0l}\| < \frac{\varepsilon}{3}$ for $i \geq i_0$. Then

$$\|x_{0j} - x_{0l}\| \leq \|x_{0j} - x_{ij}\| + \|x_{ij} - x_{il}\| + \|x_{il} - x_{0l}\| \leq \varepsilon$$

for $i \geq i_0$. This proves that $(x_{0j})_j$ is a Cauchy sequence and, similarly, $(x_{i0})_i$ is also a Cauchy sequence. Let us observe that, by (a), $(x_{ij})_i$ converges to x_{0j} uniformly in i and, similarly, $(x_{ij})_j$ converges to x_{i0} uniformly in j. The remaining results are now easy to be obtained.

Theorem 3.2. *Let $(x_{ij})_{i,j}$ be a matrix in a normed space X, where $(x_{ij})_i$ is a Cauchy sequence for every $j \in \mathbb{N}$, and $\sum_{j\in\mathbb{N}} x_{ij}$ is, for $i \in \mathbb{N}$, either uca or verifies the condition* (ii) *in Theorem* 2.2. *The following statements are equivalent:*

(1) *If $[(A_n)_n, (B_n)_n]$ is a pair of disjoint sequences of mutually disjoint elements of $\phi_0(\mathbb{N})$ such that A_i is a singleton for $i \in \mathbb{N}$, there exists $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ such that* (i) *$A_i \subseteq B$ and $B_i \subseteq B^c$ for $i \in M$;* (ii) *For every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that if $p, q \geq k$, there exists l_0 with $\Big\| \sum_{j\in B\cap[1,l]} (x_{pj} - x_{qj})\Big\| < \varepsilon$ for $l \geq l_0$.*

(2) *The sequence $(x_{ij})_i$ is of Cauchy uniformly in $j \in \mathbb{N}$.*

(3) *For any infinite set $P \subseteq \mathbb{N}$, there exists an infinite set $B \subseteq P$ such that for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that if $p, q \geq k$, there exists l_0 with $\Big\| \sum_{j\in B\cap[1,l]} (x_{pj} - x_{qj})\Big\| < \varepsilon$ for $l \geq l_0$.*

Proof. We first prove that (1)⇒(2). If (2) is false, let $\varepsilon > 0$ be such that for every $k \in \mathbb{N}$, there exists $i > k$ and j with $\|x_{ij} - x_{kj}\| > \varepsilon$.

We can inductively obtain three sequences of natural numbers $(j_r)_r$, $(m_r)_r$, with $j_1 < m_1 < j_2 < m_2 < \cdots$, and $(k_r)_r$, $(i_r)_r$, with $k_1 < i_1 < k_2 < i_2 < \cdots$, such that for $r \in \mathbb{N}$, the following properties hold:

(a) $\|x_{k_r j_r} - x_{i_r j_r}\| > \varepsilon$.

(b) $\Big\| \sum_{j\in C} (x_{k_r j} - x_{i_r j})\Big\| < \dfrac{\varepsilon}{8}$ for $C \subseteq \{1, 2, \cdots, m_{r-1}\}$ and $r > 1$.

(c) $\Big\| \sum_{j\in B} (x_{k_r j} - x_{i_r j})\Big\| < \dfrac{\varepsilon}{8}$ for a finite set $B \subseteq \{m_r, m_r + 1, \cdots\}$.

For $r \in \mathbb{N}$, let $A_r = \{j_r\}$ and $B_r = \{m_{r-1} + 1, \cdots, m_r\} \setminus \{j_r\}$. By hypothesis, there exists $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ with the properties (i) and (ii) in (1). However, for every $r \in M$, $r < 1$, and every $l \geq m_r$, we have

$$\Big\| \sum_{j\in B\cap[1,l]} (x_{k_r j} - x_{i_r j})\Big\| \geq \|x_{k_r j_r} - x_{i_r j_r}\| - \Big\| \sum_{j\in B\cap[1,l],\, j\leq m_{r-1}} (x_{k_r j} - x_{i_r j})\Big\|$$
$$- \Big\| \sum_{j\in B\cap[1,l],\, j\geq m_r} (x_{k_r j} - x_{i_r j})\Big\| \geq \frac{3\varepsilon}{4}.$$

This contradicts condition (ii) in (1).

We now prove that (2)⇒(3).

It is easy to check that the sequences $(x_{ij})_j$ are uniformly convergent for $i \in \mathbb{N}$. We can inductively obtain two strictly increasing sequences of natural numbers $(m_r)_r$ and $(i_r)_r$ such that

(i) $\|x_{pj} - x_{qj}\| < \dfrac{1}{(r+1)2^{r+1}}$ for $p, q \geq i_r$ and $j \in \mathbb{N}$.

(ii) $\|x_{ij}\| < \dfrac{1}{(r+1)2^{r+1}}$ for $j \geq m_r$ and $i \in \mathbb{N}$.

Without loss of generality, we can assume that $P = \mathbb{N}$. For any $r \in \mathbb{N}$, we choose $j_r \in P \cap [m_r, m_{r+1})$ and define $B = \{j_r, r \in \mathbb{N}\}$. Let $\varepsilon > 0$. If $r \in \mathbb{N}$ is such that $\dfrac{1}{2^r} < \varepsilon$

and $p, q \geq i_r$, then

$$\Big\| \sum_{j \in B \cap [1,l]} (x_{pj} - x_{qj}) \Big\| \leq \Big\| \sum_{j \in B \cap [1,l],\, j < m_r} (x_{pj} - x_{qj}) \Big\| + \Big\| \sum_{j \in B \cap [1,l],\, j \geq m_r} (x_{pj} - x_{qj}) \Big\|$$
$$< \frac{1}{2^{r+1}} + \frac{1}{2^{r+2}} < \varepsilon,$$

where $l \in \mathbb{N}$.

It is obvious that (3)⇒(1). This completes the proof.

Corollary 3.2. *Let $(x_{ij})_{i,j}$ be a matrix in a normed space X, where $(x_{ij})_i$ is a Cauchy sequence, for $j \in \mathbb{N}$, and $\sum_j x_{ij}$ is subseries convergent, for $i \in \mathbb{N}$. The following statements are equivalent:*

(1) *If $[(A_n)_n, (B_n)_n]$ is a pair of disjoint sequences of mutually disjoint elements of $\phi_0(\mathbb{N})$ and, for $i \in \mathbb{N}$, A_i is a singleton, then there exists $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ with the properties:* (i) *$A_i \subseteq B$ and $B_i \subseteq B^c$ for $i \in M$,* (ii) *$\Big(\sum_{j \in B} x_{ij}\Big)_i$ is a Cauchy sequence.*

(2) *The sequence $(x_{ij})_i$ is of Cauchy, uniformly in $j \in \mathbb{N}$.*

(3) *For any infinite set $P \subset \mathbb{N}$, there exists an infinite set $B \subset P$ such that $\Big(\sum_{j \in B} x_{ij}\Big)_i$ is a Cauchy sequence.*

Remark 3.1. Let $\mathcal{L}$ be a family in $P(\mathbb{N})$ such that $\phi_0(\mathbb{N}) \subset \mathcal{L}$ and has the following property:

(c) For any pair $[(A_i), (B_i)]$ of disjoint sequences of mutually disjoint elements of $\phi_0(\mathbb{N})$ and such that A_i is a singleton, for $i \in \mathbb{N}$, there exists an infinite set $M \subset \mathbb{N}$ and a $B \in \mathcal{L}$ such that $A_i \subset B$ and $B_i \subset B^c$ for $i \in M$.

Corollary 3.2 is also true if we substitute (1) by the following statement: There exists a family $\mathcal{L}$ with the property (c) such that $\Big(\sum_{j \in B} x_{ij}\Big)_i$ is a Cauchy sequence, for $B \in \mathcal{L}$.

Let $\mathcal{L}$ be a family in $P(\mathbb{N})$ such that $\phi_0(\mathbb{N}) \subset \mathcal{L}$ and with the following property:

(d) for every sequence of mutually different singleton sets $(A_i)_i$, there exists an infinite set $M \subset \mathbb{N}$ such that $\bigcup_{i \in M} A_i \in \mathcal{L}$.

In [3], it has been proved that (1)⇒(2) (in Corollary 3.2) remains true if (1) is replaced by the following statement: there exists a family $\mathcal{L}$ with property (d) such that $\Big(\sum_{j \in B} x_{ij}\Big)_i$ is a Cauchy sequence, for $B \in \mathcal{L}$.

In the following example we obtain a family $\mathcal{L}$ with the property (c) that lacks the properties (d), (a) and (b) mentioned in Remark 2.2.

We consider the family $\mathcal{B} \subset P(\mathbb{N})$ of the $A \subset \mathbb{N}$ such that

(i) A and A^c have an infinite number of even numbers and an infinite number of odd numbers.

(ii) The set $\{n \in \mathbb{N}; \{4n, 4n-1\} \subseteq A\}$ is finite.

Let $\mathcal{L} = \mathcal{B} \cup \phi_0(\mathbb{N})$. We consider the pair of disjoint sequences $[(A_i)_i, (B_i)_i]$ of mutually disjoint sets, where $A_i = \{4i-1, 4i\}$ and $B_i = \{4i+2\}$ for $i \in \mathbb{N}$. It is obvious that if there exists $A \in \mathcal{L}$ such that $\{i \in \mathbb{N}; A_i \subseteq A\}$ is infinite, then A does not satisfy (ii), which is a contradiction. However, it is easy to check that $\mathcal{L}$ satisfies the separation property where A_i is a singleton; i.e. satisfies (d). We also have that the union of the terms on any subsequence of $(\{2n\})_{n\in\mathbb{N}}$ does not belong to $\mathcal{L}$.

As a consequence of the former theorem, we can obtain an easy proof of the Basic Matrix Theorem of Antosik and Swartz. We also prove that the converse result is also valid.

Corollary 3.3. *Let $(x_{ij})_{i,j}$ be a matrix in a normed space X such that, for $j \in \mathbb{N}$, $(x_{ij})_i$ is a Cauchy sequence. The following statements are equivalent:*

(1) *For any infinite set $P \subset \mathbb{N}$, there exists an infinite set $B \subset P$ such that $\Big(\sum\limits_{j\in B} x_{ij}\Big)_i$ is a Cauchy sequence.*

(2) *The matrix $(x_{ij})_{i,j}$ satisfies the following properties:*

(i) *The sequence $(x_{ij})_j$ is convergent for every $i \in \mathbb{N}$.*

(ii) *$(x_{ij})_i$ is a Cauchy sequence, uniformly in $j \in \mathbb{N}$.*

(iii) *For any infinite set $P \subseteq \mathbb{N}$, there exists an infinite set $B \subseteq P$ such that $\sum\limits_{j\in B} x_{ij}$ is convergent, for $i \in \mathbb{N}$.*

Proof. We first prove that (1)⇒(2). If (2) is false, then there exist $\varepsilon > 0$ and three strictly increasing sequences $(k_r)_r$, $(i_r)_r$ and $(j_r)_r$ of natural numbers such that

(a) $k_1 < i_1 < k_2 < i_2 < \cdots$,

(b) $\|x_{k_r j_r} - x_{i_r j_r}\| > \varepsilon$ for $r \in \mathbb{N}$.

For $p, q \in \mathbb{N}$, we define $z_{2p-1,q} = x_{k_p j_q}$ and $z_{2p,q} = x_{i_p j_q}$. The sequence $(z_{ij})_j$ converges to zero for $i \in \mathbb{N}$. We can inductively obtain a sequence $(M_i)_i$ of infinite subsets of $\mathbb{N}$, $M_i = \{\alpha_{ij}, j \in \mathbb{N}\}$ for $i \in \mathbb{N}$, such that, for $i \in \mathbb{N}$,

(1) $M_{i+1} \subseteq M_i$,

(2) $\alpha_{i1} < \alpha_{(i+1)1}$,

(3) $\sum\limits_{j\in M_i} z_{ij}$ is uca.

It is obvious that if $M = \{\alpha_{ii}, i \in \mathbb{N}\}$, then the matrix $(z_{ij})_{i\in\mathbb{N}, j\in M}$ satisfies the conditions of Theorem 3.2. This contradicts that $\|z_{2i-1,i} - z_{2i,i}\| > \varepsilon$ for $i \in \mathbb{N}$.

Let us prove that (2)⇒(1). Let P be an infinite set and, as before, let $P' \subseteq P$ be such that $\sum\limits_{j\in P'} x_{ij}$ is uca, for $i \in \mathbb{N}$. By (iii), let $B \subseteq P'$ be an infinite set such that $\sum\limits_{j\in B} x_{ij}$ is convergent. Theorem 3.2 let us conclude the proof.

By using arguments similar to those we have used in the former proof, the following result can be proved.

Corollary 3.4. *Let $(x_{ij})_{i,j}$ be a matrix in a Banach space X such that $(x_{ij})_i$ is a Cauchy sequence for $i \in \mathbb{N}$. The following statements are equivalent:*

(1) *For any infinite set $P \subset \mathbb{N}$, there exists an infinite set $B \subset P$ such that $\Big(\sum\limits_{j\in B} x_{ij}\Big)_i$ is a Cauchy sequence.*

(2) *The matrix $(x_{ij})_{i,j}$ satisfies the following properties:*

(a) *The sequence* $(x_{ij})_j$ *converges to zero, for* $i \in \mathbb{N}$.

(b) $(x_{ij})_i$ *is of Cauchy, uniformly in* $j \in \mathbb{N}$.

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