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IRREDUCIBLE REPRESENTATIONS FOR THE AFFINE-VIRASORO LIE ALGEBRA OF TYPE B_l^{***}

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Abstract

An explicit construction of irreducible representations for the affine-Virasoro Lie algebra of type B_l, through the use of vertex operators and certain oscillator representations of the Virasoro algebra, is given.

Keywords Affine-Virasoro Lie algebra, Vertex operator, Irreducible module
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§ 1. Introduction

The theory of vertex operator representations of affine Lie algebras has many important and interesting applications in mathematics and physics. The first vertex construction of basic representations for affine Lie algebras, which is usually called the principal vertex representation, was discovered by Lepowsky and Wilson [13] and it has been generalized in [12] to all the affine Lie algebras of ADE type. Frenkel and Kac [8], and Segal [15], have given another construction of the same basic modules of all ADE type Lie algebras by using the vertex operators $X(\alpha, z)$ which are very useful for the study of the dual resonance theory. A suitable chosen Heisenberg algebra (or a Heisenberg system) is crucial to these constructions. By enlarging the Fock spaces, one can construct vertex representations for other affine Lie algebras. These representations, however, are no longer irreducible.

It was also observed in [8] that the basic given modules afford representations of the Virasoro algebra too. This makes the basic modules into irreducible modules of larger Lie algebras—the semi-direct products of affine Lie algebras and the Virasoro algebra, which play an important role in the representation theory of affine and Virasoro algebras and in the quantum field theory.

This paper is based on the work of [10], where vertex representations for the toroidal Lie algebras of type B_l were constructed. A toroidal Lie algebras is the universal central extension of the iterated multi-loop algebra, $\dot{\mathcal{G}} \otimes \mathbf{C}[t_0^{\pm 1}, t_1^{\pm 1}, \cdots, t_\nu^{\pm 1}]$ ($\nu \geq 1$), where $\dot{\mathcal{G}}$ is a simple finite-dimensional Lie algebra over the complex field $\mathbf{C}$. Toroidal Lie algebras can be considered as the natural generalization of affine Lie algebras. Unlike the affine case, however, the universal central extension has infinite-dimensional center. By adding the full derivation algebra together with a 2-cocycle to the toroidal Lie algebra, we can get a larger

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toroidal Lie algebra which has finite-dimensional center. Over the past decade, the vertex representations of affine Lie algebras have been generalized to vertex representations of the toroidal Lie algebras (e.g., see [14, 6, 16, 3]). For more results on toroidal Lie algebras, one can see [1, 2, 4, 5], etc.

The modules constructed in [10] are not irreducible, since the corresponding modules of affine Lie algebra are not irreducible. In this paper, we extend the affine Lie algebra by the Virasoro algebra thus constructing a semi-direct product algebra containing the affine Lie algebra as an ideal and the Virasoro algebra as a subalgebra. By introducing Virasoro operators which, different from the previous ones, do not preserve the irreducible submodules of the affine Lie algebra, we prove that the given reducible module not only becomes a module of the larger algebra but also becomes irreducible.

Through out the paper, we denote by $\mathbf{Z}^+$ and $\mathbf{Z}^-$ the set of positive integers and the set of negative integers respectively.

§ 2 . Preliminaries

Let $\mathfrak{g}$ be the finite-dimensional simple Lie algebra of type B_l over the complex field $\mathbf{C}$, and $(\cdot \mid \cdot)$ the Killing form on $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and Δ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$. Then there exists the normal orthogonal basis $\{e_1, \cdots, e_l\}$ of $\mathfrak{h}^*$ such that the short root system is $\Delta_S = \{\pm e_i \mid 1 \le i \le l\}$ and the long root system is $\Delta_L = \{\pm e_i + e_j, \pm e_i - e_j \mid 1 \le i < j \le l\}$. The simple root system is $\sqcap = \{e_1 - e_2, e_2 - e_3, \cdots, e_{l-1} - e_l, e_l\}$. Let $\alpha_i = e_i - e_{i+1}, 1 \le i \le l-1, \alpha_l = e_l$. Then the root lattice is

$$Q = \Big\{ \sum_{i=1}^{l} k_i \alpha_i \mid k_i \in \mathbf{Z}, 1 \le i \le l \Big\} = \Big\{ \sum_{i=1}^{l} k_i e_i \mid k_i \in \mathbf{Z}, 1 \le i \le l \Big\}.$$

It is clear that $(\alpha_i \mid \alpha_i) = 2, 1 \le i \le l-1, (\alpha_l \mid \alpha_l) = 1$. Let $\alpha^\vee \in \mathfrak{h}$ be such that $\alpha(\alpha^\vee) = 2$. Define $\epsilon : Q \times Q \to \{\pm 1\}$ by

$$\epsilon(e_i, e_j) = 1, \quad 1 \le i \le j \le l, \qquad \epsilon(e_j, e_i) = -\epsilon(e_i, e_j), \quad i < j; \tag{2.1}$$

$$\epsilon(a+b, c) = \epsilon(a, c)\epsilon(b, c), \qquad \epsilon(a, b+c) = \epsilon(a, b)\epsilon(a, c). \tag{2.2}$$

Proposition 2.1. (cf. [10]) *The Lie algebra* $\mathfrak{g}$ *has a basis* $\{e_\alpha, \alpha_i^\vee \mid \alpha \in \Delta, 1 \le i \le l\}$ *such that*

$$[\alpha_i^\vee, \alpha_j^\vee] = 0, \quad [\alpha_i^\vee, e_\alpha] = \alpha(\alpha_i^\vee) e_\alpha, \qquad \alpha \in \Delta,\ 1 \le i \le l; \tag{2.3}$$

$$[e_\alpha, e_{-\alpha}] = \epsilon(\alpha, -\alpha)\alpha^\vee, \qquad \alpha \in \Delta; \tag{2.4}$$

$$[e_\alpha, e_\beta] = \epsilon(\alpha, \beta) e_{\alpha+\beta}, \qquad \alpha \in \Delta_L,\ \beta,\ \alpha+\beta \in \Delta; \tag{2.5}$$

$$[e_\alpha, e_\beta] = 2\epsilon(\alpha, \beta) e_{\alpha+\beta}, \qquad \alpha, \beta \in \Delta_S,\ \alpha+\beta \in \Delta; \tag{2.6}$$

$$[e_\alpha, e_\beta] = 0, \qquad \alpha, \beta \in \Delta,\ 0 \ne \alpha+\beta \notin \Delta. \tag{2.7}$$

Proposition 2.2. $\{e_{\alpha_i}, -e_{-\alpha_i}, e_{\alpha_l}, e_{-\alpha_l} \mid 1 \le i \le l-1\}$ *are Chevalley generators of* $\mathfrak{g}$, *and* $\{e_\alpha, -e_{-\alpha}, e_\beta, e_{-\beta}, \alpha_i^\vee \mid \alpha \in \Delta_L^+, \beta \in \Delta_S^+, 1 \le i \le l\}$ *is a Chevalley basis of* $\mathfrak{g}$.

Proof. The results can be checked directly by Proposition 2.1 and the definition of ϵ.

Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c$ be the associated affine Lie algebra with the following Lie bracket

$$[x \otimes t^m + \lambda_1 c, y \otimes t^n + \lambda_2 c] = [x, y] \otimes t^{m+n} + m\delta_{m+n,0}(x \mid y)c, \tag{2.8}$$

where $x, y \in \mathfrak{g}, \lambda_1, \lambda_2 \in \mathbf{C}, m, n \in \mathbf{Z}$. For $k \in \mathbf{Z}$, let $\mathfrak{h}^*(k)$ be an isomorphic copy of $\mathfrak{h}^*$, and $e_l(k+\frac{1}{2})$ an isomorphic copy of e_l. Then

$$\widetilde{\mathcal{H}} = \underset{k\in\mathbf{Z}}{\oplus} \mathfrak{h}^*(k) \underset{k\in\mathbf{Z}}{\oplus} \mathbf{C}e_l\Big(k+\frac{1}{2}\Big) \oplus \mathbf{C}c_0$$

becomes a Lie algebra with the following Lie bracket

$$\begin{aligned}
&[a(m), b(n)] = m(a \mid b)\delta_{m+n,0}c_0,\\
&\Big[e_l\Big(m+\frac{1}{2}\Big), e_l\Big(n-\frac{1}{2}\Big)\Big] = \Big(m+\frac{1}{2}\Big)\delta_{m+n,0}c_0,\\
&[\widetilde{\mathcal{H}}, c_0] = \Big[\underset{k\in\mathbf{Z}}{\oplus} \mathfrak{h}^*(k), e_l\Big(k+\frac{1}{2}\Big)\Big] = 0,
\end{aligned}$$

where $a, b \in \mathfrak{h}^*, m, n, k \in \mathbf{Z}$. It is easy to see that $\widehat{\mathcal{H}} = \underset{k\in\mathbf{Z}\setminus\{0\}}{\oplus} \mathfrak{h}^*(k) \underset{k\in\mathbf{Z}}{\oplus} \mathbf{C}e_l(k+\frac{1}{2}) \oplus \mathbf{C}c_0$ is a Heisenberg Lie algebra. Let

$$\widehat{\mathcal{H}}^- = \underset{k\in\mathbf{Z}^-}{\oplus} \mathfrak{h}^*(k) \underset{k\in\mathbf{Z}^-}{\oplus} \mathbf{C}e_l\Big(k+\frac{1}{2}\Big),$$

and $S(\widehat{\mathcal{H}}^-)$ the symmetric algebra generated by $\widehat{\mathcal{H}}^-$. We define the Fock space $V = S(\widehat{\mathcal{H}}^-)\otimes \mathbf{C}[Q]$, where $\mathbf{C}[Q] = \underset{\alpha\in Q}{\oplus} \mathbf{C}e^\alpha$ is the group algebra on the additive group Q. Then V has a natural module structure for the Lie algebra $\widetilde{\mathcal{H}}$ and the group algebra $\mathbf{C}[Q]$ with the actions defined by assigning c_0 acting as 1, $a(-n)$ and $e_l(-n+\frac{1}{2})$ acting as multiplications and $a(n)$ and $e_l(n-\frac{1}{2})$ acting as partial differential operators, for $n > 0, a \in \mathfrak{h}^*$, and $a(0)$ acting as a partial differential operator on $\mathbf{C}[Q]$ for which $a(0)e^r = (a \mid r)e^r$, where $a \in \mathfrak{h}^*$.

Let z be a complex variable. We define elements of $\mathrm{End}V[[z, z^{-1}]]$ as follows:

$$\epsilon_a(v\otimes e^r) = \epsilon(a, r)v\otimes e^r, \qquad a \in Q, \tag{2.9}$$

$$z^a(v\otimes e^r) = z^{(a|r)}(v\otimes e^r), \qquad a \in Q, \tag{2.10}$$

$$Y(\alpha, z) = \exp\Big(\sum_{n=1}^{\infty}\frac{z^{2n}}{n}\alpha(-n)\Big)\exp\Big(-\sum_{n=1}^{\infty}\frac{z^{-2n}}{n}\alpha(n)\Big), \qquad \alpha \in \Delta_L, \tag{2.11}$$

$$\begin{aligned}
Y(\alpha, z) = \sqrt{-1}&\exp\Big(\sum_{n=1}^{\infty}\frac{z^{2n}}{n}\alpha(-n)\Big)\exp\Big(-\sum_{n=1}^{\infty}\frac{z^{-2n}}{n}\alpha(n)\Big)\\
&\cdot\exp\Big(\sum_{n=0}^{\infty}\frac{2z^{2n+1}}{2n+1}e_l\Big(-n-\frac{1}{2}\Big)\Big)\\
&\cdot\exp\Big(-\sum_{n=0}^{\infty}\frac{2z^{-2n-1}}{2n+1}e_l\Big(n+\frac{1}{2}\Big)\Big), \qquad \alpha \in \Delta_S,
\end{aligned} \tag{2.12}$$

$$X(\alpha, z) = Y(\alpha, z)z^{(\alpha|\alpha)}e^\alpha z^{2\alpha}\epsilon_\alpha = \sum_{n\in\frac{1}{2}\mathbf{Z}} X_n(\alpha)z^{-2n}. \tag{2.13}$$

For $v\otimes e^r = a_1(-n_1)\cdots a_k(-n_k)e_l(-m_1+\frac{1}{2})\cdots e_l(-m_s+\frac{1}{2})\otimes e^r$, where $a_i, r \in Q, n_i, m_j \in \mathbf{Z}^+$, define

$$\deg(v\otimes e^r) = -\sum_{i=1}^{k} n_i - \sum_{j=1}^{s}\Big(m_j - \frac{1}{2}\Big) - \frac{1}{2}(r \mid r). \tag{2.14}$$

Let V_0 and $V_{\frac{1}{2}}$ be the subspaces of V spanned by $\{v\otimes e^r \mid \deg(v\otimes e^r)\in \mathbf{Z}\}$ and $\{v\otimes e^r \mid \deg(v\otimes e^r)\in \mathbf{Z}+\frac{1}{2}\}$ respectively. Then

$$V = V_0 + V_{\frac{1}{2}}.$$

Proposition 2.3. (cf. [10]) *V_0 and $V_{\frac{1}{2}}$ are completely reducible modules of the affine Lie algebra $\hat{\mathfrak{g}}$. Particularly, we have*

$$(\alpha^\vee\otimes t^n)(v\otimes e^r) = \frac{2}{(\alpha\mid\alpha)}\alpha(n)(v\otimes e^r), \tag{2.15}$$

$$(e_\alpha\otimes t^n)(v\otimes e^r) = X_n(\alpha)(v\otimes e^r), \tag{2.16}$$

$$c(v\otimes e^r) = v\otimes e^r, \tag{2.17}$$

where $\alpha\in\Delta, n\in\mathbf{Z}$.

§ 3 . Irreducible Modules of the Affine-Virasoro Lie Algebra $\hat{\mathfrak{g}}_{\mathrm{vir}}$

Let $d_i = -t^{i+1}\frac{d}{dt}$ $(i\in\mathbf{Z})$ be a derivation of $\mathbf{C}[t,t^{-1}]$. Extend it to the operators on $\hat{\mathfrak{g}}$ by

$$d_i(x\otimes t^m + \lambda c) = x\otimes d_i(t^m) = -mx\otimes t^{m+i}. \tag{3.1}$$

Then d_i is a derivation of $\hat{\mathfrak{g}}$. Let $\mathfrak{d}$ be the Lie algebra spanned by $\{d_i \mid i\in\mathbf{Z}\}$. Then the universal central extension of $\mathfrak{d}$ by a 1-dimensional center $\mathbf{C}K$, is the Virasoro algebra, denoted by Vir, with the following Lie bracket

$$[d_m,d_n] = (m-n)d_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}K. \tag{3.2}$$

Now we consider the Lie algebra $\hat{\mathfrak{g}}_{\mathrm{vir}} = \mathfrak{g}\oplus \mathrm{Vir}$ with the Lie bracket defined by (2.8), (3.1), (3.2) and $[\hat{\mathfrak{g}}_{\mathrm{vir}}, \mathbf{C}c\oplus\mathbf{C}K] = 0$. The goal of this section is to prove that V_0 and $V_{\frac{1}{2}}$ are irreducible $\hat{\mathfrak{g}}_{\mathrm{vir}}$-modules. Define operators on V as follows:

$$L_0 = \frac{1}{2}\sum_{i=1}^{l} e_i(0)e_i(0) + \sum_{j>0}\sum_{i=1}^{l} e_i(-j)e_i(j) + \sum_{j\geq 0} e_l\Big(-j-\frac{1}{2}\Big)e_l\Big(j+\frac{1}{2}\Big) + \frac{1}{16}, \tag{3.3}$$

$$L_n = \frac{1}{2}\sum_{j\in\mathbf{Z}}\sum_{i=1}^{l} e_i(-j)e_i(j+n) + \frac{1}{2}\sum_{j\in\mathbf{Z}} e_l\Big(-j-\frac{1}{2}\Big)e_l(j+\frac{1}{2}+n), \qquad n\in\mathbf{Z}\backslash\{0\}. \tag{3.4}$$

Lemma 3.1. *For $v\otimes e^r\in V$, we have*

$$L_0(v\otimes e^r) = \Big(-\deg(v\otimes e^r) + \frac{1}{16}\Big)(v\otimes e^r). \tag{3.5}$$

Lemma 3.2. *For $m,n\in\mathbf{Z}, \alpha\in\Delta$, we have*

$$[L_m, X_n(\alpha)] = -nX_{n+m}(\alpha). \tag{3.6}$$

Proof. First it is easy to see that

$$\deg(X_n(\alpha)(v\otimes e^r)) = n + \deg(v\otimes e^r), \tag{3.7}$$

$$\deg(L_n(v\otimes e^r)) = n + \deg(v\otimes e^r). \tag{3.8}$$

If $m = 0$, then by (3.7) and (3.5) we have

$$[L_0, X_n(\alpha)] = -nX_n(\alpha). \tag{3.9}$$

We can deduce (3.6) by using (3.7)–(3.9).

Lemma 3.3. *For $m, n \in \mathbf{Z}$, we have*

$$[e_i(m), L_n] = me_i(m+n), \qquad 1 \le i \le l; \tag{3.10}$$

$$\Big[e_l\Big(m+\frac{1}{2}\Big), L_n\Big] = \Big(m+\frac{1}{2}\Big)e_l\Big(m+n+\frac{1}{2}\Big). \tag{3.11}$$

Proof. It is straightforward to check the results.

Lemma 3.4. *For $m, n \in \mathbf{Z}$, one has*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}(l+1)\delta_{m+n,0}. \tag{3.12}$$

From Lemmas 3.2–3.4 and Proposition 2.3, we deduce

Theorem 3.1. *V_0 and $V_{\frac{1}{2}}$ are $\hat{\mathfrak{g}}_{\mathrm{vir}}$-modules with the action defined by* (2.15)–(2.17) *and*

$$d_i(v \otimes e^r) = L_i(v \otimes e^r), \qquad i \in \mathbf{Z}, \tag{3.13}$$

$$K(v \otimes e^r) = (l+1)(v \otimes e^r). \tag{3.14}$$

Now, define a Hermitian form $H(\cdot,\cdot)$ on V as follows:

$$\begin{aligned} &H\Big(\prod_{i=1}^{l}\Big(\prod_{j=1}^{s_i} e_i(-n_{ij})^{k_{ij}}\Big)\prod_{j=1}^{s} e_l\Big(-n_j-\frac{1}{2}\Big)^{k_j} \otimes e^{r_1}, \\ &\quad \prod_{i=1}^{l}\Big(\prod_{j=1}^{q_i} e_i(-m_{ij})^{l_{ij}}\Big)\prod_{j=1}^{p} e_l\Big(-m_j-\frac{1}{2}\Big)^{l_j} \otimes e^{r_2}\Big) \\ &= \delta_{r_1-r_2,0}\delta_{s,p}\prod_{j=1}^{s}\delta_{n_j,m_j}\delta_{k_j,l_j}\prod_{i=1}^{l}\Big(\delta_{s_i,q_i}\prod_{j=1}^{s_i}\delta_{n_{ij},m_{ij}}\delta_{k_{ij},l_{ij}}\Big) \\ &\quad \cdot\Big(\prod_{i=1}^{l}\prod_{j=1}^{s_i} k_{ij}!n_{ij}^{k_{ij}}\Big)\prod_{j=1}^{s} k_j!\Big(n_j+\frac{1}{2}\Big)^{k_j}, \end{aligned}$$

and for $\lambda \in \mathbf{C}, v, w \in V$,

$$(\lambda v, w) = \lambda(v, w) = (v, \bar{\lambda} w),$$

where we assume that $n_{i1} < n_{i2} < \cdots < n_{is_i},\ m_{i1} < m_{i2} < \cdots < m_{iq_i},\ 1 \le i \le l,\ n_1 < n_2 < \cdots < n_s,\ m_1 < m_2 \cdots < m_p$.

Lemma 3.5. *The Hermitian form $H(\cdot,\cdot)$ is positive definite.*

Let $e_\alpha(\alpha \in \Delta),\ \alpha_i^\vee\ (1 \le i \le l)$ be the same as in Proposition 2.1. Let

$$e'_{\pm\alpha} = \pm e_{\pm\alpha}, \quad e'_{\pm\beta} = e_{\pm\beta}, \qquad \alpha \in \Delta_L^+,\ \beta \in \Delta_S^+. \tag{3.15}$$

Then by Proposition 2.2, $\{e'_\alpha, \alpha_i^\vee \mid \alpha \in \Delta, 1 \le i \le l\}$ is a Chevalley basis of $\mathfrak{g}$. Now define an antilinear operator ω_0 on $\hat{\mathfrak{g}}_{\rm vir}$ by

$$\omega_0(\lambda x) = \bar{\lambda}\omega_0(x), \qquad x \in \hat{\mathfrak{g}}_{\rm vir}; \tag{3.16}$$

$$\omega_0(e'_\alpha \otimes t^m) = -e'_{-\alpha} \otimes t^{-m}, \qquad m \in \mathbf{Z},\ \alpha \in \Delta; \tag{3.17}$$

$$\omega_0(\alpha_i^\vee \otimes t^m) = -\alpha_i^\vee \otimes t^{-m}, \qquad m \in \mathbf{Z},\ 1 \le i \le l; \tag{3.18}$$

$$\omega_0(c) = -c, \quad \omega_0(K) = -K, \quad \omega_0(d_m) = -d_{-m}. \tag{3.19}$$

It is easy to see that ω_0 is an antilinear automorphism of $\hat{\mathfrak{g}}_{\rm vir}$.

Lemma 3.6. *For $n \in \mathbf{Z}$, $1 \le i \le l$ and $x = v \otimes e^{r_1}, y = w \otimes e^{r_2} \in V$, we have*

$$H(e_i(n)x, y) = H(x, e_i(-n)y); \tag{3.20}$$

$$H\Big(e_l\Big(n + \frac{1}{2}\Big)x, y\Big) = H\Big(x, e_l\Big(-n - \frac{1}{2}\Big)y\Big). \tag{3.21}$$

Proof. We can assume that

$$v = \prod_{i=1}^{l}\Big(\prod_{j=1}^{s_i} e_i(-n_{ij})^{k_{ij}}\Big)\prod_{j=1}^{s} e_l\Big(-n_j - \frac{1}{2}\Big)^{k_j} \otimes e^{r_1},$$

$$w = \prod_{i=1}^{l}\Big(\prod_{j=1}^{t_i} e_i(-m_{ij})^{l_{ij}}\Big)\prod_{j=1}^{p} e_l\Big(-m_j - \frac{1}{2}\Big)^{l_j} \otimes e^{r_2}.$$

For $n \in \mathbf{Z}^+$, let

$$v = e_i(-n)^k v_1, \qquad w = e_i(-n)^q w_1,$$

where $e_i(n)v_1 = e_i(n)w_1 = 0$. Then

$$\begin{aligned} H(e_i(n)v \otimes e^{r_1}, w \otimes e^{r_2}) &= knH(e_i(-n)^{k-1}v_1 \otimes e^{r_1}, e_i(-n)^q w_1 \otimes e^{r_2}) \\ &= \delta_{k-1,q}k!n^k H(v_1 \otimes e^{r_1}, w_1 \otimes e^{r_2}) \\ &= H(e_i(-n)^k v_1 \otimes e^{r_1}, e_i(-n)^{q+1}w_1 \otimes e^{r_2}) \\ &= H(v \otimes e^{r_1}, e_i(-n)w \otimes e^{r_2}). \end{aligned}$$

If $n = 0$, we have

$$H(e_i(0)(v \otimes e^{r_1}), w \otimes e^{r_2}) = (r_1 \mid e_i)H(v \otimes e^{r_1}, w \otimes e^{r_2}),$$

$$H(v \otimes e^{r_1}, e_i(0)(w \otimes e^{r_2})) = (r_2 \mid e_i)H(v \otimes e^{r_1}, w \otimes e^{r_2}).$$

By the definition of $H(\cdot,\cdot)$, we have

$$H(e_i(0)x, y) = H(x, e_i(0)y).$$

The proof of (3.21) is similar.

Lemma 3.7. *Let $Y(\alpha, z)$ be defined by* (2.11) *and* (2.12)*, and let*

$$Y(\alpha, z) = \sum_{n \in \frac{1}{2}\mathbf{Z}} Y_n(\alpha)z^{-2n}.$$

Then for $v, w \in S(\widehat{\mathcal{H}}^-), r \in Q, n \in \mathbf{Z}$, we have

$$H((Y_n(\alpha)v) \otimes e^r, w \otimes e^r) = H(v \otimes e^r, (Y_{-n}(-\alpha)w) \otimes e^r), \qquad \alpha \in \Delta_L; \tag{3.22}$$

$$H((Y_{n+\frac{1}{2}}(\alpha)v) \otimes e^r, w \otimes e^r) = H(v \otimes e^r, (Y_{-n-\frac{1}{2}}(-\alpha)w) \otimes e^r), \qquad \alpha \in \Delta_S. \tag{3.23}$$

Proof. Let

$$E^{\mp}(\alpha, z) = \exp\Big(\pm \sum_{n=1}^{\infty} \frac{z^{\pm 2n}}{n} \alpha(\mp n)\Big) = \sum_{n \in \mathbf{Z}} E_n^{\mp}(\alpha) z^{-2n},$$

$$F^{\mp}(e_l, z) = \exp\Big(\pm \sum_{n=0}^{\infty} \frac{2z^{\pm(2n+1)}}{2n+1} (e_l)\Big(\mp \frac{2n+1}{2}\Big)\Big) = \sum_{n \in \frac{1}{2}\mathbf{Z}} F_n^{\mp}(e_l) z^{-2n},$$

$$F^{\mp}(-e_l, z) = \exp\Big(\pm \sum_{n=0}^{\infty} \frac{2z^{\pm(2n+1)}}{2n+1} (-e_l)\Big(\mp \frac{2n+1}{2}\Big)\Big) = \sum_{n \in \frac{1}{2}\mathbf{Z}} F_n^{\mp}(-e_l) z^{-2n}.$$

Note that $E_n^-(\alpha) = E_{-n}^+(\alpha) = F_{n-\frac{1}{2}}^-(\pm e_l) = F_{-n+\frac{1}{2}}^+(\pm e_l) = 0$ for $n \in \mathbf{Z}^+$. By Lemma 3.6, for any $v_1, v_2 \in S(\widehat{\mathcal{H}}^-), r \in Q$, we have

$$H((E_n^{\mp}(\alpha)v_1) \otimes e^r, v_2 \otimes e^r) = H(v_1 \otimes e^r, (E_{-n}^{\pm}(-\alpha)v_2) \otimes e^r), \qquad \alpha \in \Delta;\ n \in \mathbf{Z}, \tag{3.24}$$

$$H((F_n^{\mp}(e_l)v_1) \otimes e^r, v_2 \otimes e^r) = H(v_1 \otimes e^r, (F_{-n}^{\pm}(-e_l)v_2) \otimes e^r), \qquad n \in \frac{1}{2}\mathbf{Z}. \tag{3.25}$$

If $\alpha \in \Delta_L$, by the definition of $H(\cdot,\cdot)$, we can assume that $\deg v = \deg w - n = -k$ $(k \geq n)$. Then

$$(Y_n(\alpha)v) \otimes e^r = \Big(\sum_{i=0}^{k} E_{n-i}^-(\alpha) E_i^+(\alpha) v\Big) \otimes e^r,$$

$$(Y_{-n}(-\alpha)w) \otimes e^r = \Big(\sum_{i=0}^{k-n} E_{-n-i}^-(-\alpha) E_i^+(-\alpha) w\Big) \otimes e^r.$$

Therefore by (3.24) we have

$$\begin{aligned}
H((Y_n(\alpha)v) \otimes e^r, w \otimes e^r) &= \sum_{i=0}^{k} H(E_{n-i}^-(\alpha) E_i^+(\alpha) v) \otimes e^r, w \otimes e^r) \\
&= \sum_{i=0}^{k} H((E_i^+(\alpha)v) \otimes e^r, (E_{-n+i}^+(-\alpha)w) \otimes e^r) \\
&= H\Big(v \otimes e^r, \Big(\sum_{i=0}^{k} E_{-i}^-(-\alpha) E_{i-n}^+(-\alpha) w\Big) \otimes e^r\Big) \\
&= H\Big(v \otimes e^r, \Big(\sum_{i=0}^{k-n} E_{-n-i}^-(-\alpha) E_i^+(-\alpha) w\Big) \otimes e^r\Big) \\
&= H(v \otimes e^r, (Y_{-n}(-\alpha)w) \otimes e^r).
\end{aligned}$$

Therefore (3.22) holds. We can deduce (3.23) similarly, for $\beta \in \Delta_S$, though we still point out that

$$Y(\pm\beta, z) = \sqrt{-1} E^-(\pm\beta, z) E^+(\pm\beta, z) F^-(e_l, z) F^+(e_l, z),$$

and

$$H(\sqrt{-1}\,Y_{n+\frac{1}{2}}(\beta)v\otimes e^r, w\otimes e^r) = \sqrt{-1}\,H(Y_{n+\frac{1}{2}}(\beta)v\otimes e^r, w\otimes e^r)$$
$$= -H(v\otimes e^r, \sqrt{-1}\,Y_{-n-\frac{1}{2}}(-\beta)w\otimes e^r).$$

Theorem 3.2. *$H(\cdot,\cdot)$ is a contravariant Hermitian form on V with respect to ω_0, i.e.,*

$$H(g(x), y) = -H(x, \omega_0(g)y) \tag{3.26}$$

for all $g\in\hat{\mathfrak{g}}_{\rm vir}, x, y\in V$. Therefore by Lemma 3.5, V_0 and $V_{\frac{1}{2}}$ are unitary modules of $\hat{\mathfrak{g}}_{\rm vir}$.

Proof. By Lemma 3.6 and the definition of ω_0, (3.26) holds for all $g\in\mathfrak{h}\otimes\mathbf{C}[t,t^{-1}]\oplus\mathbf{C}c\oplus\mathbf{C}K$, and it is easy to deduce by Lemma 3.6 that

$$H(d_n(x), y) = -H(x, \omega_0(d_n)y), \qquad n\in\mathbf{Z}.$$

Let $x = v\otimes e^{r_1},\ y = w\otimes e^{r_2}\in V$. By (3.15), (3.17) and (2.16), we have to prove

$$H(X_n(\alpha)(v\otimes e^{r_1}), w\otimes e^{r_2}) = -H(v\otimes e^{r_1}, X_{-n}(-\alpha)(w\otimes e^{r_2})), \qquad \alpha\in\Delta_L; \tag{3.27}$$

$$H(X_n(\beta)(v\otimes e^{r_1}), w\otimes e^{r_2}) = H(v\otimes e^{r_1}, X_{-n}(-\beta)(w\otimes e^{r_2})), \qquad \beta\in\Delta_S. \tag{3.28}$$

We first prove (3.27). By the definition of $H(\cdot,\cdot)$, we can assume that $r_1+\alpha = r_2$. Then

$$X_n(\alpha)(v\otimes e^{r_1}) = \epsilon(\alpha, r_1)Y_{n+(\alpha|r_1)+1}(\alpha)v\otimes e^{r_2},$$
$$X_{-n}(-\alpha)(w\otimes e^{r_2}) = \epsilon(-\alpha, r_1+\alpha)Y_{-n-(\alpha|r_1)-1}(-\alpha)w\otimes e^{r_1}.$$

By (2.1) and (2.2), we know that $\epsilon(\alpha, r_1) = \epsilon(-\alpha, r_1)$, $\epsilon(-\alpha,\alpha) = -1$. Therefore

$$X_{-n}(-\alpha)(w\otimes e^{r_2}) = -\epsilon(\alpha, r_1)Y_{-n-(\alpha|r_1)-1}(-\alpha)w\otimes e^{r_1}.$$

Then by (3.22) we have

$$H(X_n(\alpha)(v\otimes e^{r_1}), w\otimes e^{r_2}) = \epsilon(\alpha, r_1)H(v\otimes e^{r_2}, Y_{-n-(\alpha|r_1)-1}(-\alpha)w\otimes e^{r_2})$$
$$= \epsilon(-\alpha, r_1)H(v\otimes e^{r_1}, Y_{-n-(\alpha|r_1)-1}(-\alpha)w\otimes e^{r_1})$$
$$= -H(v\otimes e^{r_1}, X_{-n}(-\alpha)w\otimes e^{r_2}).$$

For $\beta\in\Delta_S$, we can assume that $r_1+\beta = r_2$. Then

$$H(X_n(\beta)(v\otimes e^{r_1}), w\otimes e^{r_2}) = \epsilon(\beta, r_1)H(Y_{n+(\beta|r_1)+\frac{1}{2}}(\beta)v\otimes e^{r_2}, w\otimes e^{r_2})$$
$$= \epsilon(\beta, r_1)H(Y_{n+(\beta|r_1)+\frac{1}{2}}(\beta)v\otimes e^{r_1}, w\otimes e^{r_1}),$$
$$H(v\otimes e^{r_1}, X_{-n}(-\beta)(w\otimes e^{r_2})) = \epsilon(\beta, r_1+\beta)H(v\otimes e^{r_1}, Y_{-n-(\beta|r_1)-\frac{1}{2}}(-\beta)w\otimes e^{r_1}).$$

Now (3.28) follows from (3.23) and (2.1).

Corollary 3.1. *V_0 and $V_{\frac{1}{2}}$ are completely reducible $\hat{\mathfrak{g}}_{\rm vir}$-modules.*

Let $\mathfrak{h}_{\rm vir} = \mathfrak{h}\oplus\mathbf{C}c\oplus\mathbf{C}d_0\oplus\mathbf{C}K$, $\hat{\mathfrak{g}}^+_{\rm vir}$ and $\hat{\mathfrak{g}}^-_{\rm vir}$ be the linear spans of $\{e_\alpha, d_n \mid \alpha\in\Delta^+, n\in\mathbf{Z}^+\}$ and $\{e_\alpha, d_n \mid \alpha\in\Delta^-, n\in\mathbf{Z}^-\}$ respectively.

Definition 3.1. *A highest weight representation of $\hat{\mathfrak{g}}_{\rm vir}$ is a representation in a vector space M which admits a non-zero vector v, such that for given $\Lambda \in \mathfrak{h}_{\rm vir}^*$,*

$$h(v) = \Lambda(h)v, \qquad h \in \mathfrak{h}_{\rm vir},$$
$$\mathcal{U}(\hat{\mathfrak{g}}_{\rm vir}^+)v = 0, \qquad M = \mathcal{U}(\hat{\mathfrak{g}}_{\rm vir}^-)v,$$

where $\mathcal{U}(\hat{\mathfrak{g}}_{\rm vir}^+)$ and $\mathcal{U}(\hat{\mathfrak{g}}_{\rm vir}^-)$ are the universal evenloping algebras of $\hat{\mathfrak{g}}_{\rm vir}^+$ and $\hat{\mathfrak{g}}_{\rm vir}^-$ respectively.

Remark. By Theorem 3.2 and the fact that V is graded, we know that any highest weight submodule of V is irreducible. Conversely, since V is a graded and restricted module, any irreducible submodule of V is a highest weight module. Therefore, as $\hat{\mathfrak{g}}_{\rm vir}$-module, V is a direct sum of highest weight modules.

Theorem 3.3. *V_0 and $V_{\frac{1}{2}}$ are irreducible $\hat{\mathfrak{g}}_{\rm vir}$-modules.*

Proof. We only prove V_0 is irreducible, since the proof for $V_{\frac{1}{2}}$ is quite similar. Let $v \otimes e^r \in V_0$ be such that

$$h(v \otimes e^r) = \lambda(h)v \otimes e^r, \qquad h \in \mathfrak{h}_{\rm vir}, \tag{3.29}$$
$$\mathcal{U}(\hat{\mathfrak{g}}_{\rm vir}^+)(v \otimes e^r) = 0. \tag{3.30}$$

Then $\mathcal{U}(\hat{\mathfrak{g}}^+)(v \otimes e^r) = 0$. Therefore $r = 0$ (see [10]). Let S_k be such that

$$S(e_l, z) = F^-(e_l, z)F^+(e_l, z) = \sum_{k \in \frac{1}{2}\mathbf{Z}} S_k z^{-2k}.$$

Then similarly to the proof of (3.6), one can deduce that

$$[L_n, S_m] = \Big(-m - \frac{1}{2}n\Big)S_{m+n}, \qquad m, n \in \mathbf{Z}. \tag{3.31}$$

If $v \notin \mathbf{C}(1 \otimes e^0)$, then we can assume that

$$v = \sum_{i=1}^{q} a_{k_{i1},\cdots,k_{is_i}} S_{-k_{i1}} S_{-k_{i2}} \cdots S_{-k_{is_i}},$$

where $a_{k_{i1},\cdots,k_{is_i}} \in \mathbf{C}^* = \mathbf{C}\backslash\{0\}, k_{i1} + k_{i2} + \cdots + k_{is_i} = m$ $(1 \le i \le q), k_{i1} > k_{i2} > \cdots > k_{is_i} \ge 1$, and $(k_{i1}, k_{i2}, \cdots, k_{is_i}) > (k_{i+1,1}, k_{i+1,2}, \cdots, k_{i+1,s_{i+1}})$. We say $(k_{i1}, k_{i2}, \cdots, k_{is_i}) > (k_{j1}, k_{j2}, \cdots, k_{js_j})$, if there exists $p \ge 1$ such that $k_{i1} = k_{j1}, \cdots, k_{i,p-1} = k_{j,p-1}, k_{ip} > k_{jp}$. By (3.31), $q \ge 2$, since $L_1 v = 0$. Let $p \ge 1$ be such that $k_{1j} = k_{2j}, 1 \le j \le p-1, k_{1p} > k_{2p}$. Then by the fact that $L_1 v = 0$ and the use of (3.31), we have $p = s_1, k_{2p} = k_{1p} - 1$ and $k_{2,p+1} = 1$. Therefore $k_{1s_1} \ge 3$ and

$$\Big(k_{1s_1} - \frac{1}{2}\Big)a_{k_{11},\cdots,k_{1s_1}} + \frac{1}{2}a_{k_{21},\cdots,k_{2s_2}} = 0. \tag{3.32}$$

Note that $L_2 v = 0$. If $k_{1s_1} > 3$, then

$$(k_{1s_1} - 1)a_{k_{11},\cdots,k_{1s_1}} S_{-k_{11}} S_{-k_{12}} \cdots S_{-k_{1,s_1-1}} S_{-k_{1s_1}+2} = 0.$$

This means that $a_{k_{11},\cdots,k_{1s_1}} = 0$, which contradicts the assumption. Therefore $k_{1s_1} = 3$ and

$$2a_{k_{11},\cdots,k_{1s_1}} - a_{k_{21},\cdots,k_{2s_2}} = 0. \tag{3.33}$$

From (3.32) and (3.33), we have

$$a_{k_{11},\cdots,k_{1s_1}} = a_{k_{21},\cdots,k_{2s_2}} = 0,$$

which is also in contradiction with the assumption. Therefore $v \in \mathbf{C}(1 \otimes e^0)$. This proves that V_0 is irreducible.

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