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LOCAL SPECTRA OF UNILATERAL OPERATOR WEIGHTED SHIFTS**

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Abstract

In this note, the local spectral properties of unilateral operator weighted shifts are studied.

Keywords Operator weighted shifts, Local spectrum, Single-valued extension property, Dunford's condition (C), Bishop's property (β)
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§ 1. Introduction

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$, and let $\mathcal{A} := (A_n)_{n\geq 0}$ be a sequence of uniformly bounded invertible operators of $\mathcal{L}(\mathcal{H})$. Let

$$\widehat{\mathcal{H}} := \sum_{n=0}^{+\infty} \oplus \mathcal{H}_n,$$

where $\mathcal{H}_n = \mathcal{H}$ for each $n \geq 0$. It is a Hilbert space when equipped with the inner product

$$\langle (x_n)_n, (y_n)_n \rangle_{\widehat{\mathcal{H}}} = \sum_{n=0}^{+\infty} \langle x_n, y_n \rangle_{\mathcal{H}}, \qquad (x_n)_n,\ (y_n)_n \in \widehat{\mathcal{H}}.$$

Therefore, the corresponding norm is given by

$$\|(x_n)_n\|_{\widehat{\mathcal{H}}} = \Big(\sum_{n=0}^{+\infty} \|x_n\|_{\mathcal{H}}^2 \Big)^{\frac{1}{2}}, \qquad (x_n)_n \in \widehat{\mathcal{H}}.$$

The unilateral operator weighted shift, S_u, with the weight sequence $\mathcal{A} = (A_n)_{n\geq 0}$ is the operator on $\widehat{\mathcal{H}}$ defined by

$$S_u(x_0, x_1, x_2, \cdots) = (0, A_0x_0, A_1x_1, A_2x_2, \cdots), \qquad (x_n)_n \in \widehat{\mathcal{H}}.$$

Operator weighted shifts were first introduced by A. Lambert [19], and have been studied by many authors (see for example, [4, 12–18]). In the case when $\dim \mathcal{H} = 1$, they are exactly the scalar weighted shifts which have been widely studied. An excellent survey

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of the investigation of the spectral theory of such operators was given by A. L. Shields [27]. Moreover, several known results for the scalar case have been generalized and extended to the setting of operator weighted shifts. However, the question of determining the local spectral properties for operator weighted shifts is natural and the investigation of these properties for scalar weighted shifts has been studied recently in [6] and [24]. The main goal of the present note is to study and examine whether or not the results obtained in [6, 24] remain valid for unilateral operator weighted shifts. We give necessary conditions for a unilateral operator weighted shift to satisfy Dunford's condition (C) or Bishop's property (β). Unlike the scalar weighted shift operators, we show that there are examples of unilateral operator weighted shifts possessing Bishop's property (β) with large approximate point spectrum and without fat local spectra.

For an operator $T \in \mathcal{L}(\mathcal{H})$, let, as usual, T^*, $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_p(T)$, and $r(T)$ denote the adjoint, the spectrum, the approximate point spectrum, the point spectrum, and the spectral radius of T, respectively. Let $m(T) := \inf\{\|Tx\| : \|x\| = 1\}$ stand for the lower bound of T. Just as the case of the spectral radius, it is shown in [22] that the sequence $\left(m(T^n)^{\frac{1}{n}}\right)_{n\geq 1}$ converges and its limit, denoted by $r_1(T)$, equals its supremum.

We need to review some notions and basic facts from local spectral theory; we refer the reader to the monographs [8] and [20] for details. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the single-valued extension property provided that for every open subset U of $\mathbb{C}$ there exists no nonzero analytic function $\phi : U \to \mathcal{H}$ such that $(T-\lambda)\phi(\lambda) = 0,\ \lambda \in U$. A local version of this property was first introduced by J. K. Finch [11] and has been recently studied and investigated in the local spectral theory and Fredolhm theory by several authors (see for instance, [1–3]). Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the single-valued extension property at a point $\lambda_0 \in \mathbb{C}$ if for every open disc U centered at λ_0, the only analytic solution of the equation $(T-\lambda)\phi(\lambda) = 0,\ \lambda \in U$, is the zero function $\phi \equiv 0$. The set of all $\lambda \in \mathbb{C}$ on which T fails to have the single-valued extension property, denoted by $\Re(T)$, is clearly an open subset of $\mathbb{C}$ contained in the interior of $\sigma_p(T)$. It is empty precisely when T has the single-valued extension property.

Let $T \in \mathcal{L}(\mathcal{H})$ be a given operator. The local resolvent set, $\rho_T(x)$, of T at point $x \in \mathcal{H}$ is defined to be the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $\phi : U \to \mathcal{H}$ which satisfies $(T-\lambda)\phi(\lambda) = x,\ \lambda \in U$. It is evidently an open subset of $\mathbb{C}$ which contains $\rho(T)$; therefore, the local spectrum, $\sigma_T(x) := \mathbb{C}\backslash\rho_T(x)$, of T at x is a closed subset of $\sigma(T)$. If $\sigma_T(x) = \sigma(T)$ for all nonzero $x \in \mathcal{H}$, the operator T is said to have fat local spectra. For a closed subset F of $\mathbb{C}$, let $\mathcal{H}_T(F) := \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ stand for the corresponding local spectral subspace. It is a T-hyperinvariant subspace, but generally not closed. If the local spectral subspaces, $\mathcal{H}_T(\cdot)$, are closed, the operator T is said to satisfy Dunford's condition (C). It is well known that Dunford's condition (C) implies the single-valued extension property and it is clear that Dunford's condition (C) follows from fat local spectra. The local spectral radius of T at a point $x \in \mathcal{H}$ is defined by $r_T(x) := \limsup\limits_{n\to+\infty} \|T^n x\|^{\frac{1}{n}}$. It should be noted that if T has the single-valued extension property, then for any $x \in \mathcal{H}$ there exists a unique $\mathcal{H}$-valued analytic function, $\widetilde{x}(\cdot)$, defined on $\rho_T(x)$ such that

$$(T-\lambda)\widetilde{x}(\lambda) = x, \qquad \lambda \in \rho_T(x).$$

This function is called the local resolvent function of T at x, and satisfies

$$\widetilde{x}(\lambda) = -\sum_{n\geq 0} \frac{T^n x}{\lambda^{n+1}} \qquad \text{for all } \ \lambda \in \mathbb{C},\ |\lambda| > r_T(x).$$

Therefore, if T has the single-valued extension property, then

$$r_T(x) = \max\{|\lambda| : \lambda \in \sigma_T(x)\} \qquad \text{for all } \ x \in \mathcal{H}.$$

For information on local spectral radii of Banach space operators, we refer the reader to [10, 31].

Throughout this note, let S_u be a unilateral operator weighted shift with weight sequence $\mathcal{A} := (A_n)_{n\geq 0}$, and let $(B_n)_{n\geq 0}$ be the sequence given by

$$B_n = \begin{cases} A_{n-1}A_{n-2}\cdots A_1A_0, & \text{if } n>0,\\ 1, & \text{if } n=0. \end{cases}$$

Define

$$r_2(S_u) := \frac{1}{\limsup\limits_{n\to+\infty} \|B_n{}^{-1}\|^{\frac{1}{n}}}, \qquad r_3(S_u) := \frac{1}{\liminf\limits_{n\to+\infty} \|B_n{}^{-1}\|^{\frac{1}{n}}},$$

$$R_2^+(S_u) := \sup_{x\in\mathcal{H},\ x\neq 0}\left\{\frac{1}{\limsup\limits_{n\to+\infty}\|B_n^*{}^{-1}x\|^{\frac{1}{n}}}\right\} = \sup_{x\in\mathcal{H},\ \|x\|=1}\left\{\frac{1}{\limsup\limits_{n\to+\infty}\|B_n^*{}^{-1}x\|^{\frac{1}{n}}}\right\},$$

$$R_2^-(S_u) := \inf_{x\in\mathcal{H},\ x\neq 0}\left\{\frac{1}{\limsup\limits_{n\to+\infty}\|B_n^*{}^{-1}x\|^{\frac{1}{n}}}\right\} = \inf_{x\in\mathcal{H},\ \|x\|=1}\left\{\frac{1}{\limsup\limits_{n\to+\infty}\|B_n^*{}^{-1}x\|^{\frac{1}{n}}}\right\},$$

$$R_3^+(S_u) := \sup_{x\in\mathcal{H},\ x\neq 0}\left\{\limsup_{n\to+\infty}\|B_nx\|^{\frac{1}{n}}\right\} = \sup_{x\in\mathcal{H},\ \|x\|=1}\left\{\limsup_{n\to+\infty}\|B_nx\|^{\frac{1}{n}}\right\},$$

$$R_3^-(S_u) := \inf_{x\in\mathcal{H},\ x\neq 0}\left\{\limsup_{n\to+\infty}\|B_nx\|^{\frac{1}{n}}\right\} = \inf_{x\in\mathcal{H},\ \|x\|=1}\left\{\limsup_{n\to+\infty}\|B_nx\|^{\frac{1}{n}}\right\}.$$

Note that

$$r_1(S_u) \leq r_2(S_u) \leq R_2^-(S_u) \leq R_2^+(S_u) \quad \text{and} \quad r_3(S_u) \leq R_3^-(S_u) \leq R_3^+(S_u) \leq r(S_u).$$

Note also that if S_u is a scalar weighted shift, then

$$r_1(S_u) \leq r_2(S_u) = R_2^-(S_u) = R_2^+(S_u) \leq r_3(S_u) = R_3^-(S_u) = R_3^+(S_u) \leq r(S_u).$$

Finally, we would like to record and without further mention some notations that we will use repeatedly throughout this note. Wherever it is more convenient, we will write $y = \sum\limits_{n\geq 0}\oplus y_n$ instead of $y = (y_n)_n \in \widehat{\mathcal{H}}$. Moreover, for every $x\in\mathcal{H}$, we write

$$x^{(n)} = (0, \cdots, 0, x, 0, \cdots), \qquad n \geq 0$$

for the element of $\widehat{\mathcal{H}}$ for which all the coordinates are zero except the nth coordinate which equals x. Note that

$$r_{S_u}(x^{(k)}) = \limsup_{n\to+\infty}\|B_{n+k}B_k^{-1}x\|^{\frac{1}{n}} \qquad \text{for all } x\in\mathcal{H} \text{ and all } k\geq 0. \tag{1.1}$$

§ 2. Preliminaries and Elementary Background

In this section, we assemble some known results whose proofs are straightforward and are therefore omitted. Variants of Proposition 2.1 can be found in [20] and Corollary 2.1(b), Proposition 2.2, and Proposition 2.3 have been appeared in [19].

Proposition 2.1. *Assume that $T \in \mathcal{L}(\mathcal{H})$ is an operator for which $\bigcap_{n\geq 0} T^n\mathcal{H} = \{0\}$. The following statements hold:*

(a) *$\{\lambda \in \mathbb{C} : |\lambda| \leq r_1(T)\} \subset \sigma_T(x)$ for every nonzero element $x \in \mathcal{H}$.*

(b) *$\sigma_p(T) \subset \{0\}$.*

(c) *Each $\sigma_T(x)$ is connected.*

(d) *$\sigma(T)$ is a connected set and satisfies $\{\lambda \in \mathbb{C} : |\lambda| \leq r_1(T)\} \subset \sigma(T)$. In particular, if $\sigma(T)$ is circularly symmetric about the origin, then $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq r(T)\}$.*

Evidently, the unilateral operator weighted shift S_u satisfies the condition that

$$\bigcap_{n\geq 0} S_u^n\widehat{\mathcal{H}} = \{0\},$$

and its spectrum is rotationally symmetric. Therefore, the next result is an immediate consequence of Proposition 2.1.

Corollary 2.1. *The following statements hold:*

(a) *For every nonzero element $x \in \widehat{\mathcal{H}}$, the local spectrum, $\sigma_{S_u}(x)$, of S_u at x is connected and satisfies $\{\lambda \in \mathbb{C} : |\lambda| \leq r_1(S_u)\} \subset \sigma_{S_u}(x)$.*

(b) *The spectrum of S_u is the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq r(S_u)\}$.*

Proposition 2.2. *For every $n \geq 1$, we have*

$$\|S_u^n\| = \sup_{k\geq 0} \|B_{n+k}B_k^{-1}\|, \qquad m(S_u^n) = \inf_{k\geq 0} \left\{\frac{1}{\|B_kB_{n+k}^{-1}\|}\right\}.$$

Thus

$$r(S_u) = \lim_{n\to+\infty} \left[\sup_{k\geq 0} \|B_{n+k}B_k^{-1}\|\right]^{\frac{1}{n}}, \quad r_1(S_u) = \lim_{n\to+\infty} \left[\inf_{k\geq 0} \left\{\frac{1}{\|B_kB_{n+k}^{-1}\|}\right\}\right]^{\frac{1}{n}}.$$

Proposition 2.3. *The adjoint of S_u is given by*

$$S_u^*x = (A_0^*x_1, A_1^*x_2, A_2^*x_3, \cdots), \qquad x = (x_0, x_1, \cdots) \in \widehat{\mathcal{H}}.$$

§ 3. Local Spectra of S_u

We begin this section with the following result that gives a necessary and sufficient condition for S_u^* to enjoy the single-valued extension property.

Lemma 3.1. *The following statements hold:*

(a) *$\sigma_p(S_u) = \emptyset$. In particular, S_u has always the single-valued extension property.*

(b) *$\{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| < R_2^+(S_u)\} \subset \sigma_p(S_u^*) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq R_2^+(S_u)\}$.*

(c) *$\Re(S_u^*) = \{\lambda \in \mathbb{C} : |\lambda| < R_2^+(S_u)\}$. In particular, S_u^* has the single-valued extension property if and only if $R_2^+(S_u) = 0$.*

Proof. (a) By Proposition 2.1(b), we have $\sigma_p(S_u) \subset \{0\}$. As S_u is injective, we note that $\sigma_p(S_u) = \emptyset$.

(b) Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue for S_u^* and that $(x_n)_n$ is a corresponding eigenvector. We have

$$(A_0^*x_1, A_1^*x_2, A_2^*x_3, \cdots) = (\lambda x_0, \lambda x_1, \lambda x_2, \cdots).$$

This shows that

$$x_n = \lambda^n B_n^{*-1} x_0, \qquad n \geq 0.$$

Therefore

$$\|x\|^2 = \sum_{n \geq 0} |\lambda|^{2n} \|B_n^{*-1} x_0\|^2.$$

By the Cauchy-Hadamard formula for the radius of convergence, we get that

$$|\lambda| \leq \frac{1}{\limsup\limits_{n \to +\infty} \|B_n^{*-1} x_0\|^{\frac{1}{n}}}.$$

Thus

$$\sigma_p(S_u^*) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq R_2^+(S_u)\}.$$

Now, let us prove that

$$\{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| < R_2^+(S_u)\} \subset \sigma_p(S_u^*).$$

It is clear that for every $x \in \mathcal{H}$, we have $S_u^* x^{(0)} = 0$; hence, $0 \in \sigma_p(S_u^*)$. If $R_2^+(S_u) = 0$, then there is nothing to prove; thus, we may assume that $R_2^+(S_u) > 0$. Let $\lambda \in \mathbb{C}$ such that $|\lambda| < R_2^+(S_u)$. So, there is a nonzero $x_0 \in \mathcal{H}$ such that $|\lambda| < \frac{1}{\limsup\limits_{n \to +\infty} \|B_n^{*-1} x_0\|^{\frac{1}{n}}}$. We have $(S_u^* - \lambda) k_{x_0}(\lambda) = 0$, where $k_{x_0}(\lambda) = \sum\limits_{n \geq 0} \oplus \lambda^n B_n^{*-1} x_0$. This shows that

$$\{\lambda \in \mathbb{C} : |\lambda| < R_2^+(S_u)\} \subset \sigma_p(S_u^*),$$

and the desired statement holds.

(c) In view of the statement (b) and the fact that $\Re(S_u^*) \subset \mathrm{int}(\sigma_p(S_u^*))$, we have $\Re(S_u^*) \subset \{\lambda \in \mathbb{C} : |\lambda| < R_2^+(S_u)\}$.

Conversely, let x be a nonzero element of $\mathcal{H}$ and set

$$U_x := \Big\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{\limsup\limits_{n \to +\infty} \|B_n^{*-1} x\|^{\frac{1}{n}}}\Big\}, \qquad k_x(\lambda) := \sum_{n \geq 0} \oplus \lambda^n B_n^{*-1} x, \quad \lambda \in U_x.$$

Since $(S_u^* - \lambda) k_x(\lambda) = 0$ for all $\lambda \in U_x$, and x is an arbitrary nonzero element of $\mathcal{H}$, we have

$$\{\lambda \in \mathbb{C} : |\lambda| < R_2^+(S_u)\} = \bigcup_{x \in \mathcal{H},\ x \neq 0} U_x \subset \Re(S_u^*).$$

The proof is therefore complete.

The following result refines the local spectral inclusion given in Corollary 2.1.

Proposition 3.1. *For every nonzero $y = (y_0, y_1, y_2, \cdots) \in \widehat{\mathcal{H}}$, we have*

$$\{\lambda \in \mathbb{C} : |\lambda| \leq R_2^-(S_u)\} \subset \sigma_{S_u}(y).$$

In particular, if $r(S_u) = R_2^-(S_u)$, then S_u has fat local spectra.

Proof. As $\bigcap\limits_{n \geq 0} S_u^n \widehat{\mathcal{H}} = \{0\}$, we have $0 \in \sigma_{S_u}(y)$. Thus, we may assume that $R_2^-(S_u) > 0$. Let $O := \{\lambda \in \mathbb{C} : |\lambda| < R_2^-(S_u)\}$, and let x be a nonzero element of $\mathcal{H}$. Consider the following analytic $\widehat{\mathcal{H}}$-valued function defined on O by

$$k_x(\lambda) := \sum_{n \geq 0} \oplus \lambda^n B_n^{*-1} x.$$

We have $(S_u-\lambda)^*k_x(\overline{\lambda})=0$ for every $\lambda\in O$. Now, let $y=(y_0,y_1,y_2,\cdots)\in\widehat{\mathcal{H}}$ such that $O\cap\rho_{S_u}(y)\neq\emptyset$. So, for every $\lambda\in O\cap\rho_{S_u}(y)$, we have

$$\begin{aligned}\sum_{n\geq 0}\langle y_n, B_n^{*-1}x\rangle_{\mathcal{H}}\lambda^n &= \langle y, k_x(\overline{\lambda})\rangle_{\widehat{\mathcal{H}}} = \langle (S_u-\lambda)\widetilde{y}(\lambda), k_x(\overline{\lambda})\rangle_{\widehat{\mathcal{H}}}\\ &= \langle \widetilde{y}(\lambda), (S_u-\lambda)^*k_x(\overline{\lambda})\rangle_{\widehat{\mathcal{H}}} = 0.\end{aligned}$$

Hence, for every $n\geq 0$, we have

$$\langle y_n, B_n^{*-1}x\rangle_{\mathcal{H}} = 0.$$

Since x is an arbitrary element of $\mathcal{H}$, we have $y=0$; and the proof is therefore complete.

In view of Proposition 3.1, we note that $R_2^-(S_u)\leq r_{S_u}(x)$ for all nonzero $x=(x_0,x_1,x_2,\cdots)\in\widehat{\mathcal{H}}$. The following gives more information about local spectral radii of S_u.

Proposition 3.2. *For every nonzero element $x=(x_0,x_1,\cdots)\in\widehat{\mathcal{H}}$, we have*

$$R_3^-(S_u)\leq r_{S_u}(x)\leq r(S_u).$$

Moreover, if $x=(x_0,x_1,\cdots)$ is a nonzero finitely supported element of $\widehat{\mathcal{H}}$, then

$$R_3^-(S_u)\leq r_{S_u}(x)=\max_{k\geq 0}(r_{S_u}(x_k^{(k)}))\leq R_3^+(S_u). \tag{3.1}$$

Proof. Let $x=(x_0,x_1,\cdots)$ be a nonzero element of $\widehat{\mathcal{H}}$; so, there is an integer $k_0\geq 0$ such that $x_{k_0}\neq 0$. Since

$$\|S_u^n x\|^2=\sum_{k=0}^{+\infty}\|B_{n+k}B_k^{-1}x_k\|^2, \qquad \forall\, n\geq 0,$$

we have

$$\|B_{n+k_0}B_{k_0}^{-1}x_{k_0}\|^{\frac{1}{n+k_0}}\leq \|S_u^n x\|^{\frac{1}{n+k_0}}, \qquad \forall\, n\geq 0.$$

Now, taking lim sup as $n\to+\infty$, we get

$$R_3^-(S_u)\leq \limsup_{n\to+\infty}\|B_{n+k_0}B_{k_0}^{-1}x_{k_0}\|^{\frac{1}{n+k_0}}\leq r_{S_u}(x),$$

as desired.

Now, assume that $x=(x_0,x_1,\cdots)$ is a nonzero finitely supported element of $\widehat{\mathcal{H}}$. As above, we have

$$\|B_{n+k}B_k^{-1}x_k\|^{\frac{1}{n}}\leq \|S_u^n x\|^{\frac{1}{n}}, \qquad \forall\, n,\ k\geq 0.$$

By taking lim sup as $n\to+\infty$, we get $r_{S_u}(x_k^{(k)})\leq r_{S_u}(x),\ \forall\, k\geq 0$. Hence

$$\max_{k\geq 0}(r_{S_u}(x_k^{(k)}))\leq r_{S_u}(x).$$

As $\sigma_{S_u}(x)\subset\bigcup_{k\geq 0}\sigma_{S_u}(x_k^{(k)})$, and $r_{S_u}(y)=\max\{|\lambda|:\lambda\in\sigma_{S_u}(y)\}$ for every nonzero $y\in\widehat{\mathcal{H}}$, we obtain $r_{S_u}(x)\leq\max_{k\geq 0}(r_{S_u}(x_k^{(k)}))$. Hence $r_{S_u}(x)=\max_{k\geq 0}(r_{S_u}(x_k^{(k)}))$. On the other hand, we have $r_{S_u}(x_k^{(k)})=r_{S_u}((B_k^{-1}x_k)^{(0)}),\ \forall\, k\geq 0$. This shows that

$$r_{S_u}(x)=\max_{k\geq 0}(r_{S_u}(x_k^{(k)}))\leq R_3^+(S_u).$$

Therefore, the desired result holds.

For every $x = (x_0, x_1, \cdots) \in \widehat{\mathcal{H}}$, we set

$$R_{\mathcal{A}}(x) := \frac{1}{\limsup_{n\to+\infty} \|B_n^{-1}x_n\|^{\frac{1}{n}}}.$$

Obviously, if x is a nonzero element of $\widehat{\mathcal{H}}$, then $r_2(S_u) \le R_{\mathcal{A}}(x) \le +\infty$.

Theorem 3.1. *For every nonzero element $x = (x_0, x_1, \cdots) \in \widehat{\mathcal{H}}$, we have*

$$\{\lambda \in \mathbb{C} : |\lambda| \le \min(R_{\mathcal{A}}(x), r_3(S_u))\} \subset \sigma_{S_u}(x).$$

Moreover, if $x = (x_0, x_1, \cdots)$ is a nonzero finitely supported element of $\widehat{\mathcal{H}}$, then

$$\{\lambda \in \mathbb{C} : |\lambda| \le R_3^-(S_u)\} \subset \sigma_{S_u}(x).$$

Proof. Let $x = (x_0, x_1, \cdots)$ be a nonzero element of $\widehat{\mathcal{H}}$. If $\min(R_{\mathcal{A}}(x), r_3(S_u)) = 0$, then there is nothing to prove since $0 \in \sigma_{S_u}(x)$. Thus we may suppose that $\min(R_{\mathcal{A}}(x), r_3(S_u)) > 0$. Now, for each $n \ge 0$, let

$$\begin{aligned} F_n(\lambda) &= -\frac{B_n x_0}{\lambda^{n+1}} - \frac{B_n B_1^{-1} x_1}{\lambda^n} - \frac{B_n B_2^{-1} x_2}{\lambda^{n-1}} - \cdots - \frac{x_n}{\lambda}, \qquad && \lambda \in \mathbb{C}\backslash\{0\}, \\ G_n(\lambda) &= x_0 + \lambda B_1^{-1} x_1 + \lambda^2 B_2^{-1} x_2 + \cdots + \lambda^n B_n^{-1} x_n, && \lambda \in \mathbb{C}. \end{aligned}$$

We have

$$F_n(\lambda) = \frac{-1}{\lambda^{n+1}} B_n G_n(\lambda), \qquad \lambda \in \mathbb{C}\backslash\{0\}. \tag{3.2}$$

By writing $\widetilde{x}(\lambda) := (f_0(\lambda), f_1(\lambda), f_2(\lambda), \cdots)$, $\lambda \in \rho_{S_u}(x)$, we get from the equation

$$(S_u - \lambda)\widetilde{x}(\lambda) = x, \qquad \lambda \in \rho_{S_u}(x)$$

that

$$\begin{cases} -\lambda f_0(\lambda) = x_0, \\ A_n f_n(\lambda) - \lambda f_{n+1}(\lambda) = x_{n+1} \qquad \text{for every } \ n \ge 0 \end{cases}$$

for all $\lambda \in \rho_{S_u}(x)$. Therefore, for every $n \ge 0$ and for every $\lambda \in \rho_{S_u}(x)$, we have

$$f_n(\lambda) = -\frac{B_n x_0}{\lambda^{n+1}} - \frac{B_n B_1^{-1} x_1}{\lambda^n} - \frac{B_n B_2^{-1} x_2}{\lambda^{n-1}} - \cdots - \frac{x_n}{\lambda} = F_n(\lambda).$$

Since $\|\widetilde{x}(\lambda)\|^2 = \sum_{n\ge 0} \|f_n(\lambda)\|^2 < +\infty$ for every $\lambda \in \rho_{S_u}(x)$, it then follows that

$$\lim_{n\to+\infty} F_n(\lambda) = \lim_{n\to+\infty} f_n(\lambda) = 0 \qquad \text{for every } \ \lambda \in \rho_{S_u}(x). \tag{3.3}$$

We shall show that (3.3) is not satisfied for most of the points in the open disc $V(x) := \{\lambda \in \mathbb{C} : |\lambda| < \min(R_{\mathcal{A}}(x), r_3(S_u))\}$. It is clear that the sequence $(G_n)_{n\ge 0}$ converges uniformly on compact subsets of $V(x)$ to the nonzero power series $G(\lambda) = \sum_{n\ge 0} \lambda^n B_n^{-1} x_n$. Now, let $\lambda_0 \in V(x)\backslash\{0\}$ such that $G(\lambda_0) \ne 0$; there is $\epsilon > 0$ and an integer n_0 such that $\epsilon < \|G_n(\lambda_0)\|$ for every $n \ge n_0$. On the other hand, $|\lambda_0| < r_3(S_u)$, then there is a subsequence $(n_k)_{k\ge 0}$

of integers greater than n_0 such that $|\lambda_0|^{n_k}\|B_{n_k}^{-1}\| < 1$. Thus, it follows from (3.2) that for every $k \geq 0$, we have

$$\|F_{n_k}(\lambda_0)\| = \Big|\frac{-1}{\lambda_0^{n_k+1}}\Big|\|B_{n_k}G_{n_k}(\lambda_0)\| \geq \frac{1}{|\lambda_0^{n_k+1}|\|B_{n_k}^{-1}\|}\|G_{n_k}(\lambda_0)\| \geq \frac{\epsilon}{|\lambda_0|}.$$

And so, by (3.3), $\lambda_0 \notin \rho_{S_u}(x)$. Since the set of zeros of G is at most countable, we have $\{\lambda \in \mathbb{C} : |\lambda| \leq \min(R_{\mathcal{A}}(x), r_3(S_u))\} \subset \sigma_{S_u}(x)$.

Now, assume that $x = (x_0, x_1, \cdots)$ is a nonzero finitely supported element of $\widehat{\mathcal{H}}$, and k_0 is the largest integer $n \geq 0$ for which $x_n \neq 0$. Conserve the same notations as above and note that, for every $n \geq k_0$, we have

$$F_n(\lambda) = \frac{-1}{\lambda^{n+1}}B_nG(\lambda), \qquad \lambda \in \mathbb{C}\backslash\{0\},$$

where

$$G(\lambda) := x_0 + \lambda B_1^{-1}x_1 + \lambda^2 B_2^{-1}x_2 + \cdots + \lambda^{k_0}B_{k_0}^{-1}x_{k_0}, \qquad \lambda \in \mathbb{C}.$$

Let $W(x) := \{\lambda \in \mathbb{C} : |\lambda| < R_3^-(S_u)\}$, and let $\lambda_0 \in W(x)\backslash\{0\}$ such that $G(\lambda_0) \neq 0$. As $|\lambda_0| < R_3^-(S_u) \leq \limsup\limits_{n\to+\infty}\|B_nG(\lambda_0)\|^{\frac{1}{n}}$, we note that the series $\sum\limits_{n\geq 0}\|F_n(\lambda_0)\|^2$ diverges. Hence, $\lambda_0 \in \sigma_{S_u}(x)$, and therefore $\{\lambda \in \mathbb{C} : |\lambda| \leq R_3^-(S_u)\} \subset \sigma_{S_u}(x)$.

For every $x \in \mathcal{H}$, we write

$$\widehat{\mathcal{H}}(x) := \bigvee\{(B_nx)^{(n)} : n \geq 0\},$$

where "$\bigvee$" denotes the closed linear span. It is shown in Proposition 4.3.5 of [30] that for every nonzero $x \in \mathcal{H}$, we have

$$\sigma_{S_u}(x^{(n)}) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{S_u}(x^{(n)})\}, \qquad n \geq 0.$$

We refine this result as follows; our proof is inspired by an argument of [6].

Proposition 3.3. *Let x be a nonzero element of $\mathcal{H}$, and let $y \in \widehat{\mathcal{H}}(x)$. The following statements hold:*

(a) *If $R_{\mathcal{A}}(y) > r_{S_u}(x^{(0)})$, then $\sigma_{S_u}(y) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{S_u}(x^{(0)})\}$.*

(b) *If $R_{\mathcal{A}}(y) \leq r_{S_u}(x^{(0)})$, then $\{\lambda \in \mathbb{C} : |\lambda| \leq R_{\mathcal{A}}(y)\} \subset \sigma_{S_u}(y)$.*

Proof. Let x be a nonzero element of $\mathcal{H}$, and let us first show that

$$\sigma_{S_u}(x^{(0)}) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{S_u}(x^{(0)})\}.$$

To do this it suffices to prove that $\{\lambda \in \mathbb{C} : |\lambda| \leq r_{S_u}(x^{(0)})\} \subset \sigma_{S_u}(x^{(0)})$. Since $0 \in \sigma_{S_u}(x)$, we may and shall assume that $r_{S_u}(x^{(0)}) > 0$. As in the proof of Theorem 3.1, we trivially have

$$\widetilde{x^{(0)}}(\lambda) = \Big(-\frac{x}{\lambda}, -\frac{B_1x}{\lambda^2}, -\frac{B_2x}{\lambda^3}, \cdots\Big), \qquad \lambda \in \rho_{S_u}(x^{(0)}).$$

In particular, we have $\|\widetilde{x^{(0)}}(\lambda)\|_{\widehat{\mathcal{H}}}^2 = \sum\limits_{k=0}^{+\infty}\frac{\|B_kx\|_{\mathcal{H}}^2}{|\lambda|^{2(k+1)}}$, $\lambda \in \rho_{S_u}(x^{(0)})$. This implies that $\rho_{S_u}(x^{(0)}) \subset \{\lambda \in \mathbb{C} : r_{S_u}(x^{(0)}) \leq |\lambda|\}$. Or, equivalently

$$\{\lambda \in \mathbb{C} : |\lambda| < r_{S_u}(x^{(0)})\} \subset \sigma_{S_u}(x^{(0)}).$$

As $\sigma_{S_u}(x^{(0)})$ is a closed set and $r_{S_u}(x^{(0)}) > 0$, the desired identity holds.

(a) Assume that $y = \sum_{n=0}^{+\infty} a_n \big(B_n x\big)^{(n)}$ is a nonzero element of $\widehat{\mathcal{H}}(x)$ for which $R_{\mathcal{A}}(y) > r_{S_u}(x^{(0)})$. In this case the function $f(\lambda) := \sum_{n\geq 0} a_n \lambda^n$ is analytic on the open disc $\{\lambda \in \mathbb{C} : |\lambda| < R_{\mathcal{A}}(y)\}$ which is a neighborhood of $\sigma_{S_u}(x^{(0)})$. Let r be a real number such that $r_{S_u}(x^{(0)}) < r < R_{\mathcal{A}}(y)$, we have

$$f(S_u, x^{(0)}) := \frac{-1}{2\pi i}\oint_{|\lambda|=r} f(\lambda)\widetilde{x^{(0)}}(\lambda)d\lambda = \frac{-1}{2\pi i}\oint_{|\lambda|=r} f(\lambda)\Big(-\sum_{n\geq 0}\frac{S_u^n x^{(0)}}{\lambda^{n+1}}\Big)d\lambda = y.$$

And so, by Theorem 2.12 of [29], we have

$$\sigma_{S_u}(y) = \sigma_{S_u}(f(S_u, x^{(0)})) = \sigma_{S_u}(x^{(0)}) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{S_u}(x^{(0)})\}.$$

(b) The proof of the second statement is similar to the one of Theorem 3.1 if, for every integer $n \geq 0$, we take

$$\begin{aligned} F_n(\lambda) &:= -\big(\frac{a_n}{\lambda^{n+1}} + \frac{a_1}{\lambda^n} + \frac{a_2}{\lambda^{n-1}} + \cdots + \frac{a_n}{\lambda}\big)B_n x, && \lambda \in \mathbb{C}\backslash\{0\},\\ G_n(\lambda) &:= a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n, && \lambda \in \mathbb{C}. \end{aligned}$$

§ 4. Dunford's Condition (C) and Bishop's Property (β) for S_u

Before outlining the statement of the main results of this section, let us recall a few more notions and properties from the local spectral theory which will be needed in the sequel. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be hyponormal if $\|T^*x\| \leq \|Tx\|$ for all $x \in \mathcal{H}$. It is said be subnormal if it has a normal extension which means that there is a normal operator N on a Hilbert space $\mathcal{K}$, containing $\mathcal{H}$, such that $\mathcal{H}$ is a closed invariant subspace of N and the restriction $N_{|\mathcal{H}}$ coincides with T. Note that every subnormal operator is hyponormal, but the converse is false (see [9]). For an open subset U of $\mathbb{C}$, let $\mathcal{O}(U, \mathcal{H})$ denote, as usual, the Fréchet space of all analytic $\mathcal{H}$-valued functions on U. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to possess Bishop's property (β) if the continuous mapping

$$\begin{aligned} T_U \quad : \quad \mathcal{O}(U, \mathcal{H}) &\longrightarrow \mathcal{O}(U, \mathcal{H})\\ f &\longmapsto (T - z)f \end{aligned}$$

is injective with closed range for each open subset U of $\mathbb{C}$. It is known that hyponormal operators possess Bishop's property (β) and it turns out that Dunford's condition (C) follows from Bishop's property (β) (see [20, 25]). Let $\lambda_0 \in \mathbb{C}$; recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to possess Bishop's property (β) at λ_0 if there is an open neighbourhood V of λ_0 such that for every open subset U of V, the mapping T_U is injective and has a closed range. Note that if T possesses Bishop's property (β) at any point $\lambda \in \mathbb{C}$, then T possesses Bishop's classical property (β). Finally, for any operator $T \in \mathcal{L}(\mathcal{H})$, we shall denote

$$\sigma_\beta(T) := \big\{\lambda \in \mathbb{C} : T \text{ fails to possess Bishop's property } (\beta) \text{ at } \lambda\big\}.$$

It is a closed subset of $\sigma_{ap}(T)$ (see for instance [6, Proposition 2.1]).

The following result gives necessary conditions for the operator weighted shift, S_u, to enjoy Dunford's condition (C).

Theorem 4.1. *If S_u satisfies Dunford's condition* (C)*, then $r(S_u) = R_3^+(S_u)$. Moreover, for every nonzero $x \in \mathcal{H}$, we have*

$$\limsup_{n\to+\infty} \|B_n x\|^{\frac{1}{n}} = \lim_{n\to+\infty} \Big[\sup_{k\geq 0} \frac{\|B_{n+k}x\|}{\|B_k x\|}\Big]^{\frac{1}{n}}. \tag{4.1}$$

Proof. To prove $R_3^+(S_u) = r(S_u)$, it suffices to show that $r(S_u) \leq R_3^+(S_u)$. Since each B_k is an invertible operator, we note that

$$R_3^+(S_u) = \sup_{x\in\mathcal{H},\ x\neq 0} (r_{S_u}(x^{(k)})), \qquad \forall\, k \geq 0.$$

Now, assume that S_u satisfies Dunford's condition (C), and let

$$F := \{\lambda \in \mathbb{C} : |\lambda| \leq R_3^+(S_u)\}.$$

It follows from (3.1) that $\widehat{\mathcal{H}}_{S_u}(F)$ contains a dense subspace of $\widehat{\mathcal{H}}$. As the subspace $\widehat{\mathcal{H}}_{S_u}(F)$ is closed, we have $\widehat{\mathcal{H}}_{S_u}(F) = \widehat{\mathcal{H}}$; therefore, $\sigma_{S_u}(y) \subset F$ for every $y \in \widehat{\mathcal{H}}$. And so, $\sigma(S_u) = \bigcup_{y\in\widehat{\mathcal{H}}} \sigma_{S_u}(y) \subset F$ (see [20, Proposition 1.3.2]). Hence, $r(S_u) \leq R_3^+(S_u)$, as desired.

Let x be a nonzero element of $\mathcal{H}$ and let us now establish the identity (4.1). Since S_u satisfies Dunford's condition (C), we note that S_u restricted to $\widehat{\mathcal{H}}(x)$ also satisfies Dunford's condition (C) (see [20, Proposition 1.2.21]). Now, note that $(v_n)_{n\geq 0}$ is an orthonormal basis of $\widehat{\mathcal{H}}(x)$, where

$$v_n := \frac{(B_n x)^{(n)}}{\|B_n x\|}, \qquad n \geq 0.$$

We have

$$S_u v_n = \frac{\|B_{n+1}x\|}{\|B_n x\|} v_{n+1}, \qquad n \geq 0.$$

This shows that $S_{u|\widehat{\mathcal{H}}(x)}$ is an injective scalar unilateral weighted shift with weight sequence $\Big(\frac{\|B_{n+1}x\|}{\|B_n x\|}\Big)_{n\geq 0}$. Therefore, the identity (4.1), follows from Theorem 3.8 of [6].

Unlike the scalar weighted shift operators, generally we do not have $r_1(S_u) = r(S_u)$ if the unilateral operator weighted shift S_u possesses Bishop's property (β) (see Example 4.2). But, of course, if $r_1(S_u) = r(S_u)$, then either S_u possesses Bishop's property (β), or $\sigma_\beta(S_u) = \{\lambda \in \mathbb{C} : |\lambda| = r(S_u)\}$. In [30], H. Zguitti represented, just as in [16], a unilateral operator weighted shift as operator multiplication by z on a Hilbert space of formal power series whose coefficients are in $\mathcal{H}$. He therefore adapted T. L. Miller and V. G. Miller's arguments given in [23] to show that if S_u possesses Bishop's property (β), then $r_2(S_u) = R_1(S_u)$, where $R_1(S_u) = \liminf_{n\to+\infty} \Big[\inf_{k\geq 0} \|B_{n+k}B_k^{-1}\|\Big]^{\frac{1}{n}}$. Here, we refine this result as follows and provide a direct proof.

Theorem 4.2. *If S_u possesses Bishop's property (β), then $r_2(S_u) = r_1(S_u)$, and $r(S_u) = R_3^+(S_u)$. Moreover, for every nonzero $x \in \mathcal{H}$, we have*

$$\lim_{n\to+\infty} \Big[\inf_{k\geq 0} \frac{\|B_{n+k}x\|}{\|B_k x\|}\Big]^{\frac{1}{n}} = \lim_{n\to+\infty} \Big[\sup_{k\geq 0} \frac{\|B_{n+k}x\|}{\|B_k x\|}\Big]^{\frac{1}{n}}. \tag{4.2}$$

Proof. Suppose that S_u possesses Bishop's property (β) and note that, since S_u satisfies Dunford's condition (C), $r(S_u) = R_3^+(S_u)$ (see Theorem 4.1). If $r_2(S_u) = 0$, then, since $r_1(S_u) \leq r_2(S_u)$, there is nothing to prove. Thus, we may assume that $0 < r_2(S_u)$. Now, recall that it is shown in [22] that

$$r_1(T) = \min\{|\lambda| : \lambda \in \sigma_{ap}(T)\}$$

for any operator $T \in \mathcal{L}(\mathcal{H})$. And so, in order to show that $r_2(S_u) = r_1(S_u)$, it suffices to prove that $\mathcal{U} \cap \sigma_{ap}(S_u) = \emptyset$, where $\mathcal{U} := \{\lambda \in \mathbb{C} : |\lambda| < r_2(S_u)\}$. Assume for the sake of contradiction that there is $\lambda_0 \in \overline{\mathcal{U}} \cap \sigma_{ap}(S_u)$. Since $\sigma_p(S_u) = \emptyset$, there is $y = (y_0, y_1, y_2, \cdots) \in \mathrm{cl}(\mathrm{ran}(S_u - \lambda_0)) \backslash \mathrm{ran}(S_u - \lambda_0)$. For every $x \in \mathcal{H}$, set $k_x(\lambda) := \sum_{i \geq 0} \oplus \overline{\lambda}^i B_i^{*-1} x,\ \lambda \in \mathcal{U}$, and note that

$$(S_u - \lambda)^* k_x(\lambda) = 0, \qquad \forall\, \lambda \in \mathcal{U}.$$

In particular, we have

$$\langle y, k_x(\lambda_0)\rangle_{\widehat{\mathcal{H}}} = 0 \qquad \text{for all } \ x \in \mathcal{H}. \tag{4.3}$$

And so, for every $x \in \mathcal{H}$, we have

$$\Big\langle \sum_{i \geq 0} \lambda_0^i B_i^{-1} y_i, x \Big\rangle_{\mathcal{H}} = \sum_{i \geq 0} \langle y_i, \overline{\lambda}_0^i B_i^{*-1} x\rangle_{\mathcal{H}} = \langle y, k_x(\lambda_0)\rangle_{\widehat{\mathcal{H}}} = 0.$$

This implies that

$$\sum_{i \geq 0} \lambda_0^i B_i^{-1} y_i = 0. \tag{4.4}$$

Now, for every integer $n \geq 0$, we define on $\mathcal{U}$ the following analytic $\widehat{\mathcal{H}}$-valued functions by

$$f(\lambda) := y - \Big(\sum_{i \geq 0} \lambda^i B_i^{-1} y_i\Big)^{(0)} \quad \text{and} \quad f_n(\lambda) := y^n - \Big(\sum_{i=0}^{n} \lambda^i B_i^{-1} y_i\Big)^{(0)},$$

where $y^n := (y_0, \cdots, y_n, 0, 0, \cdots)$. Noting that for every integer $n \geq 0$, we have

$$f_n(\lambda) = \sum_{i=0}^{n} (S_u^i - \lambda^i)(B_i^{-1} y_i)^{(0)}, \qquad \lambda \in \mathcal{U}.$$

This implies that each f_n is in $\mathrm{ran}((S_u)_{\mathcal{U}})$. But $f \notin \mathrm{ran}((S_u)_{\mathcal{U}})$ since, in view of (4.4), we have $f(\lambda_0) = y \notin \mathrm{ran}(S_u - \lambda_0)$. On the other hand, for every compact subset K of $\mathcal{U}$, we have

$$\begin{aligned}
\sup_{\lambda \in K} \|f_n(\lambda) - f(\lambda)\|_{\widehat{\mathcal{H}}} &\leq \|y - y^n\|_{\widehat{\mathcal{H}}} + \sup_{\lambda \in K} \Big\| \Big(\sum_{i > n} \lambda^i B_i^{-1} y_i\Big)^{(0)} \Big\|_{\widehat{\mathcal{H}}} \\
&= \|y - y^n\|_{\widehat{\mathcal{H}}} + \sup_{\lambda \in K} \Big\| \sum_{i > n} \lambda^i B_i^{-1} y_i \Big\|_{\mathcal{H}} \\
&\leq \|y - y^n\|_{\widehat{\mathcal{H}}} + \sup_{\lambda \in K} \Big\{ \sum_{i > n} |\lambda|^i \|B_i^{-1}\| \|y_i\|_{\mathcal{H}} \Big\} \\
&\leq \Big(1 + \sup_{\lambda \in K} \Big(\sum_{i \geq 0} |\lambda|^{2i} \|B_i^{-1}\|^2\Big)^{\frac{1}{2}}\Big) \|y - y^n\|_{\widehat{\mathcal{H}}}.
\end{aligned}$$

Therefore, $f_n \to f$ in $\mathcal{O}(\mathcal{U}, \widehat{\mathcal{H}})$. As each $f_n \in \mathrm{ran}((S_u)_{\mathcal{U}})$ and $f \notin \mathrm{ran}((S_u)_{\mathcal{U}})$, we note that $\mathrm{ran}((S_u)_{\mathcal{U}})$ is not closed. We have a contradiction to the fact that S_u possesses Bishop's property (β). And so, $\overline{\mathcal{U}} \cap \sigma_{ap}(S_u) = \emptyset$, as desired.

Now, let x be a nonzero element of $\mathcal{H}$. Since S_u possesses Bishop's property (β), the injective scalar unilateral weighted shift $S_{u|\widehat{\mathcal{H}}(x)}$ possesses also Bishop's property (β). Thus, applying Theorem 3.9 of [6] gives the identity (4.2).

Remark 4.1. Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator, and assume that $A_n = T$ for all $n \geq 0$. The corresponding unilateral operator weighted shift, S_u, satisfies the following identities

$$r(S_u) = R_3^+(S_u) = r(T),$$
$$r_1(S_u) = r_2(S_u) = R_2^-(S_u) = r_1(T) = \frac{1}{r(T^{-1})}.$$

Indeed, we clearly have $r(S_u) = r(T)$ and $r_1(S_u) = r_2(S_u) = r_1(T) = \frac{1}{r(T^{-1})}$. Since, $R_3^+(S_u) = \sup\{r_T(x) : x \in \mathcal{H},\ x \neq 0\}$, it follows from Proposition 3.3.14 of [20] that $R_3^+(S_u) = r(T)$; therefore, the first identity holds. On the other hand, we have

$$R_2^-(S_u) = \inf\left\{\frac{1}{r_{T^{*-1}}(x)} : x \in \mathcal{H},\ x \neq 0\right\} = \frac{1}{\sup\{r_{T^{*-1}}(x) : x \in \mathcal{H},\ x \neq 0\}}.$$

Again, by Proposition 3.3.14 of [20], we have $R_2^-(S_u) = \frac{1}{r(T^{-1})}$; and the second identity follows.

Assume that $T \in \mathcal{L}(\mathcal{H})$ is an invertible operator and that $A_n = T$ for all $n \geq 0$. So, one may think that the corresponding unilateral operator weighted shift, S_u, satisfies Dunford's condition (C). It turns out that this is not true in general as the next example shows.

Example 4.1. Let $(e_n)_{n\in\mathbb{Z}}$ be an orthonormal basis of $\mathcal{H}$, and let $(\omega_n)_{n\in\mathbb{Z}}$ be a positive two-sided sequence for which

(a) $0 < \inf\limits_{n\in\mathbb{Z}} \omega_n \leq \sup\limits_{n\in\mathbb{Z}} \omega_n < +\infty$.

(b) $\limsup\limits_{n\to+\infty}[\omega_0\omega_1\cdots\omega_{n-1}]^{\frac{1}{n}} < \lim\limits_{n\to+\infty}\left[\sup\limits_{k\geq 0}(\omega_k\omega_{k+1}\cdots\omega_{n+k-1})\right]^{\frac{1}{n}}$.

Let T be the scalar invertible bilateral weighted shift on $\mathcal{H}$, defined by

$$Te_n = \omega_n e_{n+1}, \qquad n \in \mathbb{Z}.$$

If $A_n = T$ for all $n \geq 0$, then, in view of (b), neither the identity (4.1) nor the identity (4.2) is satisfied for e_0. Hence, S_u is without Dunford's condition (C).

For the construction of a specific example of a positive two-sided sequence satisfying the above conditions, we refer the reader to [26].

It is shown in Theorem 2.5 of [28] that a nonnormal hyponormal scalar (unilateral or bilateral) weighted shift has fat local spectra (see also [7, Theorem 3.7]). The next example shows that this result is not valid for hyponormal operator weighted shifts.

Example 4.2. Assume that $(e_n)_{n\geq 0}$ is an orthonormal basis of $\mathcal{H}$, and let $(\alpha_n)_{n\geq 0}$ be an increasing positive sequence such that $\lim\limits_{n\to+\infty} \alpha_n = 1$. The diagonal operator, T, with the diagonal sequence $(\alpha_n)_{n\geq 0}$ (i.e., $Te_n = \alpha_n e_n,\ \forall\, n \geq 0$) is invertible and satisfies $r_1(T) = \alpha_0 < r(T) = 1$. If $A_n = T$ for all $n \geq 0$, then the unilateral operator weighted shift S_u is subnormal. Indeed, let $\mathcal{H}_n = \mathcal{H}$ for all $n \in \mathbb{Z}$ and let

$$\widetilde{\mathcal{H}} = \sum_{n\in\mathbb{Z}} \oplus \mathcal{H}_n$$

be the Hilbert space of the two-sided sequences $(x_n)_{n\in\mathbb{Z}}$ such that

$$\|(x_n)_{n\in\mathbb{Z}}\|_{\widetilde{\mathcal{H}}} := \Big(\sum_{n\in\mathbb{Z}} \|x_n\|^2_{\mathcal{H}}\Big)^{\frac{1}{2}} < +\infty.$$

Let S_b be the bilateral operator weighted shift defined on $\widetilde{\mathcal{H}}$ by

$$S_b(\cdots, x_{-2}, x_{-1}, [x_0], x_1, x_2, \cdots) = (\cdots, Tx_{-2}, [Tx_{-1}], Tx_0, Tx_1, \cdots),$$

where for an element $x = (\cdots, x_{-2}, x_{-1}, [x_0], x_1, x_2, \cdots) \in \widetilde{\mathcal{H}}$, $[x_0]$ denotes the central $(0th)$ term of x. Note that, since T is an hermitian operator, S_b is a normal extension of S_u. This shows that S_u is a subnormal operator. Now, we note that for every $k \geq 0$, we have

$$r_{S_u}(e_k^{(0)}) = \limsup_{n\to+\infty} \|T^n e_k\|^{\frac{1}{n}} = \alpha_k < r(S_u) = 1.$$

This shows, on the one hand, that S_u is without fat local spectra and, on the other hand, that

$$r_1(S_u) = r_2(S_u) = R_2^{\mp}(S_u) = r_3(S_u) = R_3^{-}(S_u) = \alpha_0 < R_3^{+}(S_u) = r(S_u) = 1.$$

Therefore, in view of the fact that $\sigma(S_u) = \sigma_{ap}(S_u) \cup \overline{\sigma_p(S_u^*)}$, Corollary 2.1 and Lemma 3.1, we have

$$\sigma_{ap}(S_u) = \{\lambda \in \mathbb{C} : \alpha_0 \leq |\lambda| \leq 1\}.$$

Remark 4.2. Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator. If $A_n = T$ for all $n \geq 0$, then S_u is hyponormal if and only if T is hyponormal. Therefore, to construct an example of the kind given in Example 4.2, it suffices to take T a hyponormal operator for which there is a nonzero element $x \in \mathcal{H}$ with $r_T(x) < r(T)$.

Finally, we would like to point out that

(a) Proposition 3.10 of [6] remain valid for the general setting of operator weighted shift. This is not the case for Proposition 3.12 of [6] as it is shown in Example 4.1.

(b) One can show that, for every nonzero $x \in \mathcal{H}$, the identity

$$\sigma_{S_u}(y) = \sigma_{S_u|\widehat{\mathcal{H}}(x)}(y)$$

holds for all $y \in \widehat{\mathcal{H}}(x)$.

(c) After the present note was completed, we began to study the local spectra of bilateral operator weighted shifts; this case is quite difficult. However, the question of which bilateral operator weighted shift has the single-valued extension property, even when $\mathcal{H}$ is an infinite dimensional Hilbert space, was recently settled in [5].

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