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POSITIVE SOLUTIONS FOR SINGULAR BOUNDARY VALUE PROBLEM OF SECOND ORDER***

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Abstract

Some results of existence of positive solutions for singular boundary value problems

$$\begin{cases} -u''(t) = p(t)f(u(t)), & t \in (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

are given, where the function $p(t)$ may be singular at $t = 0, 1$.

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§ 1 . Introduction

We consider the following problem

$$\begin{cases} -u''(t) = p(t)f(u(t)), & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $p \in C(0,1)$, which may be singular at $t = 0$ or $t = 1$. We are looking for positive classical solutions for (1.1). A function u is called a classical solution of (1.1), if $u \in C[0,1] \cap C^2(0,1)$ and satisfies both the equation and the boundary value condition.

The basic assumption on p is

$$\text{(P)} \qquad p \geq 0, \qquad p \in C(0,1), \qquad p(t) \not\equiv 0, \qquad \lim_{s \to 0} s \int_s^{1-s} p(\tau)d\tau = 0. \tag{1.2}$$

In many papers about the singular boundary value problem (1.1) (see [1] and the references therein), the following assumption is made:

$$\int_0^1 p(\tau)\tau(1-\tau)d\tau < +\infty. \tag{1.3}$$

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Notice that (1.3) implies (1.2). Another sufficient condition to make (1.2) true is

$$\lim_{s\to 0} s^2 p(s) = \lim_{s\to 1}(1-s)^2 p(s) = 0. \tag{1.4}$$

In the following we shall use the weighted L^2-space

$$L^2_p = \Big\{ u \Big| \int_0^1 p(s)u^2(s)ds < +\infty \Big\}$$

equipped with the quasi-norm

$$|u|_p = \Big\{ \int_0^1 p(s)u^2(s)ds \Big\}^{\frac{1}{2}}.$$

We also need the Sobolev space H^1_0. The norm of H^1_0 is denoted by $\|\cdot\|$:

$$\|u\| = \Big\{ \int_0^1 |u'(s)|^2 ds \Big\}^{\frac{1}{2}}.$$

H^1_0 is the completion of $C^\infty_0(0,1)$ with respect to this norm. It turns out that the condition (P) is closely related to the imbedding $H^1_0 \hookrightarrow L^2_p$. In fact we have the following theorem, which is of its own interest.

Theorem A. *Let L^2_p be the weighted L^2-space defined as above. Then*
(1) *H^1_0 is continuously imbedded into L^2_p if and only if*

$$\lim_{s\to 0} s \int_s^{1-s} p(\tau)d\tau < \infty. \tag{1.5}$$

(2) *H^1_0 is compactly imbedded into L^2_p if and only if*

$$\lim_{s\to 0} s \int_s^{1-s} p(\tau)d\tau = 0. \tag{1.6}$$

If we set

$$\lambda = \inf_{u\in H^1_0} \frac{\int_0^1 |u'(t)|^2 dt}{\int_0^1 p(t)u^2(t)dt}, \tag{1.7}$$

then as a consequence of Theorem A, $\lambda > 0$ is achieved, provided the assumption (P) holds.

Now we turn to the nonlinear function f. We assume
(F) $f \geq 0,\ f \in C[0,\infty)$ and

$$f_0^- = \varliminf_{t\to 0}\frac{f(t)}{t}, \qquad f_0^+ = \varlimsup_{t\to 0}\frac{f(t)}{t};$$
$$f_\infty^- = \varliminf_{t\to +\infty}\frac{f(t)}{t}, \qquad f_\infty^+ = \varlimsup_{t\to +\infty}\frac{f(t)}{t}.$$

Our main theorem is

Theorem B. *Assume that* (P) *and* (F) *hold. Assume moreover that either*

$$f_0^+ < \lambda < f_\infty^- \leq f_\infty^+ \leq +\infty \tag{1.8}$$

or

$$f_\infty^+ < \lambda < f_0^- \le f_0^+ < +\infty. \tag{1.9}$$

Then the problem (1.1) *has a positive classical solution.*

If $p \in C[0,1]$ and there exist the limits

$$f_0 = \lim_{t\to 0}\frac{f(t)}{t}, \qquad f_\infty = \lim_{t\to +\infty}\frac{f(t)}{t}, \tag{1.10}$$

then Theorem B is well known (see [4]), that is, the equation (1.1) has a positive solution provided either $f_0 < \lambda < f_\infty \le +\infty$ or $f_\infty < \lambda < f_0 < +\infty$. The case $f_\infty < +\infty$ is the asymptotically linear case, while the case $f_\infty = +\infty$ is called superlinear case. The variational method is used to prove Theorem B. Define $f(t) = 0$, as $t < 0$. Let F be the primitive function of f:

$$F(t) = \int_0^t f(s)ds.$$

Define a functional I on H_0^1 :

$$I(u) = \frac{1}{2}\int_0^1 |u'(t)|^2 dt - \int_0^1 p(t)F(u(t))dt.$$

Then I is well defined on H_0^1. Any critical point u of the functional I is a weak solution of the equation (1.1) :

$$\int_0^1 u'(t)\varphi'(t)dt - \int_0^1 p(t)f(u(t))\varphi(t)dt = 0, \qquad \forall\, \varphi \in H_0^1. \tag{1.11}$$

It is easy to prove that such a weak solution is a positive classical solution of (1.1).

This paper is organized as follows. In Section 2, we prove the imbedding theorem A. In Section 3, we prove the existence theorem B under the assumption $f_\infty^+ < +\infty$. In Section 4, we deal with the case $f_\infty^+ = +\infty$. In the last section, we make some remarks to indicate possible improvement and further extension.

§ 2. Proof of Theorem A

In this section, we prove the imbedding theorem A. To simplify the presentation, we assume that p has only a unique singular point $t = 0,\ p \in C(0,1]$. By integrating by parts, we have for $\epsilon > 0$,

$$\int_\epsilon^1 p(t)u^2(t)dt = u^2(\epsilon)\int_\epsilon^1 p(\tau)d\tau + 2\int_\epsilon^1 u'(t)u(t)\Big(\int_t^1 p(\tau)d\tau\Big)dt. \tag{2.1}$$

As $\lim\limits_{t\to 0} t\int_t^1 p(\tau)d\tau < +\infty$, we have

$$u^2(t) = \Big(\int_0^t u'(s)ds\Big)^2 \le t\int_0^t |u'(s)|^2 ds \le t\|u\|^2.$$

The first term of (2.1)

$$I_1 = u^2(\epsilon)\int_\epsilon^1 p(\tau)d\tau \le \|u\|^2\Big(\epsilon\int_\epsilon^1 p(\tau)d\tau\Big) \le c_1\|u\|^2. \tag{2.2}$$

And the second term of (2.1)

$$\begin{aligned} I_2 &\le c_2 \int_\epsilon^1 \Big|u'(t)\frac{u(t)}{t}\Big| dt \le c_2\Big(\int_\epsilon^1 |u'(t)|^2 dt\Big)^{\frac12}\Big(\int_\epsilon^1 \Big|\frac{u(t)}{t}\Big|^2 dt\Big)^{\frac12} \\ &\le c_2\Big(\int_0^1 |u'(t)|^2 dt\Big)^{\frac12}\Big(\int_0^1 \Big|\frac{u(t)}{t}\Big|^2 dt\Big)^{\frac12} \le c_3\|u\|^2. \end{aligned} \tag{2.3}$$

In the last inequality, we have used the Hardy inequality

$$\int_0^\infty \frac{u^2(t)}{t^2} dt \le 4\int_0^\infty |u'(t)|^2 dt.$$

Hence we have the continuous imbedding $H_0^1 \hookrightarrow L_p^2$, provided the quantity $s\int_s^1 p(\tau)d\tau$ keeps bounded. Now suppose that $s\int_s^1 p(\tau)d\tau \to 0$, as $s \to 0$. We are going to show that this embedding is compact. Let $\{u_n\}$ be a bounded subset of H_0^1, say $\|u_n\| = 1$. We can assume that $\{u_n\}$ uniformly converges to a function u in $C[0,1]$. We have by (2.1),

$$\int_0^1 p(t)|u_n(t)-u(t)|^2 dt = 2\int_0^1 (u_n(t)-u(t))(u_n'(t)-u'(t))\Big(\int_t^1 p(\tau)d\tau\Big)dt. \tag{2.4}$$

Since $\lim\limits_{s\to 0} s\int_s^1 p(t)dt = 0$ for all $\epsilon > 0$, there exists $\delta > 0$ such that $0 < s\int_s^1 p(t)dt < \epsilon$ for $0 < s < \delta$. So the right hand of (2.4)

$$\begin{aligned} I_3 &\le 2\int_0^1 |u_n(t)-u(t)||u_n'(t)-u'(t)|\Big(\int_t^1 p(\tau)d\tau\Big)dt \\ &= 2\int_0^\delta |u_n'(t)-u'(t)|\Big|\frac{u_n(t)-u(t)}{t}\Big|\Big(t\int_t^1 p(\tau)d\tau\Big)dt \\ &\quad + 2\int_\delta^1 |u_n(t)-u(t)||u_n'(t)-u'(t)|\Big(\int_t^1 p(\tau)d\tau\Big)dt \\ &\le 2\epsilon\Big(\int_0^\delta |u_n'(t)-u'(t)|^2 dt\Big)^{\frac12}\Big(\int_0^\delta \Big|\frac{u_n(t)-u(t)}{t}\Big|^2 dt\Big)^{\frac12} \\ &\quad + C_\epsilon\int_0^1 |u_n(t)-u(t)||u_n'(t)-u'(t)|dt \\ &\le 2\epsilon\|u_n-u\|^2 + C_\epsilon\|u_n-u\|\|u_n-u\|_c \\ &\le \mu(\epsilon) + C_\epsilon\|u_n-u\|_c, \end{aligned} \tag{2.5}$$

where $\|\cdot\|_c$ is the norm of $C[0,1]$, $\mu(\epsilon)$ denotes a small quantity which tends to zero as $\epsilon \to 0$ and C_ϵ denotes constants dependent on ϵ. From (2.5), we have $\int_0^1 p(t)|u_n(t)-u(t)|^2 dt \to 0$, as $n \to \infty$. This complete the sufficient part of Theorem A.

Now suppose that H_0^1 is continuously imbedded into L_p^2. For $0 < \epsilon < \frac12$, define a function u_ϵ by

$$u_\epsilon(t) = \begin{cases} \dfrac{1}{\sqrt{\epsilon}}t, & 0 \le t \le \epsilon, \\ \sqrt{\epsilon}, & \epsilon \le t \le 1-\epsilon, \\ \dfrac{1}{\sqrt{\epsilon}}(1-t), & 1-\epsilon \le t \le 1. \end{cases} \tag{2.6}$$

Then $\|u_\epsilon\|^2 = 2$. We get

$$c \geq \int_0^1 p(t)u_\epsilon^2(t)dt \geq \int_\epsilon^{1-\epsilon} p(t)u_\epsilon^2(t)dt = \epsilon \int_\epsilon^{1-\epsilon} p(t)dt. \tag{2.7}$$

Suppose now that H_0^1 is compactly imbedded into L_p^2. Since the function u_ϵ defined by (2.6) uniformly converges to zero, hence by compactness, u_ϵ converges to zero in L_p^2. And from (2.7) we have

$$\epsilon \int_\epsilon^{1-\epsilon} p(t)dt \leq \int_0^1 p(t)u_\epsilon^2(t)dt \to 0 \qquad \text{as } \ \epsilon \to 0.$$

The proof of Theorem A is completed.

Lemma 2.1. *Let*

$$\lambda = \inf_{u \in H_0^1} \frac{\int_0^1 |u'(t)|^2 dt}{\int_0^1 p(t)u^2(t)dt}.$$

Then there exists a function $\varphi \in H_0^1$ such that $\varphi > 0, \ \forall\, t \in (0,1), \ \frac{\int_0^1 |\varphi'(t)|^2 dt}{\int_0^1 p(t)\varphi^2(t)dt} = \lambda$, and φ satisfies

$$\begin{cases} -\varphi''(t) = \lambda p(t)\varphi(t), & t \in (0,1), \\ \varphi(0) = \varphi(1) = 0. \end{cases} \tag{2.8}$$

Moreover the eigenvalue λ is of multiplicity one, that is, if ψ is another minimizer, then $\psi = c\varphi$ for a constant c.

Proof. The existence of a minimizer φ is a consequence of Theorem A(2). If we set $\widetilde{\varphi} = |\varphi|$, then $\widetilde{\varphi}$ is a minimizer too. Hence we can assume that $\varphi \geq 0, \varphi$ satisfies the weak form of (2.8) :

$$\int_0^1 \varphi'(t)v'(t)dt = \lambda \int_0^1 p(t)\varphi(t)v(t)dt, \qquad \forall\, v \in H_0^1. \tag{2.9}$$

By the regularity theorem, (2.8) follows (2.9). Notice that the right hand side of (2.8) is nonnegative, hence φ is concave, and $\varphi(t) > 0$ for all $t \in (0,1)$.

Now suppose that $\psi \in H_0^1$ is another minimizer and there is a constant c such that the function $\psi - c\varphi$ changes sign, then the positive part of $\psi - c\varphi$, say χ, is a minimizer too. This contradicts the above fact that $\chi > 0$ for all $t \in (0,1)$.

§ 3. Proof of Theorem B for the Case $f_\infty^+ < +\infty$

In this section, we will prove Theorem B for the case $f_\infty^+ < +\infty$ by variational method, especially the Mountain Pass Lemma. Firstly we verify the P.S condition.

Lemma 3.1. *Suppose that* (P), (F) *hold and $f_\infty^+ < +\infty, \ f_0^+ < +\infty$. Then $I(u)$ is well defined on H_0^1 and C^1-continuous, and the Frechét derivative of $I(u)$ has the form*

$$\langle I'(u), \varphi \rangle = \int_0^1 u'(t)\varphi'(t)dt - \int_0^1 p(t)f(u(t))\varphi(t)dt, \qquad \forall\, \varphi \in H_0^1. \tag{3.1}$$

Proof. If $P(x)$ is bounded, this lemma is a well-known fact. Now with the conditions (P) and (F), the proof is similar, so we omit it.

Lemma 3.2. *Suppose that* (P), (F) *hold and either* $f_0^+ < \lambda < f_\infty^- \le f_\infty^+ < +\infty$ *or* $f_\infty^+ < \lambda < f_0^- \le f_0^+ < +\infty$, *then* I *satisfies the P.S condition.*

Proof. Let $\{u_n\}$ be a P.S sequence. Then

$$\langle I'(u_n), \varphi\rangle = \int_0^1 u_n'\varphi' dt - \int_0^1 p(t)f(u_n)\varphi dt = o(\|\varphi\|), \qquad \forall\, \varphi \in H_0^1. \tag{3.2}$$

Suppose $|u_n^+|_p$ is bounded, where $u^+ = \max\{u, 0\}$. Taking $\varphi = u_n$ in (3.2) and noticing that $|f(u_n)| \le cu_n^+$, we have the bound of $\|u_n\|$. Let $u_n \rightharpoonup u$ in H_0^1. By the embedding theorem A, we can assume that $u_n \to u$ in L_p^2, hence $f(u_n) \to f(u)$ in L_p^2. It follows from (3.2) that $u_n \to u$ in H_0^1.

We prove that any P.S sequence $\{u_n\}$ is $|u_n^+|_p < +\infty$. Otherwise suppose $|u_n^+|_p \to +\infty$. Set $v_n = \dfrac{u_n}{|u_n^+|_p}$, $|v_n^+|_p = 1$. Taking $\varphi = v_n$ in (3.2), we have that $\|v_n\|$ is bounded. Assume that $v_n \rightharpoonup v$ in H_0^1, $v_n \to v$ in $C[0,1]$ and L_p^2, $|v^+|_p = 1$. Set $t_n = |u_n^+|_p$. From (3.2), we have

$$\int_0^1 v_n'\varphi' dt = \int_0^1 p(t)\frac{f(u_n)}{|u_n^+|_p}\varphi dt + \frac{o(\|\varphi\|)}{|u_n^+|_p}. \tag{3.3}$$

We firstly consider the case $f_0^+ < \lambda < f_\infty^- \le f_\infty^+ < +\infty$. Let $\epsilon > 0$ and $f_\infty^- - \epsilon > \lambda$, choose a constant $M > 0$ such that $f(t) > (f_\infty^- - \epsilon)t$, $\forall\, t > M$. Let $\varphi \in H_0^1$ and $\varphi \ge 0$. From (3.3), we have

$$\begin{aligned}\int_0^1 v_n'\varphi' dt &= \int_0^1 p(t)\frac{f(t_nv_n)}{t_n}\varphi dt + o(1)\\ &= \int_{t_nv_n<M} p(t)\frac{f(t_nv_n)}{t_n}\varphi dt + \int_{t_nv_n\ge M} p(t)\frac{f(t_nv_n)}{t_n}\varphi dt + o(1)\\ &\ge \int_{t_nv_n<M} p(t)\frac{f(t_nv_n)}{t_n}\varphi dt + (f_\infty^- - \epsilon)\int_{t_nv_n\ge M} p(t)v_n^+\varphi dt + o(1)\\ &\ge (f_\infty^- - \epsilon)\int_0^1 p(t)v_n^+\varphi dt - c\int_{0\le t_nv_n\le M} p(t)v_n^+\varphi dt + o(1).\end{aligned} \tag{3.4}$$

Letting $n \to \infty$, we have

$$\int_0^1 v'\varphi' dt \ge (f_\infty^- - \epsilon)\int_0^1 p(t)v^+\varphi dt, \qquad \forall\, \varphi \in H_0^1,\ \varphi \ge 0. \tag{3.5}$$

In particular, let φ be the minimizer in Lemma 2.1. We have

$$\lambda\int_0^1 p(t)v^+\varphi dt \ge \lambda\int_0^1 p(t)v\varphi dt = \int_0^1 v'\varphi' dt \ge (f_\infty^- - \epsilon)\int_0^1 p(t)v^+\varphi dt, \tag{3.6}$$

which implies that $pv^+ \equiv 0$, hence $|v^+|_p = 0$, a contradiction.

Suppose now that $f_\infty^+ < \lambda < f_0^- \le f_0^+ < +\infty$. Let $\epsilon > 0$ and $f_\infty^+ + \epsilon < \lambda$. As in (3.4), we have

$$\int_0^1 v_n'\varphi' dt \le (f_\infty^+ + \epsilon)\int_0^1 p(t)v_n^+\varphi dt + c\int_{0\le t_nv_n\le M} p(t)v_n^+\varphi dt + o(1).$$

Letting $n \to \infty$, we have

$$\int_0^1 v'\varphi' dt \le (f_\infty^+ + \epsilon)\int_0^1 p(t)v^+\varphi dt, \qquad \forall\, \varphi \in H_0^1,\ \varphi \ge 0. \tag{3.7}$$

Taking $\varphi = v^+$ in (3.7), we have by the definition of λ,

$$\lambda\int_0^1 p(t)(v^+)^2 dt \le \int_0^1 |(v^+)'|^2 dt = \int_0^1 v'(v^+)' dt \le (f_\infty^+ + \epsilon)\int_0^1 p(t)(v^+)^2 dt.$$

Since $\int_0^1 p(t)(v^+)^2 dt = 1$, we arrive at a contradiction. Hence I satisfies the P.S condition.

Proof of Theorem B. The case $f_\infty^+ < \lambda < f_0^- \le f_0^+ < +\infty$.

We prove that in this case the functional I is coercive, that is, $I(u) \to \infty$ as $\|u\| \to \infty$. We use an indirect argument. Suppose that there is a sequence $\{u_n\} \subset H_0^1$ such that $\|u_n\| \to \infty$. Set $t_n = |u_n^+|_p,\ v_n = \frac{u_n}{t_n}$. Then

$$+\infty > c \ge I(u_n) = \frac{1}{2}\int_0^1 |u_n'|^2 dt - \int_0^1 p(t)F(u_n(t))dt.$$

Dividing the above inequality by t_n^2, we have

$$\int_0^1 p(t)\frac{F(u_n(t))}{t_n^2}dt \ge o(1) + \frac{1}{2}\frac{\int_0^1 |u_n'|^2 dt}{t_n^2} \ge o(1) + \frac{1}{2}\lambda.$$

Let $\epsilon > 0$ and $f_\infty^+ + \epsilon < \lambda$, choose a constant $M > 0$ such that $F(t) \le \frac{1}{2}(f_\infty^+ + \epsilon)t^2$, $\forall\, t > M$. Hence we have

$$\int_0^1 p(t)\frac{F(u_n(t))}{t_n^2}dt \le \frac{1}{2}(f_\infty^+ + \epsilon)\int_0^1 p(t)(v_n^+)^2 dt + c\int_{0\le t_n v_n \le M} p(t)(v_n^+)^2 dt.$$

Assume $v_n \to v$ in L_p^2, we have $\int_0^1 pv_+^2 dt = 1$ and $\lambda \le (f_\infty^+ + \epsilon)$, a contradiction.

Since I is bounded from below and satisfies the P.S condition, I has a minimizer u. We need only to show that the trivial solution $u \equiv 0$ is not a local minimizer, then the minimizer should be a nontrivial positive solution. Let φ be the eigenfunction in Lemma 2.1, $\int_0^1 |\varphi'|^2 dt = \lambda\int_0^1 p\varphi^2 dt$. For $\sigma > 0$ very small, we have

$$I(\sigma\varphi) = \frac{1}{2}\sigma^2\int_0^1 |\varphi'|^2 dt - \int_0^1 p(t)F(\sigma\varphi)dt.$$

Let $\epsilon > 0$ and $f_0^- - \epsilon > \lambda$, choose a constant $\sigma > 0$ such that $F(t) \ge \frac{1}{2}(f_0^- - \epsilon)t^2$ for all $0 < t < \sigma$. Therefore we have

$$\begin{aligned} I(\sigma\varphi) &\le \frac{1}{2}\sigma^2\int_0^1 |\varphi'|^2 dt - \frac{1}{2}(f_0^- - \epsilon)\sigma^2\int_0^1 p(t)\varphi^2 dt \\ &= \frac{1}{2}\sigma^2\|\varphi\|^2 - \frac{1}{2\lambda}(f_0^- - \epsilon)\sigma^2\|\varphi\|^2 \\ &= \frac{1}{2}\sigma^2\|\varphi\|^2\Big(1 - \frac{f_0^- - \epsilon}{\lambda}\Big) < 0. \end{aligned}$$

The case $f_0^+ < \lambda < f_\infty^- \le f_\infty^+ < +\infty$.

In this case we use the Mountain Pass Lemma. We need to verify

(a) there are constants $\alpha, \rho > 0$ such that $I(u) \geq \alpha,\ \forall\, u, \|u\| = \rho$,

(b) there is an element e such that $I(e) \leq 0$ and $\|e\| > \rho$.

Take $\epsilon > 0, f_0^+ + \epsilon < \lambda$. For $\rho \ll 1,\ \|u\| = \rho$, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_0^1 (u')^2 dt - \int_0^1 p(t)F(u)dt \\ &\geq \frac{1}{2}\int_0^1 (u')^2 dt - \frac{1}{2}(f_0^+ + \epsilon)\int_0^1 p(t)u^2 dt \\ &\geq \frac{1}{2}\Big(1 - \frac{f_0^+ + \epsilon}{\lambda}\Big)\int_0 1(u')^2 dt \\ &= \frac{1}{2}\Big(1 - \frac{f_0^+ + \epsilon}{\lambda}\Big)\rho^2 = \alpha > 0. \end{aligned}$$

On the other hand, let φ be the eigenfunction in Lemma 2.1, $\epsilon > 0$ and $f_\infty^- - \epsilon > \lambda$. Choosing T large enough, we have $f(t) \geq (f_\infty^- - \epsilon)t,\ F(t) \geq \frac{1}{2}(f_\infty^- - \epsilon)t^2,\ \forall\, t > T$ and

$$\begin{aligned} I(T\varphi) &= \frac{1}{2}T^2\int_0^1 (\varphi')^2 dt - \int_0^1 p(t)F(T\varphi)dt \\ &\leq \frac{1}{2}T^2\|\varphi\|^2 - \frac{1}{2}T^2(f_\infty^- - \epsilon)\int_0^1 p(t)\varphi^2 dt \\ &= \frac{1}{2}T^2\|\varphi\|^2\Big(1 - \frac{f_\infty^- - \epsilon}{\lambda}\Big) < 0. \end{aligned}$$

Now by the Mountain Pass Lemma, we define

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \mid \gamma \in C([0,1], H_0^1),\ \gamma(0) = \theta,\ \gamma(1) = T\varphi\}.$$

Then $c \geq \alpha$ is a critical value of I, and I has a critical point u with $I(u) = c$. u is a classical positive solution of our problem (1.1).

§ 4. Proof of Theorem B for the Case $f_\infty^+ = +\infty$

In this section we deal with the case $f_0^+ < \lambda < f_\infty^- \leq f_\infty^+ = \infty$. Take $\lambda < \lambda_\infty < f_\infty^-$. Define a function

$$f_M(t) = \begin{cases} \lambda_\infty(t - M) + f(M), & t \geq M, \\ f(t), & t \leq M, \end{cases} \tag{4.1}$$

where M is a constant to be chosen. For this truncated function, by the result in Section 3, we have a solution u satisfying

$$\begin{cases} -u''(t) = p(t)f_M(u(t)), & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{4.2}$$

or the weak form

$$\int_0^1 u'\varphi' dt = \int_0^1 p(t) f_M(u(t))\varphi dt, \qquad \forall\, \varphi \in H_0^1. \tag{4.3}$$

We will choose the constant M so that the solution u of (4.2) satisfies $\|u\|_c \leq M$. Hence u is in fact a solution for our original problem.

For $\delta > 0$, define

$$\lambda_\delta = \inf_{u\in H_0^1} \frac{\int_0^1 |u'(t)|^2 dt}{\int_\delta^{1-\delta} p(t)u^2(t)dt}. \tag{4.4}$$

Then $\lambda_\delta \to \lambda$ as $\delta \to 0$. There is a function $\varphi_\delta \in H_0^1$ such that $\varphi_\delta(t) > 0,\ t \in (0,1),\ \lambda_\delta = \frac{\int_0^1 |\varphi_\delta'|^2 dt}{\int_\delta^{1-\delta} p(t)u^2(t)dt}$ and $\int_0^1 \varphi_\delta' v' dt = \lambda_\delta \int_\delta^{1-\delta} p(t)\varphi_\delta v dt,\ \forall\ v \in H_0^1$. The existence of such λ_δ and φ_δ is obvious. To show that $\lambda_\delta \to \lambda$, let φ be the eigenfunction in Lemma 2.1. We have

$$\lambda \leq \lambda_\delta \leq \frac{\int_0^1 |\varphi'(t)|^2 dt}{\int_\delta^{1-\delta} p(t)\varphi^2(t)dt} \to \frac{\int_0^1 |\varphi'(t)|^2 dt}{\int_0^1 p(t)\varphi^2(t)dt} = \lambda \qquad \text{as } \ \delta \to 0.$$

Choose δ so small that $\lambda_\delta < f_\infty^-$ and $\exists\, t \in [\delta, 1-\delta],\ p(t) \neq 0$. Choose λ_∞ and M_0 such that $\lambda_\delta < \lambda_\infty < f_\infty^-$ and $f(t) \geq \lambda_\infty t$ for $t \geq M_0$. So we have $f_M(t) \geq \lambda_\infty t$ for $t \geq M_0$ and $M \geq M_0$. Set $M = \frac{M_0}{\delta}$ as the constant in (4.1). We prove that the solution u of (4.2) satisfies $0 \leq u \leq M$. Suppose that this is not true, then $\max u = \widetilde{M} > M$. Since u is concave, $u(t) \geq \widetilde{M}\delta \geq M\delta \geq M_0$ and $f_M(u(t)) \geq \lambda_\infty u(t)$ for $t \in [\delta, 1-\delta]$. Now for every $v \in H_0^1, v \geq 0$, we have

$$\int_0^1 u'v' dt \geq \int_\delta^{1-\delta} p(t) f_M(u) v dt \geq \lambda_\infty \int_\delta^{1-\delta} p(t) u v dt. \tag{4.5}$$

In particular, take $v = \varphi_\delta$, the minimizer for the eigenvalue λ_δ, then

$$\lambda_\delta \int_\delta^{1-\delta} p(t) u \varphi_\delta dt = \int_0^1 u'\varphi_\delta' dt \geq \lambda_\infty \int_\delta^{1-\delta} p(t) u \varphi_\delta dt, \tag{4.6}$$

which implies that $\int_\delta^{1-\delta} p(t)u\varphi_\delta dt = 0$ and $pu\varphi_\delta \equiv 0$ in $[\delta, 1-\delta]$. We have at least a point $t_0 \in [\delta, 1-\delta]$ with $u(t_0) = 0$. Since u is concave, we have $u \equiv 0$ for all $t \in [\delta, 1-\delta]$, a contradiction. We have proved that $\|u\|_c \leq M$ and u is the desired solution for our original problem.

§ 5. Some Remarks

We make some remarks. A few papers discussed the problem (1.1) by transforming it into an integral equation and using fixed point theorem on a cone, usually the positive cone of the space $C[0,1]$. For such a setting the condition (1.2) is essential. It seems that H_0^1, the Sobolev space, is a more natural working space, as shown in this paper. We are able to weaken the assumption on p. Theorem B describes what is actually needed.

By estimating the eigenvalue λ, one can give some sufficient condition, apparent but not very precise. For example we have the following estimate (see [5]):

$$m = \Big(\int_0^1 G(s,s)p(s)ds\Big)^{-1} \leq \lambda \leq 4\Big(\max_{0\leq t\leq 1} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)p(s)ds\Big)^{-1} = M, \tag{5.1}$$

where $G(t,s)$ is the Green function in dimension one:

$$G(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{5.2}$$

By Theorem B, we have the following conclusion: if

$$f_0^+ < m \qquad \text{and} \qquad M < f_\infty^-, \tag{5.3}$$

then the problem has a positive solution. This conclusion appeared in some references.

Finally we indicate some parallel results for other kinds of problems. One is the boundary value problem of fourth order, another is the p-Laplacian equation. We will present the details otherwhere.

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