Chin. Ann. Math.
25B:3(2004),393–400.

EXIT DISTRIBUTION LEAVING A BALL FROM THE CENTER AND BROWNIAN MOTION****

ZHANG Huizeng* JIN Mengwei** YING Jiangang***

Abstract

It is showed that if the first exit distribution leaving any ball from the center is the uniform distribution on the sphere, then the Lévy process is a scaled Brownian motion. The paper also gives a characterization of a continuous Hunt process by the first exit distribution from any ball.

Keywords First exit hitting distribution, Brownian motion, Lévy process
2000 MR Subject Classification 60G17, 60J65

§ 1. Introduction

Suppose that E is a locally compact space with a countable base (LCCB) and a compatible metric d. Let $\mathcal{E}$ be the Borel σ-algebra on E. For any $x \in E$ and $r > 0$, we let $B_r(x) = \{y \in E : d(x,y) \le r\}$, $B_r^\circ(x) = \{y \in E : d(x,y) < r\}$ and $S_r(x) = \{y \in E : d(x,y) = r\}$. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space and $\mathcal{B}(\mathbb{R}^d)$ be the Borel σ-algebra on $\mathbb{R}^d$. Let $b\mathcal{B}(\mathbb{R}^d)$ denote the set of all bounded Borel measurable functions. For any $x, y \in \mathbb{R}^d$, we use $\langle x, y\rangle$ and $|x|$ to denote the inner product of x and y and the length of x respectively. For any $x \in \mathbb{R}^d$ and any $r > 0$, let $\sigma_r(x) = \sigma_r(x, \cdot)$ denote the uniform probability distribution on $S_r(x)$. For convenience, we let $B_r = B_r(0)$, $B_r^\circ = B_r^\circ(0)$, $S_r = S_r(0)$ and $\sigma_r = \sigma_r(0)$. Let $\hat{\mu}$ denote the characteristic function of a probability μ on $\mathbb{R}^d$.

Let $X = (X_t, P)$ be a Lévy process on $\mathbb{R}^d$ starting at 0 with convolution semigroup $\pi = \{\pi_t : \ t > 0\}$. The Lévy exponent of $\pi = \{\pi_t\}$ is denoted by ϕ, i.e., $\hat{\pi}_t(x) = e^{-t\phi(x)}$. Let $D = \{x : \ |x| \le 1\}$, the closed unit ball. By the Lévy-Khintchine formula (see Theorem 8.1 of [6]),

$$\phi(x) = \frac{1}{2}\langle x, Ax\rangle - i\langle \gamma, x\rangle + \int_{\mathbb{R}^d} [1 - e^{i\langle x,y\rangle} + i\langle x, y\rangle 1_D(y)]\upsilon(dy),$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, υ is a measure on $\mathbb{R}^d$ satisfying $\upsilon(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|y|^2 \wedge 1)\upsilon(dy) < \infty$, and $\gamma \in \mathbb{R}^d$. The representation of ϕ by A, υ and γ is unique, and we call (A, υ, γ) the generating triplet of X or π.

It is well known that any first exit distribution of Brownian motion leaving a ball is uniformly distributed on the sphere of the ball. In this paper we shall consider the inverse

Manuscript received March 31, 2003.
*Department of Mathematics, Zhejiang University, Hangzhou 310027, China.
School of Computer and Computing Science, Zhejiang University City College, Hangzhou 310015, China.
E-mail: huizengzhang@sina.com
**Department of Mathematics, Zhejiang University, Hangzhou 310027, China.
***Institute of Mathematics, Fudan University, Shanghai 200433, China. **E-mail:** jgying@fudan.edu.cn
****Project supported by the National Natural Science Foundation of China (No.10271109).

problem. If any of such exit distributions of a Lévy process on $\mathbb{R}^d$ is uniform, then a natural question is whether it is a Brownian motion (by a time scaling). We should mention a program of an argument (which was actually suggested by R. K. Getoor). If the above exit distributions of a Markov process are uniform, then Theorem II-5.11 of [1] implies that the process has the same class of (lower semi-continuous) excessive functions as that of Brownian motion which in turn implies that it has the same class of hitting distributions as that of Brownian motion by Hunt's Balayage theorem. Thus a well-known Theorem of Blumenthal-Getoor-Mckean (see [1]) shows that the process differs from Brownian motion by a time change. However Hunt's balayage theorem needs the condition that the process is transient. Without assuming the transience, little information can be retrieved from excessive functions since they may only contain constants.

In the present article, we shall prove a Lévy process (without assumption of path continuity or transience) with the first exit distribution from a ball being uniform on surface is a Brownian motion (by a time scaling). The approach is rather elementary and does not use the Blumenthal-Getoor-Mackean Theorem. The paper also gives a characterization of a continuous Hunt process by the first exit distribution. Without loss of generality, in this paper, we shall assume that the Lévy process is genuinely d-dimensional, that is, π_1 is not supported on any proper linear subspace of $\mathbb{R}^d$, since otherwise we may consider the subspace instead.

§ 2. Continuity of Hunt Processes

Let $\{\Omega, \mathcal{F}_t^0, \mathcal{F}^0, P^x, \theta_t, X_t\}$ be a Hunt process with state space $(E, \mathcal{E})$ and transition function (P_t). Then $T_{B_r^c}(\omega) := \inf\{t > 0 : d(X_t(\omega), X_0(\omega)) > r\}$, where d is a fixed metric on E compatible with the topology, is a stopping time relative to $\{\mathcal{F}_t^0\}$. Let $q(\omega)$ be a property of ω. Then q is said to hold almost surely (a.s.) if the set Λ of ω in Ω for which $q(\omega)$ fails to hold is in $\mathcal{F}^0$ and $P^x(\Lambda) = 0$ for all $x \in E$. Now we give the main theorem of this section.

Theorem 2.1. *The first exit distribution $P^x(X_{T_{B_r^c}} \in dy, T_{B_r^c} < \infty)$ concentrates on $S_r(x)$ for all $x \in E$ and all $r > 0$ if and only if the sample path is continuous on $[0, \infty)$ almost surely.*

Before proving the theorem, we prepare a lemma.

Lemma 2.1. *Let $\epsilon > 0$. Then $P^x(X_{T_{B_r^c}} \in dy, T_{B_r^c} < \infty)$ concentrates on $B_{r+\epsilon}(x) \setminus B_r^\circ(x)$ for all $x \in E$ and all $r > 0$ if and only if almost surely, for all $t > 0$, $d(X_{t-}, X_t) \leq \epsilon$.*

Proof. The sufficiency is obvious. Now we prove the necessity. For convenience, we denote $T_{B_r^c}$ by T^r. Then T^r is a stopping time relative to $\{\mathcal{F}_t^0\}$ for all $r > 0$. Next, we define $T_0^r := 0$, $T_1^r := T^r$ and inductively for $n \geq 1$,

$$T_{n+1}^r := T_n^r + T^r \circ \theta_{T_n^r} = \inf\{t > T_n^r : d(X_t, X_{T_n^r}) > r\}.$$

Obviously, each T_n^r is a stopping time relative to $\{\mathcal{F}_t^0\}$. Since T_n^r increases with n, the limit $S^r = \lim_n T_n^r$ exists and S^r is a stopping time. On the set $\{S^r < \infty\}$, we have $\lim_n X(T_n^r) = X(S^r)$ almost surely by quasi-left continuity. On the other hand, the right continuity of paths implies that $d(X(T_{n+1}^r), X(T_n^r)) \geq r$ almost surely for all n, which precludes the existence of $\lim_n X(T_n^r)$. There would be a contradiction unless $S^r = \infty$ almost surely. In the latter event, we have $[0, \infty) = \bigcup_{n=0}^{\infty} [T_n^r, T_{n+1}^r)$ almost surely. Note that if $T_n^r = \infty$, then

$[T_n^r, T_{n+1}^r) = \emptyset$. In each interval $[T_n^r, T_{n+1}^r)$, the oscillation of $X(\cdot)$ does not exceed $2r$ by the definition of T_{n+1}^r. We have therefore proved that for each r, there exists Ω_r such that $P^x(\Omega_r) = 1$ for all $x \in E$, and for any $\omega \in \Omega_r$, $[0, \infty) = \bigcup_{n=0}^{\infty} [T_n^r(\omega), T_{n+1}^r(\omega))$ and $X(\cdot, \omega)$ does not oscillate by more than $2r$ in any interval $[T_n^r(\omega), T_{n+1}^r(\omega))$. Let $\Omega_* = \bigcap_{m=1}^{\infty} \Omega_{\frac{1}{m}}$. Then $P^x(\Omega_*) = 1$ for all $x \in E$.

By strong Markov property of Hunt process and by our condition, for any $x \in E$,

$$\begin{aligned}
&E^x\Big[\frac{1}{m} \le d(X(T_n^{\frac{1}{m}}), X(T_{n-1}^{\frac{1}{m}})) \le \frac{1}{m} + \epsilon, T_n^{\frac{1}{m}} < \infty\Big] \\
&= E^x\Big[E^{X(T_{n-1}^{\frac{1}{m}})}\Big[\frac{1}{m} \le d(X(T^{\frac{1}{m}}), X_0) \le \frac{1}{m} + \epsilon, T^{\frac{1}{m}} < \infty\Big], T_{n-1}^{\frac{1}{m}} < \infty\Big] \\
&= E^x\Big[E^{X(T_{n-1}^{\frac{1}{m}})}(T^{\frac{1}{m}} < \infty), T_{n-1}^{\frac{1}{m}} < \infty\Big] \\
&= E^x\Big(T^{\frac{1}{m}} \circ \theta_{T_{n-1}^{\frac{1}{m}}} < \infty, T_{n-1}^{\frac{1}{m}} < \infty\Big) \\
&= E^x(T_n^{\frac{1}{m}} < \infty).
\end{aligned}$$

Therefore for any positive integer m, there exists an Ω'_m with $P^x(\Omega'_m) = 1$ for all $x \in E$ such that, for any $\omega \in \Omega'_m$ and any integer n with $T_n^{\frac{1}{m}}(\omega) < \infty$, we have

$$\frac{1}{m} \le d\Big(X_{T_n^{\frac{1}{m}}}(\omega), X_{T_{n-1}^{\frac{1}{m}}}(\omega)\Big) \le \frac{1}{m} + \epsilon.$$

Let $\Omega'_* = \bigcap_{m=1}^{\infty} \Omega'_m$ and $\Omega^\circ = \Omega_* \bigcap \Omega'_*$. Obviously $P^x(\Omega^\circ) = 1$ for all $x \in E$.

We assert that if $\omega \in \Omega^\circ$, then for all $t > 0$, $d(X_{t-}(\omega), X_t(\omega)) \le \epsilon$. Otherwise there exists some $t > 0$ such that $d(X_{t-}(\omega), X_t(\omega)) > \epsilon$. Thus $d(X_{t-}(\omega), X_t(\omega)) > \frac{3}{m} + \epsilon$ for some positive integer m. Since $[0, \infty) = \bigcup_{n=0}^{\infty} [T_n^{\frac{1}{m}}(\omega), T_{n+1}^{\frac{1}{m}}(\omega))$, the proof is divided into the following two cases.

Case 1. Suppose that $t \in (T_n^{\frac{1}{m}}(\omega), T_{n+1}^{\frac{1}{m}}(\omega))$ for some n. Then by definition of $T_{n+1}^{\frac{1}{m}}$, $d(X_{t-}(\omega), X_t(\omega)) \le \frac{2}{m}$ which is impossible.

Case 2. Suppose that $t = T_n^{\frac{1}{m}}(\omega)$ for some n. Then $T_n^{\frac{1}{m}}(\omega) < \infty$. Hence

$$\begin{aligned}
d(X_{t-}(\omega), X_t(\omega)) &= d\Big(X_{T_n^{\frac{1}{m}}-}(\omega), X_{T_n^{\frac{1}{m}}}(\omega)\Big) \\
&\le d\Big(X_{T_n^{\frac{1}{m}}}(\omega), X_{T_{n-1}^{\frac{1}{m}}}(\omega)\Big) + d\Big(X_{T_{n-1}^{\frac{1}{m}}}(\omega), X_{T_n^{\frac{1}{m}}-}(\omega)\Big) \\
&\le \Big(\frac{1}{m} + \epsilon\Big) + \frac{2}{m} = \frac{3}{m} + \epsilon,
\end{aligned}$$

which is also impossible.

Therefore for any $\omega \in \Omega^\circ$, $d(X_{t-}(\omega), X_t(\omega)) \le \epsilon$ for all $t > 0$.

Proof of Theorem 2.1. We need only to prove the necessity. Since $P^x(X_{T_{B_r^c}} \in dy, T_{B_r^c} < \infty)$ concentrates on $B_r(x)$ for any $x \in E$ and any $r > 0$, for any positive integer n, $P^x(X_{T_{B_r^c}} \in dy, T_{B_r^c} < \infty)$ concentrates on $B_{r+\frac{1}{n}}(x) \setminus B_r^\circ(x)$ for any $x \in E$ and any $r > 0$. It follows from Lemma 2.1 that for any positive integer n, almost surely $d(X_{t-}, X_t) \le \frac{1}{n}$ for all $t > 0$. Therefore almost surely the sample path is continuous on $[0, \infty)$.

Remark. From the proof above, it is seen that to get the path continuity, it is enough that there exists a sequence $\{r_n\}$ with $r_n > 0$, $\lim_n r_n = 0$ such that the first exit distribution $P^x\big(X_{T_{B^c_{r_n}}} \in dy,\, T_{B^c_{r_n}} < \infty\big)$ concentrates on $S_{r_n}(x)$ for all $x \in E$ and all $r_n > 0$.

It is known that Lévy processes are spatially homogeneous Hunt processes on Euclidean space. Now suppose that $\{\Omega, \mathcal{F}, P^x, X_t\}$ is a Lévy Process on $\mathbb{R}^d$ with convolution semigroup $\{\pi_t\}$, and its generating triplet is (A, υ, γ). Suppose that $\pi_1 \neq \delta_0$. Then for any $r > 0$, $P^0(T_{B^c_r} < \infty) = 1$. By the Theorem 19.2 of [6], Theorem 2.1 and Lemma 2.1 above, we immediately have the following corollaries.

Corollary 2.1. *Let $\epsilon > 0$. Then the following three conditions are equivalent to each other:*
(1) *For all $r > 0$, $P^0(X_{T_{B^c_r}} \in dy)$ concentrates on $B_{r+\epsilon} \setminus B^\circ_r$.*
(2) *Almost surely for all $t > 0$, $d(X_{t-}, X_t) \le \epsilon$.*
(3) *That $\upsilon\{|x| > \epsilon\} = 0$.*

Corollary 2.2. *The following three conditions are equivalent to each other:*
(1) *For all $r > 0$, $P^0(X_{T_{B^c_r}} \in dy)$ concentrates on S_r.*
(2) *The sample paths are continuous on $[0, \infty)$ almost surely.*
(3) *The Lévy measure $\upsilon = 0$.*

Similarly, in both corollaries, it is enough that (1) holds for a sequence of positive numbers $\{r_n\}$ with $\lim r_n = 0$.

§ 3. Brownian Motion and the Hitting Distribution

In this section, we shall prove that a Lévy process on $\mathbb{R}^d$ whose exit distribution leaving any ball from the center is uniform on the sphere is a Brownian motion. Let $\{\Omega, \mathcal{F}, P^x, X_t\}$ be a genuinely d-dimensional Lévy process on $\mathbb{R}^d$ starting at 0 with generating triplet (A, υ, γ). If A is a subset of $\mathbb{R}^d$, we define a random variable $T_A(\omega) = \inf\{t > 0 : X_t(\omega) \in A\}$, which is the first hitting time of A. It is well known that the hitting distribution of a ball for the standard Brownian motion is given by the Poisson's kernel (see Theorem 3.1 of Chapter 4 in [5]).

Lemma 3.1. *Let $c, x \in \mathbb{R}^d$, $r > 0$ and $x \notin S_r(c)$. If X is the standard Brownian motion, then for any $f \in b\mathcal{B}(\mathbb{R}^d)$,*

$$E^x\{f[X(T_{S_r(c)})]\} = \int_{S_r} \frac{r^{d-2}|r^2 - |x|^2|}{|y - x|^d} f(y)\sigma_r(c, dy).$$

For $c_1, \cdots, c_d \in \mathbb{R}$, denote by $\mathrm{diag}\{c_1, \cdots, c_d\}$ the diagonalized matrix with diagonal entries $c_1, \cdots, c_d$.

Lemma 3.2. *Suppose that $\upsilon = 0$, $\gamma = 0$ and $A = \mathrm{diag}\{\lambda_1^2, \lambda_2^2, \cdots, \lambda_d^2\}$, where $\lambda_i \ge 1, 1 \le i \le d$. If there exist $1 \le i_0, i_1 \le d$ such that $\lambda_{i_0} = 1$ and $\lambda_{i_1} > 1$, then $P^0(X_{T_{S_r}} \in dy)$ is not the uniform distribution on the sphere $S_r = \{x : |x| = r\}$ for any $r > 0$.*

Proof. Without loss of generality, we assume that $\lambda_1 = 1$ and $\lambda_d > 1$. Let $C = \mathrm{diag}\{1, \lambda_2, \cdots, \lambda_d\}$. For any $t > 0$, let $Y_t = C^{-1}X_t$. Then $\{Y_t\}$ is the standard Brownian motion on $\mathbb{R}^d$ since

$$\widehat{P}_{Y_1}(z) = \widehat{P}_{C^{-1}X_1}(z) = \exp\Big\{ -\frac{1}{2}\langle C^{-1}z, AC^{-1}z\rangle\Big\} = \exp\Big\{ -\frac{1}{2}\langle z, z\rangle\Big\}.$$

Now let $n > 1$ and

$$E_1 = \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : {x_1}^2 + (\lambda_2 x_2)^2 + \cdots + (\lambda_d x_d)^2 = 1\}.$$

Obviously, $S_1 = \{Cx \in \mathbb{R}^d : x \in E_1\}$. Let $T_{S_1},\ T_{S_n}$ and T_{E_1} be the hitting times of S_1, S_n and E_1 for $\{X_t\}$ respectively. Let $\widehat{T}_{S_1},\ \widehat{T}_{S_n}$ and $\widehat{T}_{E_1}$ be the hitting times of $S_1,\ S_n$ and E_1 for $\{Y_t\}$ respectively. We assert that $P^0(X_{T_{S_1}} \in dy)$ is not the uniform distribution on S_1. Otherwise $E^0[f(Y_{\widehat{T}_{E_1}})] = \int_{S_1} f(C^{-1}x)\sigma_1(dx)$ for any $f \in b\mathcal{B}(\mathbb{R}^d)$. By the strong Markov property of $\{Y_t\}$ and Lemma 3.1, we have, for any $f \in b\mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}
E^0[f(Y_{\widehat{T}_{S_n}})] &= E^0\{(f(Y_{\widehat{T}_{S_n}})) \circ \theta_{\widehat{T}_{E_1}}\} \\
&= E^0\{E^{Y(\widehat{T}_{E_1})}[f(Y_{\widehat{T}_{S_n}})]\} = \int_{E_1} E^x(f(Y_{\widehat{T}_{S_n}}))\mu(dx) \\
&= \int_{E_1} \mu(dx) \int_{S_n} \frac{n^{d-2}|n^2 - |x|^2|}{|y-x|^d} f(y)\sigma_n(dy) \\
&= \int_{S_n} f(y)\sigma_n(dy) \int_{E_1} \frac{n^{d-2}|n^2 - |x|^2|}{|y-x|^d} \mu(dx) \\
&= \int_{S_n} f(y)\sigma_n(dy) \int_{S_1} \frac{n^{d-2}|n^2 - |C^{-1}x|^2|}{|y-C^{-1}x|^d} \sigma_1(dx),
\end{aligned}$$

where μ is the distribution of $Y(\widehat{T}_{E_1})$ under P^0. For any $y \in S_n$, let

$$g(y) = \int_{S_1} \frac{n^{d-2}|n^2 - |C^{-1}x|^2|}{|y-C^{-1}x|^d} \sigma_1(dx).$$

Then $E^0[f(Y_{\widehat{T}_{S_n}})] = \int_{S_n} f(y)g(y)\sigma_n(dy)$. Thus by Lemma 3.1, we get that for any arbitrary $f \in b\mathcal{B}(\mathbb{R}^d)$,

$$\int_{S_n} f(y)\sigma_n(dy) = \int_{S_n} f(y)g(y)\sigma_n(dy).$$

It follows that $g(y) = 1$ for all $y \in S_n$.

Now let $y_0 = (n, 0, 0, \cdots, 0)$. Then $y_0 \in S_n$ and

$$\begin{aligned}
g(y_0) &= \int_{S_1} \frac{n^{d-2}|n^2 - |C^{-1}x|^2|}{|y_0 - C^{-1}x|^d} \sigma_1(dx) \\
&= \int_{S_1} \frac{n^{d-2}|n^2 - |x|^2 + |x|^2 - |C^{-1}x|^2|}{|y_0 - C^{-1}x|^d} \sigma_1(dx) \\
&= \int_{S_1} \frac{n^{d-2}|n^2 - |x|^2 + (x_2^2 - (\frac{x_2}{\lambda_2})^2) + \cdots + (x_d^2 - (\frac{x_d}{\lambda_d})^2)}{|(n-x_1)^2 + x_2^2 + \cdots + x_n^2 + (\frac{x_2}{\lambda_2})^2 - x_2^2 + \cdots + (\frac{x_d}{\lambda_d})^2 - x_d^2|^{\frac{d}{2}}} \sigma_1(dx) \\
&> \int_{S_1} \frac{n^{d-2}|n^2 - |x|^2|}{|(n-x_1)^2 + x_2^2 + \cdots + x_d^2|^{\frac{d}{2}}} \sigma_1(dx) \\
&= \int_{S_1} \frac{n^{d-2}|n^2 - |x|^2|}{|y_0 - x|^d} \sigma_1(dx).
\end{aligned}$$

But on the other hand, for any arbitrary $f \in b\mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}
E^0[f(Y_{\widehat{T}_{S_n}})] &= E^0\{(f(Y_{\widehat{T}_{S_n}})) \circ \theta_{\widehat{T}_{S_1}}\} \\
&= E^0\{E^{Y(\widehat{T}_{S_1})}[f(Y_{\widehat{T}_{S_n}})]\} \\
&= \int_{S_1} E^x(f(Y_{\widehat{T}_{S_n}}))\sigma_1(dx) \\
&= \int_{S_1} \sigma_1(dx) \int_{S_n} \frac{n^{d-2}|n^2-|x|^2|}{|y-x|^d} f(y)\sigma_n(dy) \\
&= \int_{S_n} f(y)\sigma_n(dy) \int_{S_1} \frac{n^{d-2}|n^2-|x|^2|}{|y-x|^d}\sigma_1(dx)
\end{aligned}$$

which follows that $\displaystyle\int_{S_1} \frac{n^{d-2}|n^2-|x|^2|}{|y-x|^d}\sigma_1(dx) = 1$ for any $y \in S_n$. Therefore

$$g(y_0) > \int_{S_1} \frac{n^{d-2}|n^2-|x|^2|}{|y_0-x|^d}\sigma_1(dx) = 1.$$

This is a contradiction which yields that $P^0(X_{T_{S_1}} \in dy)$ is not the uniform distribution on S_1. Similarly, for any $r > 0$, $P^0(X_{T_{S_r}} \in dy)$ is not the uniform distribution on the sphere S_r.

Proposition 3.1. *Suppose that $\upsilon = 0$, $\gamma = 0$, and $A \neq \alpha I$ for any $\alpha > 0$. Then $P^0(X_{T_{S_r}} \in dy)$ is not the uniform distribution on the sphere S_r for any $r > 0$.*

Proof. Since X is genuinely d-dimensional, A is a symmetric positive-definite $d\times d$ matrix. Thus the eigenvalues of A, namely, $\lambda_1, \lambda_2, \ldots, \lambda_d$ are positive. Without loss of generality, we can assume that $\lambda_1 = \min\{\lambda_i : 1 \leq i \leq d\}$. There is an orthogonal matrix Q such that $QAQ^T = \operatorname{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_d\}$. If $\lambda_1 = \cdots = \lambda_d$, then $A = Q^T \operatorname{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_d\}Q = \lambda_1 I$ which contradicts our condition. Therefore there exists some $1 \leq i_0 \leq d$ such that $\lambda_{i_0} > \lambda_1$. For any $t > 0$, let $Y_t = \frac{QX_t}{\sqrt{\lambda_1}}$. Then for any $z \in R^d$, $\widehat{P}_{Y_1}(z) = \exp\{-\frac{1}{2\lambda_1}\langle Q^T z, AQ^T z\rangle\} = \exp\{-\frac{1}{2}\langle z, Bz\rangle\}$, where $B = \operatorname{diag}\{1, \frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1}, \ldots, \frac{\lambda_d}{\lambda_1}\}$. Consequently, $\{Y_t\}$ is a Lévy process on R^d with generating triplet $(B, 0, 0)$. For any $r > 0$, let T_{S_r} and $\widehat{T}_{S_r}$ denote the hitting times of S_r for $\{X_t\}$ and for $\{Y_t\}$ respectively. By Lemma 3.2, for any $r > 0$, $P^0(Y_{\widehat{T}_{S_r}} \in dy)$ is not the uniform distribution on S_r. Since Q is an orthogonal matrix, $|x| = |Qx|$ holds for all $x \in R^d$. Thus for any $r > 0$,

$$T_{S_r} = \inf\{t > 0 : |X_t| = r\} = \inf\left\{t > 0 : |Y_t| = \frac{r}{\sqrt{\lambda_1}}\right\} = \widehat{T}_{S_{r/\sqrt{\lambda_1}}}.$$

Hence $X_{T_{S_r}} = \sqrt{\lambda_1} Q^T Y_{\widehat{T}_{S_{r/\sqrt{\lambda_1}}}}$ and from the fact that Q is an orthogonal matrix it follows that $P^0(X_{T_{S_r}} \in dy)$ is not the uniform distribution on S_r.

Lemma 3.3. *If $\upsilon = 0$ and $\gamma = 0$, then $P^0(T_{S_r} > a) > 0$ for any $r > 0$ and any $a > 0$.*

Proof. Since X is genuinely d-dimensional, A is a symmetric positive-definite $d \times d$ matrix. There is a symmetric positive-definite $d \times d$ matrix B such that $A = B^2$. Let $Y_t = B^{-1}X_t$. Then $\{Y_t\}$ is the standard Brownian motion on R^d. There is a positive constant $c > 0$ such that $|X_t| = |BY_t| < c|Y_t|$ holds for all $t \geq 0$. For any $r > 0$, let $T^Y_{S_r}$ be the first hitting time of S_r for $\{Y_t\}$. Then $T_{S_{cr}} \geq T^Y_{S_r}$ for all $r > 0$ holds. Therefore we need only to show that $P^0(T^Y_{S_r} > a) > 0$ for any $r > 0$ and any $a > 0$. By [2], this is true when the dimension $d \geq 3$. Now we can suppose that $\{Y_t\}$ is the Brownian motion on R^3. Then $\{Y_1(t)\}$ and $\{(Y_1(t), Y_2(t))\}$ are the Brownian motion on R^1 and on R^2 respectively. Since $|Y_1(t)| \leq |Y(t)|$ and $|(Y_1(t), Y_2(t))| \leq |Y(t)|$ for all $t > 0$, our statement is also true when $d = 1$ or $d = 2$.

Now we come to our main result in this section.

Theorem 3.1. *Let $X = \{\Omega, \mathcal{F}, P^x, X_t\}$ be a genuinely d-dimensional Lévy Process on $\mathbb{R}^d$ with generating triplet (A, υ, γ). Then $P^0(X_{T_{B_r^c}} \in dy)$ is the uniform distribution on S_r for all $r > 0$ if and only if $\upsilon = 0$, $\gamma = 0$ and $A = \alpha I$ for some $\alpha > 0$ (that is , X is a scaled Brownian motion).*

Proof. The sufficiency can be deduced by the rotational invariance of Brownian motion directly. Now we prove the necessity. Suppose that $P^0(X_{T_{B_r^c}} \in dy)$ is the uniform distribution on S_r for all $r > 0$. By Corollary 2.2, we get $\upsilon = 0$. Thus X is a continuous Lévy process. So $P^0[T_{B_r^c} = T_{S_r}] = 1$ and then $P^0(X_{T_{S_r}} \in dy)$ is the uniform distribution on S_r for all $r > 0$. If $\gamma = 0$, then from Proposition 3.1, it follows that $A = \alpha I$ for some $\alpha > 0$. Thus we need only to show that $\gamma = 0$.

Let $H = A(R^d)$. It is easily seen that $\gamma = \gamma_1 + \gamma_2$ for some $\gamma_1 \in H$ and $\gamma_2 \in H^\perp$. If $H \neq R^d$, then $\gamma_2 \neq 0$ since otherwise $\{X_t\}$ is not genuinely d-dimensional. Let $Y_t = X_t - \gamma_2 t$. Then $\{Y_t\}$ is a Lévy process on R^d with generating triplet $(A, 0, \gamma_1)$. Since the linear space generated by $\operatorname{supp} P_{Y_1}$ is H, $P^0(Y_t \in H, t > 0) = 1$. Note that $X_t = Y_t + \gamma_2 t$. We have $P^0(\langle X_t, \gamma_2\rangle > 0, t > 0) = 1$. Hence $P^0(X_{T_{S_r}} \in \{z : \langle z, \gamma_2\rangle > 0\}) = 1$ for any $r > 0$. It follows that $P^0(X_{T_{S_r}} \in dy)$ is not the uniform distribution on S_r. Therefore $A(R^d) = R^d$, that is, A is symmetric positive-definite.

Without loss of generality, we may assume that $\gamma = (1, 0, \cdots, 0)$. Let $Y_t := X_t - t\gamma$ for $t \geq 0$. Then $\{Y_t\}$ is a symmetric Lévy Process with generating triplet $(A, 0, 0)$. Let $S_1^+ = \{x \in S_1 : x_1 > 0\}$, $S_1^- = \{x \in S_1 : x_1 \leq 0\}$, $B_1^{\circ+} = \{x \in B_1^\circ : x_1 > 0\}$ and $B_1^{\circ-} = \{x \in B_1^\circ : x_1 \leq 0\}$. Let $T_X = \inf\{t > 0 : X_t \in S_1\}$ and $T_Y = \inf\{t > 0 : Y_t \in S_1\}$. Since $\{x \in S_1 : x_1 = 0\}$ is a polar set for $\{Y_t\}$, $P^0(Y_{T_Y} \in S_1^+) = P^0(Y_{T_Y} \in S_1^-) = \frac{1}{2}$.

If $Y_{T_X}(\omega) \in B_1^{\circ+}$, then $X_{T_X}(\omega) = Y_{T_X}(\omega) + (T_X(\omega), 0, \cdots, 0) \in S_1^+$. If $Y_{T_X}(\omega) \in B_1^{\circ-}$ and $T_X(\omega) < T_Y(\omega)$, then $X_{T_X}(\omega) \in S_1^+$. Otherwise $X_{T_X}(\omega) \in S_1^-$ and then

$$|Y_{T_X}(\omega)| = |X_{T_X}(\omega) - (T_X(\omega), 0, \cdots, 0)| > |X_{T_X}(\omega)| = 1.$$

It follows that $T_X(\omega) \geq T_Y(\omega)$ which is a contradiction. Therefore $X_{T_X}(\omega) \in S_1^+$ whenever $T_X(\omega) < T_Y(\omega)$.

If $Y_{T_Y}(\omega) \in S_1^+$, then

$$|X_{T_Y}(\omega)| = |Y_{T_Y}(\omega) + (T_Y(\omega), 0, \cdots, 0)| > |Y_{T_Y}(\omega)| = 1.$$

Thus $T_X(\omega) < T_Y(\omega)$ and then $X_{T_X}(\omega) \in S_1^+$. Hence

$$P^0(X_{T_X} \in S_1^+, Y_{T_Y} \in S_1^+) = P^0(Y_{T_Y} \in S_1^+) = \frac{1}{2}.$$

By Lemma 3.3, we have $P^0(T_Y > 2) > 0$. It follows that

$$P^0(T_Y > 2, Y_{T_Y} \in S_1^-) = \frac{1}{2} P^0(T_Y > 2) > 0.$$

If $Y_{T_Y} \in S_1^-$ and $T_Y > 2$, then

$$|X_{T_Y}| = |Y_{T_Y} + (T_Y, 0, \cdots, 0)| \geq T_Y - |Y_{T_Y}| = T_Y - 1 > 1$$

and hence $T_X < T_Y$. It implies that

$$\begin{aligned}P^0(X_{T_X} \in S_1^+, Y_{T_Y} \in S_1^-) &\geq P^0(X_{T_X} \in S_1^+, Y_{T_Y} \in S_1^-, T_Y > 2)\\ &= P^0(X_{T_X} \in S_1^+, Y_{T_Y} \in S_1^-, T_Y > 2, T_X < T_Y)\\ &= P^0(Y_{T_Y} \in S_1^-, T_Y > 2, T_X < T_Y)\\ &= P^0(Y_{T_Y} \in S_1^-, T_Y > 2) > 0.\end{aligned}$$

Therefore we have

$$\begin{aligned}P^0(X_{T_X} \in S_1^+) &= P^0(X_{T_X} \in S_1^+, Y_{T_Y} \in S_1^+) + P^0(X_{T_X} \in S_1^+, Y_{T_Y} \in S_1^-)\\ &= \frac{1}{2} + P^0(X_{T_X} \in S_1^+, Y_{T_Y} \in S_1^-) > \frac{1}{2}.\end{aligned}$$

This contradicts the fact that that $P^0(X_{T_X} \in dy)$ is the uniform distribution on S_1. Consequently, $\gamma = 0$. That completes the proof.

Actually it is shown that a continuous Lévy process is a (scaled) Brownian motion if and only if the first exit distribution of X leaving some (then all) ball from the center is uniformly distributed on the sphere.

Acknowledgement. The author would like to thank Prof. R. K. Getoor for helpful comments and suggestions.

References

[1] Blumenthal, R. M. & Getoor, R. K., Markov Processes and Potential Analysis, Academic Press, 1968.

[2] Ciesielski, Z. & Taylar, J., First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of sample path, *Trans. Amer. Math. Soc.,* **103**(1962), 434–450.

[3] He, P. & Ying, J. G., Excursions and Lévy system of boundary process, *Chin. Ann. Math.,* **24B:**4(2003), 491–500.

[4] Oksendal, B. & Strook, D. W., A characterization of harmonic measure and Markov processes whose hitting distributions are preserved by rotations, translation and dilatations, *Ann. Inst. Fourier, Grenoble,* **32:**4(1982), 221–232.

[5] Port, S. C. & Stone, C. J., Brownian Motion and Classical Potential Theory, Academic Press, 1978.

[6] Sato, K., Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, 1999.