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POSITIVE PERIODIC SOLUTIONS OF FIRST AND SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS**

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Abstract

In this paper the existence results of positive ω-periodic solutions are obtained for second order ordinary differential equation $-u''(t) = f(t, u(t))$ $(t \in \mathbb{R})$, and also for first order ordinary differential equation $u'(t) = f(t, u(t))$ $(t \in \mathbb{R})$, where $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function which is ω-periodic in t. The discussion is based on the fixed point index theory in cones.

Keywords Positive periodic solution, Cone, Fixed point index
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§ 1. Introduction

The existence problems of periodic solutions for nonlinear ordinary differential equations, especially second order ordinary differential equations, have attracted many authors' attention and concern (see [1–10]). Many theorems and methods of nonlinear functional analysis have been applied to these problems. These theorems and methods are mainly the upper and lower solutions method and monotone iterative technique (see [1–4]), the continuation method of topological degree (see [5–7]), variational method and critical point theory (see [8–10]), etc.

In recent years the fixed point theorems of cone mapping, especially the fixed point theorem of Krasnoselskii's cone expansion or compression type, have been availably applied to the two-point boundary value problems of second order ordinary differential equations, and some results of existence and multiplicity of positive solutions have been obtained (see [11–14]). Lately, the present author [15] has also applied the Krasnoselskii's fixed point theorem to the periodic boundary value problems of second order nonlinear ordinary differential equations, and obtained existence results of positive periodic solutions. In this paper, we will use more precise theory of the fixed point index in cones to discuss the existence of positive periodic solutions of second order ordinary differential equation

$$-u''(t) = f(t,\, u(t)), \qquad t \in \mathbb{R}, \tag{1.1}$$

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and first order ordinary differential equation

$$u'(t) = f(t, u(t)), \qquad t \in \mathbb{R}, \tag{1.2}$$

where $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function which is ω-periodic in t, and which is not necessarily positive. We obtain the optimal conditions on the nonlinear term f so that Equation (1.1) and Equation (1.2) have a positive periodic solution. Our new existence result for Equation (1.1) is an improvement of the result in [15].

To be convenient, we introduce the following notations

$$\underline{f}_0 = \liminf_{u\to 0+} \min_{t\in[0,\omega]} (f(t,u)/u), \qquad \overline{f}_0 = \limsup_{u\to 0+} \max_{t\in[0,\omega]} (f(t,u)/u),$$

$$\underline{f}_\infty = \liminf_{u\to +\infty} \min_{t\in[0,\omega]} (f(t,u)/u), \qquad \overline{f}_\infty = \limsup_{u\to +\infty} \max_{t\in[0,\omega]} (f(t,u)/u).$$

The main results of this paper are

Theorem 1.1. *Suppose that $f(t,u) \in C(\mathbb{R}\times\mathbb{R}^+)$ and is ω-periodic in t. If one of the following conditions is satisfied*

(H1) $-\infty < \underline{f}_0$, $\quad \overline{f}_0 < 0 < \underline{f}_\infty$,

(H2) $-\infty < \underline{f}_\infty$, $\quad \overline{f}_\infty < 0 < \underline{f}_0$,

then the second order ordinary differential equation (1.1) *has at least one positive ω-periodic solution.*

Theorem 1.2. *Suppose that $f(t,u) \in C(\mathbb{R}\times\mathbb{R}^+)$ and is ω-periodic in t. If condition* (H1) *or condition* (H2) *is satisfied, then the first order ordinary differential equation* (1.2) *has at least one positive ω-periodic solution.*

Remark 1.1. Noting that 0 is an eigenvalue of associated linear eigenvalue problems of Equation (1.1) or Equation (1.2) with periodic boundary condition, if one inequality concerning comparison with 0 in (H1) or (H2) of Theorem 1.1 or Theorem 1.2 is not true, the existence of periodic solution to Equation (1.1) or Equation (1.2) can not be guaranteed. Hence, the 0 is the optimal value in conditions (H1) and (H2).

§ 2. Preliminaries

If (H1) or (H2) is satisfied, it is easy to prove that $f(t, u)/u$ is lower-bounded for $t \in [0,\omega]$ and $u \geq 0$. By the periodicity of $f(t, u)$ in t, there exists $M > 0$ such that

$$f(t, u) \geq -Mu, \qquad \forall t \in \mathbb{R},\ u \geq 0. \tag{2.1}$$

Let $f_1(t, u) = f(t, u) + Mu$, then $f_1(t, u) \geq 0$ for $t \in \mathbb{R}, u \geq 0$. Thus Equation (1.1) is equivalent to

$$-u''(t) + Mu(t) = f_1(t, u(t)), \qquad t \in \mathbb{R}, \tag{2.2}$$

and Equation (1.2) is equivalent to

$$u'(t) + Mu(t) = f_1(t, u(t)), \qquad t \in \mathbb{R}. \tag{2.3}$$

In the following, we mainly consider the second order differential equation (2.2), and the first order differential equation (2.3) can be dealt with in a similar way. Let $r_2(t)$ be unique solution of linear second order boundary value problem

$$\begin{cases} -u''(t) + Mu(t) = 0, \qquad 0 \leq t \leq \omega, \\ u(0) = u(\omega), \quad u'(0) = u'(\omega) - 1, \end{cases}$$

which is explicitly given by

$$r_2(t) = \frac{\cosh\beta(t-\frac{\omega}{2})}{2\beta\,\sinh\frac{\beta\omega}{2}}, \qquad 0 \le t \le \omega, \tag{2.4}$$

where $\beta = \sqrt{M}$.

Let $C_\omega(\mathbb{R})$ denote the Banach space of all continuous ω-periodic function $u(t)$ with norm $\|u\| = \max\limits_{0\le t\le\omega} |u(t)|$. Let $C_\omega^+(\mathbb{R})$ be the cone of all nonnegative functions in $C_\omega(\mathbb{R})$. For $h \in C_\omega(\mathbb{R})$, we consider the associated linear differential equation of Equation (2.2),

$$-u''(t) + Mu(t) = h(t), \qquad t \in \mathbb{R}. \tag{2.5}$$

We have

Lemma 2.1. *Let $h \in C_\omega(\mathbb{R})$. The linear equation (2.5) has a unique ω-periodic solution $u(t)$ which is given by*

$$u(t) = \int_{t-\omega}^{t} r_2(t-s)h(s)\,ds, \qquad t \in \mathbb{R}. \tag{2.6}$$

Proof. Making Derivation to Equation (2.6) and using the boundary condition of $r_2(t)$, we obtain that

$$\begin{aligned} u''(t) &= (r'(0) - r'(\omega))h(t) + \int_{t-\omega}^{t} {r_2}''(t-s)h(s)\,ds \\ &= -h(t) - M\int_{t-\omega}^{t} r_2(t-s)h(s)\,ds \\ &= -h(t) - Mu(t)\,. \end{aligned}$$

Therefore, $u(t)$ satisfies the equation (2.5). Let $\tau = s + \omega$, it follows from (2.6) that

$$\begin{aligned} u(t) &= \int_{t}^{t+\omega} r_2(t+\omega-\tau)h(\tau-\omega)\,d\tau \\ &= \int_{t}^{t+\omega} r_2(t+\omega-\tau)h(\tau)\,d\tau \;=\; u(t+\omega). \end{aligned}$$

Hence, $u(t)$ is a ω-periodic solution of Equation (2.5). From the maximum principle for second order periodic boundary value problems (see [2]), it is easy to see that $u(t)$ is the unique ω-periodic solution of Equation (2.5).

Remark 2.1. Let $h \in C_\omega(\mathbb{R})$. Then the linear first order differential equation

$$u'(t) + Mu(t) = h(t), \qquad t \in \mathbb{R} \tag{2.7}$$

has a unique ω-periodic solution $u(t)$ given by

$$u(t) = \int_{t-\omega}^{t} r_1(t-s)h(s)\,ds, \qquad t \in \mathbb{R},$$

where $r_1(t)$ is the unique solution of linear first order boundary value problem

$$\begin{cases} u'(t) + Mu(t) = 0, \qquad 0 \le t \le \omega, \\ u(0) = u(\omega) + 1, \end{cases}$$

which is expressed by

$$r_1(t) = \frac{e^{-Mt}}{1 - e^{-Mt}}, \qquad 0 \le t \le \omega.$$

By Lemma 2.1, if $h \in C_\omega^+(\mathbb{R})$ and $h(t) \not\equiv 0$, then the ω-periodic solution of Equation (2.5) $u(t) > 0$ for every $t \in \mathbb{R}$, and we term it the positive ω-periodic solution.

We now define a mapping $A : C_\omega^+(\mathbb{R}) \to C_\omega^+(\mathbb{R})$ by

$$(Au)(t) = \int_{t-\omega}^{t} r_2(t-s) f_1(s,\, u(s))\, ds. \tag{2.8}$$

By Lemma 2.1, positive ω-periodic solution of Equation (1.1) is equivalent to nontrivial fixed point of A. We will find the non-zero fixed point by using the fixed point index theory in cones. Choosing the sub-cone K of $C_\omega^+(\mathbb{R})$ by

$$K = \{u \in C_\omega^+(\mathbb{R}) \mid u(t) \ge \delta \|u\|,\ \forall t \in \mathbb{R}\ \},$$

where $\delta = \left(\cosh \frac{\beta\omega}{2}\right)^{-1}$, we have

Lemma 2.2. *$A(K) \subset K$, and $A : K \to K$ is completely continuous.*

Proof. From (2.4) it is easy to see that

$$\frac{1}{2\beta \sinh \frac{\beta\omega}{2}} \ \le\ r_2(t) \ \le\ \frac{\cosh \frac{\beta\omega}{2}}{2\beta \sinh \frac{\beta\omega}{2}}\ , \qquad 0 \le t \le \omega. \tag{2.9}$$

Let $u \in K$. From (2.8) and the latter inequality above we have

$$\begin{aligned}(Au)(t) &\le \frac{\cosh \frac{\beta\omega}{2}}{2\beta \sinh \frac{\beta\omega}{2}} \int_{t-\omega}^{t} f_1(s,\, u(s))\, ds \\ &= \frac{\cosh \frac{\beta\omega}{2}}{2\beta \sinh \frac{\beta\omega}{2}} \int_{0}^{\omega} f_1(s,\, u(s))\, ds,\end{aligned}$$

and therefore

$$\|Au\| \le \frac{\cosh \frac{\beta\omega}{2}}{2\beta \sinh \frac{\beta\omega}{2}} \int_{0}^{\omega} f_1(s,\, u(s))\, ds.$$

Using (2.8) and the former inequality of (2.9), we obtain that

$$\begin{aligned}(Au)(t) &\ge \frac{1}{2\beta \sinh \frac{\beta\omega}{2}} \int_{t-\omega}^{t} f_1(s,\, u(s))\, ds \\ &= \frac{1}{2\beta \sinh \frac{\beta\omega}{2}} \int_{0}^{\omega} f_1(s,\, u(s))\, ds \\ &\ge \left(\cosh \frac{\beta\omega}{2}\right)^{-1} \|Au\|,\end{aligned}$$

which implies $Au \in K$. Thus $A(K) \subset K$.

Obviously, $A : K \to K$ is continuous. Let $D \subset K$ be a bounded set. For every $u \in D$, since

$$(Au)'(t) = \int_{t-\omega}^{t} {r_2}'(t-s) f_1(s,\, u(s))\, ds,$$

it follows that $\{(Au)' \mid u \in D\}$ is a bounded set. Consequently, $A(D)$ is an equicontinuous and bounded family of functions. Thus, by Arzela-Ascoli's theorem, $A : K \to K$ is completely continuous.

We recall some concepts and conclusions on the fixed point index in [16], which will be used in the proof of Theorem 1.1 and Theorem 1.2. Let E be a Banach space and $K \subset E$ be a closed convex cone in E. Assume Ω is a bounded open subset of E with boundary $\partial\Omega$, and $K \cap \Omega \neq \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $Au \neq u$ for any $u \in K \cap \partial\Omega$, then the fixed point index $i\,(A, K \cap \Omega, K)$ has definition. One important fact is that if $i\,(A, K \cap \Omega, K) \neq 0$, then A has a fixed point in $K \cap \Omega$.

For $r > 0$, let $K_r = \{u \in K \mid \|u\| < r\}$, and $\partial K_r = \{u \in K \mid \|u\| = r\}$, which is the relative boundary of K_r in K. The following two lemmas are needed in our argument.

Lemma 2.3. (cf. [16]) *Let $A : K \to K$ be completely continuous mapping. If $\lambda Au \neq u$ for every $u \in \partial K_r$ and $0 < \lambda \le 1$, then $i\,(A, K_r, K) = 1$.*

Lemma 2.4. (cf. [16]) *Let $A : K \to K$ be completely continuous mapping. Suppose that the following two conditions are satisfied:*

(i) $\inf\limits_{u \in \partial K_r} \|Au\| > 0$.

(ii) $\lambda Au \neq u$ *for every $u \in \partial K_r$ and $\lambda \ge 1$.*

Then, $i\,(A, K_r, K) = 0$.

§ 3. Proof of the Main Results

In this section we only prove Theorem 1.1, and Theorem 1.2 can be proved in a completely analogical way.

Proof of Theorem 1.1. We show respectively that the mapping A defined by (2.8) has a non-zero fixed point in both cases that (H1) and (H2) are satisfied.

Case (i) Assume (H1) is satisfied. From the assumption of $\overline{f}_0 < 0$ and the definition of $\overline{f}_0$, there exist $\varepsilon \in (0, M)$ and $r_0 > 0$, such that

$$f(t, u) \le -\varepsilon\, u, \qquad \forall\, t \in [0, \omega],\ 0 \le u \le r_0. \tag{3.1}$$

Let $r \in (0, r_0)$, we now prove that A satisfies the hypothesis of Lemma 2.3 in K_r, namely $\lambda\, Au \neq u$ for every $u \in \partial K_r$ and $0 < \lambda \le 1$. In fact, if there exist $u_0 \in \partial K_r$ and $0 < \lambda_0 \le 1$ such that $\lambda_0\, Au_0 = u_0$, then by the definition of A and Lemma 2.1, $u_0(t)$ satisfies the differential equation

$$-{u_0}''(t) + M u_0(t) = \lambda_0\, f_1(t, u_0(t)), \qquad t \in \mathbb{R}. \tag{3.2}$$

From (3.1), (3.2) and the definition of f_1, it follows that

$$-{u_0}''(t) + M u_0(t) \le \lambda_0 (M u_0(t) - \varepsilon\, u_0(t)\,) \le (M - \varepsilon)\, u_0(t).$$

Integrating the both sides of this inequality from 0 to ω and using the periodicity of $u_0(t)$, we get

$$M \int_0^\omega u_0(t)\, dt \le (M - \varepsilon) \int_0^\omega u_0(t)\, dt.$$

By the definition of K, $u_0(t) \ge \delta \|u_0\| = \delta\, r$, and therefore $\int_0^\omega u_0(t)\, dt > 0$. It follows that $M \le M - \varepsilon$, which is a contradiction. Hence by Lemma 2.3, we have

$$i\,(A,\, K_r,\, K) = 1. \tag{3.3}$$

On the other hand, since $\underline{f}_\infty > 0$, there exist $\varepsilon > 0$ and $H > 0$ such that

$$f(t,\, u) \geq \varepsilon\, u, \qquad \forall\, t \in [0,\, \omega],\ u \geq H. \tag{3.4}$$

Choose $R > \max\{\frac{H}{\delta},\, r_0\}$, then $0 < r < R < +\infty$. Let $u \in \partial K_R$. Since $u(t) \geq \delta\|u\| > H$ for every $t \in \mathbb{R}$, from (2.9), (3.4) and the definition of K it follows that

$$\begin{aligned} \|Au\| \geq (Au)(\omega) &= \int_0^\omega r_2(\omega - s)\, f_1(s,\, u(s))\, ds \\ &\geq \frac{1}{2\beta \sinh \frac{\beta\omega}{2}} \int_0^\omega f_1(s,\, u(s))\, ds \\ &\geq \frac{M+\varepsilon}{2\beta \sinh \frac{\beta\omega}{2}} \int_0^\omega u(s)\, ds \\ &\geq \frac{(M+\varepsilon)\, \delta\, \omega}{2\beta \sinh \frac{\beta\omega}{2}}\, \|u\|. \end{aligned} \tag{3.5}$$

Hence $\inf\limits_{u \in \partial K_R} \|Au\| > 0$, namely the hypothesis (i) of Lemma 2.4 is satisfied. Next we show that if R is large enough, then $\lambda\, Au \neq u$ for every $u \in \partial K_R$ and $\lambda \geq 1$. In fact, if there exist $u_0 \in \partial K_R$ and $\lambda_0 \geq 1$ such that $\lambda_0\, Au_0 = u_0$, then $u_0(t)$ satisfies the equation (3.2). Set $C = \max\limits_{0 \leq t \leq \omega,\, 0 \leq u \leq H} (|f(t,\, u)| + \varepsilon H)$. Then it is clear from (3.4) to see that

$$f(t, u) \geq \varepsilon\, u - C, \qquad \forall\, t \in \mathbb{R},\ u \geq 0.$$

From this and (3.2) it follows that

$$-{u_0}''(t) + Mu_0(t) \geq f_1(t,\, u_0(t)) \geq (M+\varepsilon)\, u_0(t) - C.$$

Integrating this inequality from 0 to ω, we get

$$M \int_0^\omega u_0(t)\, dt \geq (M+\varepsilon) \int_0^\omega u_0(t)\, dt - C\omega,$$

which implies that

$$\int_0^\omega u_0(t)\, dt \leq \frac{C\omega}{\varepsilon}. \tag{3.6}$$

Since $u_0 \in K$, by the definition of K, $u_0(t) \geq \delta\|u_0\|$, which implies that $\int_0^\omega u_0(t)\, dt \geq \omega\delta\, \|u_0\|$. Thus from (3.6) it follows that

$$\|u_0\| \leq \frac{1}{\delta\omega} \int_0^\omega u_0(t)\, dt \leq \frac{C}{\delta\, \varepsilon}. \tag{3.7}$$

Let $R > \max\{\frac{H}{\delta},\, r_0,\, \frac{C}{\delta\varepsilon}\}$. Then for every $u \in \partial K_R$ and $\lambda \geq 1$, $\lambda\, Au \neq u$. Hence the hypothesis (ii) of Lemma 2.4 is also satisfied. By Lemma 2.4, we obtain that

$$i\,(A,\, K_R,\, K) = 0. \tag{3.8}$$

From (3.3), (3.8) and the additivity of fixed point index, we have

$$i\,(A,\, K_R \setminus \overline{K}_r,\, K) = i(A,\, K_R,\, K) - i(A,\, K_r,\, K) = -1.$$

Therefore A has a fixed point in $K_R \setminus \overline{K}_r$, which is the positive ω-periodic solution of Equation (1.1).

Case (ii) Assume (H2) is satisfied. Since $\underline{f}_0 > 0$, there exist $\varepsilon > 0$ and $\eta > 0$ such that

$$f(t,u) \geq \varepsilon\, u, \qquad \forall\, t \in [0,\,\omega],\ 0 \leq u \leq \eta. \tag{3.9}$$

Let $r \in (0,\,\eta)$. Then for every $u \in \partial K_r$, through the argument analogous to (3.5), we have

$$\|Au\| \ \geq\ \frac{(M+\varepsilon)\,\delta\,\omega}{2\beta\,\sinh\frac{\beta\omega}{2}}\ \|u\|.$$

Hence $\inf\limits_{u\in\partial K_r} \|Au\| > 0$. we now show that $\lambda\, Au \neq u$ for every $u \in \partial K_r$ and $\lambda \geq 1$. In fact, if there exist $u_0 \in \partial K_r$ and $\lambda_0 \geq 1$ such that $\lambda_0\, Au_0 = u_0$, then $u_0(t)$ satisfies the equation (3.2), and from (3.2) and (3.9) it follows that

$$-u_0''(t) + Mu_0(t) \geq f_1(t,\,u_0(t)) \geq (M+\varepsilon)\,u_0(t).$$

Integrating this inequality from 0 to ω, we get

$$M\int_0^{\omega} u_0(t)\,dt \geq (M+\varepsilon)\int_0^{\omega} u_0(t)\,dt.$$

Since $\int_0^{\omega} u_0(t)\,dt > 0$, from the above inequality we see that $M \geq M+\varepsilon$, which is a contradiction. Hence A satisfies the hypotheses of Lemma 2.4 in K_r. By Lemma 2.4,

$$i\,(A,\,K_r,\,K) = 0. \tag{3.10}$$

Since $\overline{f}_\infty < 0$, there exist $\varepsilon \in (0,\,M)$ and $H > 0$ such that

$$f(t,\,u) \leq -\varepsilon\, u, \qquad \forall\, t \in [0,\,\omega],\ u \geq H.$$

Set $C = \max\limits_{0\leq t\leq\omega,\ 0\leq u\leq H}(|f(t,\,u)| + \varepsilon H)$. It is clear that

$$f(t,\,u) \leq\ -\varepsilon\, u + C, \qquad \forall\, t \in \mathbb{R},\ u \geq 0. \tag{3.11}$$

If $0 < \lambda_0 \leq 1$ and $u_0 \in K$ satisfy $\lambda_0\, Au_0 = u_0$, then (3.2) is valid. From (3.2) and (3.11), it follows that

$$-u_0''(t) + Mu_0(t) \leq f_1(t,\,u_0(t)) \leq (M-\varepsilon)\,u_0(t) + C.$$

Integrating this inequality, we get

$$\int_0^{\omega} u_0(t)\,dt \leq \frac{C\omega}{\varepsilon}.$$

Noticing $\int_0^{\omega} u_0(t)\,dt \geq \omega\delta\,\|u_0\|$, we see that u_0 satisfies (3.7). Choose $R > \max\{\frac{C}{\delta\varepsilon},\,\eta\}$, then $\lambda\, Au \neq u$ for any $u \in \partial K_R$ and $0 < \lambda \leq 1$. Therefore, A satisfies the hypothesis of Lemma 2.3 in K_R. By Lemma 2.3,

$$i\,(A,\,K_R,\,K) = 1. \tag{3.12}$$

From (3.10) and (3.12), it follows that

$$i\,(A,\,K_R \setminus \overline{K}_r,\,K) = i(A,\,K_R,\,K) - i(A,\,K_r,\,K) = 1.$$

Therefore A has a fixed point in $K_R \setminus \overline{K}_r$, which is the positive ω-periodic solution of Equation (1.1).

The proof is completed.

Example 3.1. Consider the second order differential equation

$$-u'' = a_1(t)u + a_2(t)u^2 + \cdots + a_n(t)u^n, \qquad t \in \mathbb{R}, \tag{3.13}$$

where $n \geq 2,\ a_i(t) \in C_\omega(\mathbb{R}),\ i = 1,\ 2,\ \cdots,\ n$. If $a_1(t) < 0,\ a_n(t) > 0$, for $t \in [0,\ \omega]$, then $f(t,\ u) = a_1(t)u + a_2(t)u^2 + \cdots + a_n(t)u^n$ satisfies the condition (H1) of Theorem 1.1. By Theorem 1.1, Equation (3.13) has at least one positive ω-periodic solution.

Example 3.2. Consider the following differential equation

$$-u'' = a(t)\sqrt{u} - b(t)\,u + h(t), \qquad t \in \mathbb{R},$$

where $a(t),\ b(t),\ h(t) \in C_\omega(\mathbb{R})$, and $a(t),\ b(t),\ h(t) > 0$ for $t \in [0,\ \omega]$. It is easy to verify that the condition (H2) is satisfied. By Theorem 1.1, this equation has at least one positive ω-periodic solution.

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