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ALMOST GLOBAL EXISTENCE OF DIRICHLET INITIAL-BOUNDARY VALUE PROBLEM FOR NONLINEAR ELASTODYNAMIC SYSTEM OUTSIDE A STAR-SHAPED DOMAIN***

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Abstract

This paper deals with the mixed initial-boundary value problem of Dirichlet type for the nonlinear elastodynamic system outside a star-shaped domain. The almost global existence of solution with small initial data to this problem is proved and a lower bound for the lifespan of solutions is given.

Keywords Nonlinear elastodynamic system, Exterior problem, Almost global existence
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§ 1 . Introduction

It is well known that the displacement $u = (u^1, u^2, u^3) = u(t, x)$ of an isotropic, homogeneous hyperelastic material without the action of external force satisfies the following quasilinear hyperbolic system (cf. [2, 13])

$$Lu = \partial_t^2 u - c_2^2 \triangle u - (c_1^2 - c_2^2)\nabla \mathrm{div} u = F(\nabla u, \nabla^2 u), \tag{1.1}$$

where $F = (F^1, F^2, F^3)$,

$$F^i(\nabla u, \nabla^2 u) = \sum_{j,l,m=1}^{3} C_{ij}^{lm}(\nabla u)\partial_l \partial_m u^j, \qquad i = 1, 2, 3 \tag{1.2}$$

with

$$C_{ij}^{lm}(\nabla u) = \sum_{h,n=1}^{3} C_{ijh}^{lmn} \partial_n u^h, \qquad \nabla = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\Big), \tag{1.3}$$

and c_1, c_2 are given by the Lamé constants λ, μ:

$$c_1^2 = \lambda + 2\mu, \qquad c_2^2 = \mu. \tag{1.4}$$

We assume that $\mu > 0,\ \lambda + \mu > 0$.

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In this paper, we consider the almost global existence of classical solutions with small initial data to the above system outside a star-shaped domain. There have been many results on the almost global existence of solutions for nonlinear wave equations. By using Lorentz invariance of the wave operator, F. John and S. Klainerman proved in [5] the almost global existence of classical solutions to the initial value problem of nonlinear wave equations. Later, the same result was shown by S. Klainerman and T. C. Sideris [12] without relying on Lorentz invariance. Recently, M. Keel, H. F. Smith and C. D. Sogge [9, 10] showed the almost global existence of classical solutions with small initial data for the Cauchy problem of semilinear and quasilinear wave equations respectively in a simpler way. Furthermore, they extended this result to the exterior problem and proved the almost global existence of solutions to the Dirichlet initial-boundary value problem for the semilinear and quasilinear wave equations outside a star-shaped domain, respectively (cf. [9, 10]). They also showed in [8] the global existence of solutions to this problem in the case satisfying the null condition (on the null condition, for example, see [14, 17]).

For the nonlinear elastodynamic system some results have been obtained only for the Cauchy problem. Combining the methods in [5] and [11] and applying the estimates on the fundamental solution of the linear elastic operator, F. John [4] proved the almost global existence of solutions to the initial value problem for the nonlinear elastodynamic system. Without relying on the estimations on the fundamental solution of the linear elastic operator and the Lorentz invariance, S. Klainerman and T. C. Sideris [12] showed the same result by applying energy estimates and Klainerman-Sobolev inequalities. In [19], we derived the same result by using an approach similar to that in [10].

In this paper, we discuss the almost global existence of solutions to the exterior problem outside an obstacle for the nonlinear elastodynamic system by applying the estimates similar to those in [10]. The key steps in the proof are pointwise estimates and weighted L^2 estimates. In order to get the pointwise estimates, we apply the exponential decay of local energy given by B. V. Kapitonov for the linear elastodynamic system (cf. [6, 7]). For getting the weighted L^2 estimates, we use the decomposition of linear elastic waves into longitudinal and transverse waves.

§ 2. Preliminaries and Main Results

The time-space gradient is denoted by

$$\partial = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_0, \nabla), \qquad \partial u = u',$$

where

$$\partial_0 = \partial_t = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i} \qquad (i = 1, 2, 3).$$

We also use the vector fields

$$\widetilde{\Omega} = \Omega I + U,$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla$ are the angular momentum operators with

$$U^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad U^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad U^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Set

$$Z = \{\partial_t I, \partial_1 I, \partial_2 I, \partial_3 I, \widetilde{\Omega}\} = \{\partial I, \widetilde{\Omega}\} = \{Z_0, Z_1, \cdots, Z_6\},$$

$$S = t\partial_t + r\partial_r = t\partial_t + x \cdot \nabla_x.$$

It is obvious that

$$[L, Z^\alpha] = 0, \qquad [S, L] = -2L. \tag{2.1}$$

By Proposition 2.1 in [1], we know that

$$C_{ij}^{lm}(\nabla u) = C_{ij}^{ml}(\nabla u) = C_{ji}^{lm}(\nabla u). \tag{2.2}$$

Assuming that the obstacle $\mathcal{K}$ is a smooth, closed and strictly star-shaped domain with respect to the origin, we consider the initial-boundary value problem of the quasilinear elastodynamic system

$$\begin{cases} Lu = F(\nabla u, \nabla^2 u), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K}, \\ u(t,\cdot)|_{\partial\mathcal{K}} = 0, \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g. \end{cases} \tag{2.3}$$

Suppose that the compatibility conditions are satisfied. Let $J_k(u) = \{\partial_x^\alpha u \mid 0 \le |\alpha| \le k\}$ denote the set of all spatial derivatives of u up to order k. For any given integer m, if u is a H^m solution of (2.3), then $\partial_t^k u$ at $t = 0$ can be uniquely defined by $J_k f$ and $J_{k-1} g$ $(0 \le k \le m-1)$. Setting $\partial_t^k u(0,\cdot) = \psi_k(J_k f,\ J_{k-1} g)$, the compatibility conditions for problem (2.3) with $(f,g) \in H^m \times H^{m-1}$ are just the requirement that $\psi_k = 0$ on $\partial\mathcal{K}$ $(0 \le k \le m-1)$. Moreover, $(f,g) \in C^\infty$ satisfies the compatibility conditions of infinite order if these conditions hold for all m.

Our main result is the following

Theorem 2.1. *Assume that $(f,g) \in C^\infty(\mathbb{R}^3 \setminus \mathcal{K})$ satisfies the compatibility conditions of infinite order. Then there are a number $\varepsilon_0 > 0$ and an integer $N > 0$ such that for any given ε with $0 < \varepsilon < \varepsilon_0$, if*

$$\sum_{|\alpha| \le N} \|(\langle x\rangle \partial_x)^\alpha f\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} + \sum_{|\alpha| \le N-1} \|(\langle x\rangle \partial_x)^\alpha g\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \le \varepsilon, \tag{2.4}$$

where $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$, then problem (2.3) admits a unique solution $u \in C^\infty([0,T_\varepsilon) \times \mathbb{R}^3 \setminus \mathcal{K})$ with

$$T_\varepsilon = \exp\Big(\frac{c}{\varepsilon}\Big),$$

where c is a positive constant.

In order to prove the above theorem, we need the almost global existence of solutions to the Cauchy problem for the elastodynamic system and some relevant estimates. The following results (Theorem 2.2) and estimates (Lemma 2.1–Lemma 2.5), given in [19], are needed in what follows.

Theorem 2.2. *There are a number $\varepsilon_0 > 0$ and an integer $N > 0$ such that for any given ε with $0 < \varepsilon < \varepsilon_0$ and any given data $(f,g) \in C^\infty(\mathbb{R}^3)$ satisfying*

$$\sum_{|\alpha| \le N} \|(\langle x\rangle \partial_x)^\alpha f\|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \le N-1} \|(\langle x\rangle \partial_x)^\alpha g\|_{L^2(\mathbb{R}^3)} \le \varepsilon, \tag{2.5}$$

the problem

$$\begin{cases} Lu = F(\nabla u, \nabla^2 u), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x) \end{cases} \tag{2.6}$$

admits a unique solution $u(t,x)\in C^\infty([0,T_\varepsilon)\times\mathbb{R}^3)$ *with*

$$T_\varepsilon=\exp\left(\frac{c}{\varepsilon}\right),$$

where c *is a positive constant.*

Lemma 2.1. *The Cauchy problem*

$$\begin{cases}Lu=F(t,x), & (t,x)\in\mathbb{R}_+\times\mathbb{R}^3,\\ u(0,\cdot)=\partial_t u(0,\cdot)=0\end{cases}\tag{2.7}$$

can be decomposed into the following two Cauchy problems of wave equations

$$\begin{cases}\partial_t^2u_1-c_1^2\triangle u_1=F_1,\\ t=0: u_1=0, \quad \partial_t u_1=0\end{cases}\qquad \text{and} \qquad \begin{cases}\partial_t^2u_2-c_2^2\triangle u_2=F_2,\\ t=0: u_2=0, \quad \partial_t u_2=0,\end{cases}$$

where $u=u_1+u_2$, $F=F_1+F_2$ *and*

$$\|F_1\|_{L^2(\mathbb{R}^3)}\le C\|F\|_{L^2(\mathbb{R}^3)}, \qquad \|F_2\|_{L^2(\mathbb{R}^3)}\le C\|F\|_{L^2(\mathbb{R}^3)},$$

here and henceforth C *denotes a positive constant.*

Lemma 2.2. *Suppose that* u *solves*

$$\begin{cases}Lu=F(t,x), & (t,x)\in\mathbb{R}_+\times\mathbb{R}^3,\\ u(0,x)=f(x), \quad \partial_t u(0,x)=g(x).\end{cases}\tag{2.8}$$

Then

$$(1+t)^{-\frac12}\|u'\|_{L^2([0,t]\times\mathbb{R}^3)}\le C\|u'(0,\cdot)\|_{L^2(\mathbb{R}^3)}+C\int_0^t\|F(s,\cdot)\|_{L^2(\mathbb{R}^3)}ds,\tag{2.9}$$

$$\|u'\|_{L^2\{[0,t],|x|<1\}}\le C\|u'(0,\cdot)\|_{L^2(\mathbb{R}^3)}+C\int_0^t\|F(s,\cdot)\|_{L^2(\mathbb{R}^3)}ds,\tag{2.10}$$

$$\|u\|_{L^2([0,t],|x|<1)}\le C\|u'(0,\cdot)\|_{L^2(\mathbb{R}^3)}+C\int_0^t\|F(s,\cdot)\|_{L^2(\mathbb{R}^3)}ds.\tag{2.11}$$

Lemma 2.3. *Suppose that* u *solves problem* (2.8). *Then*

$$(\ln(2+t))^{-\frac12}\|(1+|x|)^{-\frac12}u'\|_{L^2([0,t]\times\mathbb{R}^3)}\le C\|u'(0,\cdot)\|_{L^2(\mathbb{R}^3)}+C\int_0^t\|F(s,\cdot)\|_{L^2(\mathbb{R}^3)}ds.\tag{2.12}$$

Lemma 2.4. *Suppose that* u *solves problem* (2.7). *Then*

$$\begin{aligned}|x|\ |u(t,x)|&\le C\int_0^t\Big(\int_{|r-c_1(t-\tau)|}^{r+c_1(t-\tau)}+\int_{|r-c_2(t-\tau)|}^{r+c_2(t-\tau)}\Big)\sup_{|\theta|=1}|F(\tau,\rho\theta)|\rho d\rho ds\\ &\quad+C\int_0^t\int_{c_2}^{c_1}\int_{|r-l(t-\tau)|}^{r+l(t-\tau)}\sup_{|\theta|=1}|F(\tau,\rho\theta)|\rho d\rho dl ds\\ &\le C\int_0^t\int_{\min\limits_{i=1,2}|r-c_i(t-s)|}^{r+c_1(t-s)}\sup_{|\theta|=1}|F(s,\rho\theta)|\rho d\rho ds.\end{aligned}\tag{2.13}$$

Lemma 2.5. *Suppose that u solves problem* (2.7). *Then*

$$t|u(t,x)| \le C\int_0^t\int_{\mathbb{R}^3}\sum_{|\alpha|\le 2, m\le 1}|S^m\widetilde{\Omega}^\alpha F(s,y)|\frac{dyds}{|y|}. \tag{2.14}$$

From the proof of the almost global existence of Cauchy problem in [19], we know that if $N \ge 9$ and estimate (2.5) holds, then the solution u to problem (2.6) satisfies

$$\begin{aligned}&\sup_{0\le t\le T_\varepsilon}\sum_{|\alpha|+m\le N, m\le 1}\|\langle x\rangle^{-\frac{1}{2}}S^m Z^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3)}\\ &+(\ln(2+t))^{-\frac{1}{2}}\sum_{|\alpha|+m\le N-1, m\le 1}\|S^m Z^\alpha u'\|_{L^2([0,T_\varepsilon]\times\mathbb{R}^3)}\le C\varepsilon.\end{aligned} \tag{2.15}$$

In what follows, we need also the following local existence theorem for the exterior problem (see [18]) and the exponential decay of local energy for the linear elastodynamic system given by B. V. Kapitonov [7] (also cf. [6]).

Theorem 2.3. *Assume that $(f,g)\in C^\infty(\mathbb{R}^3\backslash\mathcal{K})$ satisfies the compatibility conditions of infinite order and $s\ge 14$ is an integer. There exists $T>0$ such that problem* (2.3) *has a unique solution u on $[0,T]\times\mathbb{R}^3\backslash\mathcal{K}$ satisfying*

$$\sup_{0\le t\le T}\sum_{j=0}^{s+1}\|\partial_t^j u(t,\cdot)\|_{H^{s+1-j}(\mathbb{R}^3\backslash\mathcal{K})}\le C(\|f\|_{H^{s+1}(\mathbb{R}^3\backslash\mathcal{K})}+\|g\|_{H^s(\mathbb{R}^3\backslash\mathcal{K})}).$$

For the proof of Theorem 2.3, see [18].

Suppose that u is the solution to the following problem

$$\begin{cases}Lu(t,x)=0, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^3\backslash\mathcal{K},\\ u(t,\cdot)|_{\partial\mathcal{K}}=0, & \\ u(0,x)=\varphi(x), & \partial_t u(0,x)=\psi(x),\end{cases} \tag{2.16}$$

where the supports of φ and ψ are contained in a ball with radius a, centered at the origin. Let

$$E(u,D,t)=\frac{1}{2}\int_D(|\partial_t u|^2+c_2^2|\nabla u|^2+(c_1^2-c_2^2)(\mathrm{div}u)^2)dx.$$

Theorem 2.4. (cf. [7]) *Let $D\subset\mathbb{R}^3\backslash\mathcal{K}$ be an arbitrary domain lying in a ball with radius d, centered at the origin. Then there exist positive constants $\beta(a)$ and $C(a,d)$ such that*

$$E(u,D,t)\le Ce^{-\beta t}E(u,\mathbb{R}^3\backslash\mathcal{K},0)\qquad \textit{for}\ \ t>(a+d)c_2^{-1}.$$

For the proof of Theorem 2.4, see [6] or [7].

We also need the following consequences of Sobolev's lemma and the embedding theorem on the spherical surface.

Lemma 2.6. *Suppose $h\in C^\infty(\mathbb{R}^3)$. Then for $R>1$ we have*

$$\|h\|_{L^\infty(\frac{R}{2}\le|x|\le R)}\le CR^{-1}\sum_{|\alpha|+|\gamma|\le 2}\|\widetilde{\Omega}^\alpha\partial_x^\gamma h\|_{L^2(\frac{R}{4}\le|x|\le 2R)}.$$

Lemma 2.7. *Suppose $h\in C^\infty(\mathbb{R}^3)$. Then*

$$\sup_{|\theta|=1}|h(\rho\theta)|\le C\sum_{|\alpha|\le 2}\int_{S^2}|(\widetilde{\Omega}^\alpha h)(\rho\theta)|d\theta. \tag{2.17}$$

The proof of Lemma 2.6 and Lemma 2.7 can be found in [11] and [3], respectively.

§ 3. Pointwise Estimates for the Linear Elastodynamic Operator Outside an Obstacle

In this section we consider the exterior problem for the linear elastodynamic system

$$\begin{cases} Lu(t,x) = F(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K}, \\ u(t,x) = 0, & x \in \partial\mathcal{K}, \\ u(t,x) = 0, & t \leq 0. \end{cases} \tag{3.1}$$

We will prove the following pointwise estimate.

Theorem 3.1. *Suppose that $u = u(t,x) \in C^\infty$ solves problem* (3.1). *Then, for $|\alpha| = N > 1$, we have*

$$\begin{aligned} t|\partial^\alpha u(t,x)| \leq{}& C\int_0^t \sum_{\substack{|\gamma|+j\leq N+3 \\ j\leq 1}} \|S^j\partial^\gamma F(s\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|\leq 4)} ds \\ &+ C\int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\gamma|\leq N+3 \\ j\leq 1, |\beta|\leq 2}} |S^j Z^\beta \partial^\gamma F(s,y)| \frac{dyds}{|y|}. \end{aligned} \tag{3.2}$$

Without loss of generality we assume that $\mathcal{K} \subset \{x \in \mathbb{R}^3 : \ |x| < 1\}$. As a first step, we prove the following

Lemma 3.1. *Suppose that $u = u(t,x) \in C^\infty$ solves problem* (3.1). *Then, for $|\alpha| = N > 1$, we have*

$$\begin{aligned} t|\partial^\alpha u(t,x)| \leq{}& C\int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{|\gamma|\leq 2, j\leq 1} |S^j Z^\gamma \partial^\alpha F(s,y)| \frac{dyds}{|y|} \\ &+ C \sup_{|y|\leq 2, 0\leq s\leq t} (1+s)(|\partial^\alpha u'(s,y)| + |\partial^\alpha u(s,y)|). \end{aligned} \tag{3.3}$$

Proof. Estimate (3.3) is obvious when $|x| < 2$.

Let $\rho \in C^\infty(\mathbb{R})$ be a cut function satisfying

$$\rho(r) = \begin{cases} 1, & r \geq 2, \\ 0, & r \leq 1. \end{cases}$$

Then $w(t,x) = \rho(|x|)\partial^\alpha u(t,x)$, as a function defined in $\mathbb{R}^3$, solves the following problem

$$\begin{cases} Lw = \rho\partial^\alpha F + G, \\ w(t,x) = 0, \qquad t \leq 0, \end{cases}$$

where

$$G = -c_2^2(\triangle\rho)\partial^\alpha u - 2c_2^2\nabla\rho\cdot\nabla\partial^\alpha u - (c_1^2 - c_2^2)\nabla(\nabla\rho\cdot\partial^\alpha u) - (c_1^2 - c_2^2)(\nabla\rho)\mathrm{div}\partial^\alpha u.$$

Set $w = w_1 + w_0$, where w_1 and w_0 solve the following problems

$$\begin{cases} Lw_1 = \rho\partial^\alpha F, \\ w_1(t,x) = 0, \qquad t \leq 0 \end{cases} \quad \text{and} \quad \begin{cases} Lw_0 = G, \\ w_0(t,x) = 0, \qquad t \leq 0, \end{cases}$$

respectively. By Lemma 2.5, we conclude that

$$t|w_1(t,x)| \leq \ C\int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}} \sum_{|\gamma|\leq 2, j\leq 1} |S^j Z^\gamma \partial^\alpha F| \frac{dyds}{|y|}.$$

On the other hand, by Lemma 2.4, we can get

$$|w_0(t,x)| \leq C\frac{1}{|x|}\frac{1}{1+\min_i |c_i t-|x|\ |} \sup_{\frac{c_1 t-|x|-2}{c_1}\leq s\leq \frac{(c_2 t-|x|)+2}{c_2}} (1+s)(|\partial^\alpha u'|+|\partial^\alpha u|). \tag{3.4}$$

This yields that (3.3) still holds when $|x|\geq 2$.

Lemma 3.2. *Suppose that $u\in C^\infty$ solves problem (3.1) and $u(t,x)=0$ when $t<0$. Suppose furthermore that $F(t,x)=0$ when $|x|>a$, where a is a constant. Then for any given constant $d>1$, there are positive constants $C(a,d)$ and $c(a)$ such that*

$$\|u'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K}:|x|<d)} \leq C\int_0^t e^{-c(t-s)}\|F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds. \tag{3.5}$$

Moreover, for any fixed nonnegative integer M, we have

$$\begin{aligned}
&\sum_{\substack{|\alpha|+j\leq M\\ j\leq 1}} \|(t\partial_t)^j\partial^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K}:|x|<d)}\\
&\leq C\sum_{\substack{|\alpha|+j\leq M-1\\ j\leq 1}} \|(t\partial_t)^j\partial^\alpha F(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&\quad + C\int_0^t e^{-\frac{c}{2}(t-s)}\sum_{\substack{|\alpha|+j\leq M\\ j\leq 1}} \|(s\partial_s)^j\partial^\alpha F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\sum_{\substack{|\alpha|+j\leq M\\ j\leq 1}} \|(t\partial_t)^j\partial^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K}:|x|<d)}\\
&\leq C\sum_{\substack{|\alpha|+j\leq M-1\\ j\leq 1}} \|S^j\partial^\alpha F(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&\quad + C\int_0^t e^{-\frac{c}{2}(t-s)}\sum_{\substack{|\alpha|+j\leq M\\ j\leq 1}} \|S^j\partial^\alpha F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds.
\end{aligned} \tag{3.7}$$

Proof. Estimate (3.5) is an immediate consequence of Theorem 2.4. Estimate (3.7) follows from (3.6). Applying the elliptic regularity, we can prove (3.6) by induction.

Proof of Theorem 3.1. By Lemma 3.1, we need only to show that the last term on the right hand side of (3.3) can be dominated by the right hand side of (3.2).

First we discuss the case: $F(s,y)\equiv 0$ when $|y|>4$.

By Sobolev's lemma we get from (3.7) that for $|\alpha|=N$,

$$
\begin{aligned}
t\sup_{|x|<2}|\partial^\alpha u(t,x)| \le C\int_0^t \sum_{\substack{|\gamma|+j\le N+2\\ j\le 1}} \|S^j\partial^\gamma F(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|\le 4)}ds \\
+C\int_0^t\int_0^s e^{-\frac{c}{2}(s-\tau)} \sum_{\substack{|\gamma|+j\le N+2\\ j\le 1}} \|S^j\partial^\gamma F(\tau,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|\le 4)}d\tau ds. \quad (3.8)
\end{aligned}
$$

Therefore

$$
t\sup_{|x|<2}|\partial^\alpha u(t,x)| \le \text{ the first term on the right hand side of (3.2).}
$$

Now we deal with the second case: $F(s,y)\equiv 0$ when $|y|<3$.

Suppose that u_0 solves the following Cauchy problem

$$
\begin{cases} Lu_0 = F, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^3,\\ u_0(t,x)=0, & t\le 0. \end{cases} \tag{3.9}
$$

Let $\eta\in C_0^\infty(\mathbb{R}^3)$ be a cut function satisfying

$$
\eta(x)=\begin{cases} 1, & |x|<2,\\ 0, & |x|\ge 3. \end{cases}
$$

Set $\tilde u=(\eta-1)u_0+u$. Then $\tilde u$ solves the following problem

$$
\begin{cases} L\tilde u = G, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^3\backslash\mathcal{K},\\ \tilde u|_{\partial\mathcal{K}}=0, & \\ \tilde u(t,x)=0, & t\le 0, \end{cases} \tag{3.10}
$$

where

$$
G=-c_2^2(\triangle\eta)u_0-2c_2^2\nabla\eta\cdot\nabla u_0-(c_1^2-c_2^2)\nabla(\nabla\eta\cdot u_0)-(c_1^2-c_2^2)\nabla\eta\,\mathrm{div}u_0
$$

vanishes unless $2<|x|<3$. Hence by the result in the first case, we obtain

$$
t\sup_{|x|<2}|\partial^\alpha u(t,x)| \le C\int_0^t \sum_{\substack{|\gamma|\le N+3\\ j\le 1}} \|S^j\partial^\gamma u_0(s,\cdot)\|_{L^\infty(2\le|x|\le 4)}ds. \tag{3.11}
$$

Set $w=S^j\partial^\gamma u_0$ $(j\le 1)$. By Lemma 2.4 and Lemma 2.7, for problem (3.9) we get

$$
\begin{aligned}
&\sum_{j\le 1}\|S^j\partial^\gamma u_0(s,\cdot)\|_{L^\infty(2\le|x|\le 4)}\\
&\le C\sup_{2\le r\le 4}\sum_{j\le 1}\Big[\int_0^s\Big(\int_{|r-c_1(s-\tau)|}^{r+c_1(s-\tau)}+\int_{|r-c_2(s-\tau)|}^{r+c_2(s-\tau)}\Big)\sup_{|\theta|=1}|S^j\partial^\gamma F(\tau,\rho\theta)|\rho d\rho d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^s\int_{c_2}^{c_1}\int_{|r-l(s-\tau)|}^{r+l(s-\tau)}\sup_{|\theta|=1}|S^j\partial^\gamma F(\tau,\rho\theta)|\rho d\rho dl d\tau\Big]\\
&\le C\sup_{2\le r\le 4}\sum_{|\beta|\le 2,j\le 1}\Big[\int_0^s\Big(\int_{|c_1(s-\tau)-|y||\le 4}+\int_{|c_2(s-\tau)-|y||\le 4}\Big)|S^j\partial^\gamma\widetilde{\Omega}^\beta F(\tau,y)|\frac{dyd\tau}{|y|}\\
&+\int_0^s\int_{c_2}^{c_1}\int_{|l(s-\tau)-|y||\le 4}|S^j\partial^\gamma\widetilde{\Omega}^\beta F(\tau,y)|\frac{dydld\tau}{|y|}\Big]. \qquad (3.12)
\end{aligned}
$$

Let $\Lambda_s(l)=\{(\tau,y):0\le\tau\le s,|l(s-\tau)-|y||\le 4\}$ $(c_2\le l\le c_1)$. It is easy to see that there exists a constant s_0 independent of $l\in[c_2,c_1]$, such that $\Lambda_s(l)\cap\Lambda_{s'}(l)=\emptyset$ when $|s-s'|>s_0$. Then, by (3.11) and (3.12), we conclude that

$$
t\sup_{|x|<2}|\partial^\alpha u(t,x)|\le C\sum_{\substack{|\gamma|\le N+3\\|\beta|\le 2,j\le 1}}\int_0^t\int_{\mathbb{R}^3\backslash\mathcal{K}}|S^j\widetilde{\Omega}^\beta\partial^\gamma F(s,y)|\frac{dyds}{|y|}.
$$

The proof of Theorem 3.1 is completed.

§ 4. Weighted $L^2_{t,x}$ Estimates for the Linear Elastodynamic Operator Outside a Star-shaped Obstacle

In this section, we prove the following

Theorem 4.1. *Suppose that $u=u(t,x)\in C^\infty$ solves problem* (3.1). *Then*

$$
\begin{aligned}
&(\ln(2+t))^{-\frac12}\sum_{|\alpha|\le N}\|\langle x\rangle^{-\frac12}\partial^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}\\
&\le C\int_0^t\sum_{|\alpha|\le N}\|L\partial^\alpha u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds\\
&\quad+C\sum_{|\alpha|\le N-1}\|L\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})},\qquad\forall\, t\ge 0. \qquad (4.1)
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&(\ln(2+t))^{-\frac12}\sum_{\substack{|\alpha|+m\le N\\m\le 1}}\|\langle x\rangle^{-\frac12}S^m\partial^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}\\
&\le C\int_0^t\sum_{\substack{|\alpha|+m\le N\\m\le 1}}\|LS^m\partial^\alpha u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds\\
&\quad+C\sum_{\substack{|\alpha|+m\le N-1\\m\le 1}}\|LS^m\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})},\qquad\forall\, t\ge 0 \qquad (4.2)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\ln(2+t))^{-\frac12}\sum_{\substack{|\alpha|+|\gamma|+m\le N\\m\le 1}}\|\langle x\rangle^{-\frac12}S^m\widetilde{\Omega}^\gamma\partial^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}\\
&\le C\int_0^t\sum_{\substack{|\alpha|+|\gamma|+m\le N\\m\le 1}}\|LS^m\widetilde{\Omega}^\gamma\partial^\alpha u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds
\end{aligned}
$$

$$+ C \sum_{\substack{|\alpha|+|\gamma|+m\le N-1 \\ m\le 1}} \|LS^m\widetilde{\Omega}^\gamma\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}, \qquad \forall\, t\ge 0. \tag{4.3}$$

We first consider the estimates in a bounded domain ($|x|<2$).

Proposition 4.1. *Suppose that u solves problem* (3.1). *Then we have*

$$\|u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<2)} \le C\int_0^t \|Lu(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds, \qquad \forall\, t\ge 0 \tag{4.4}$$

and for any given natural number N,

$$\sum_{|\alpha|\le N}\|\partial^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<2)}$$

$$\le C\int_0^t \sum_{m\le N}\|L\partial_s^m u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds + C\sum_{|\alpha|\le N-1}\|L\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}, \qquad \forall\, t\ge 0. \tag{4.5}$$

Proof. By the elliptic regularity, the estimate (4.5) is a consequence of (4.4).

First we discuss the case: $F(s,y)\equiv 0$ when $|y|>6$.

Using Schwarz inequality and estimate (3.5), we can prove

$$\int_0^t \|u'(\tau,\cdot)\|^2_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|<2)}d\tau \le C\Big(\int_0^t \|F(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds\Big)^2, \qquad \forall\, t\ge 0, \tag{4.6}$$

which completes the proof of (4.4).

Now we deal with the Case: $F(s,y)\equiv 0$ when $|y|<4$.

From the proof of (4.6), we have

$$\|u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<2)} \le C\|F\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}, \qquad \text{if } \ F(s,y)\equiv 0 \ \text{ when } \ |y|>4. \tag{4.7}$$

Let $\eta\in C^\infty(\mathbb{R}^3)$ be a cut function satisfying

$$\eta(x)=\begin{cases}1, & |x|\le 2,\\ 0, & |x|\ge 4.\end{cases}$$

Assume that u_0 solves Cauchy problem (3.9) and $\tilde u=(\eta-1)u_0+u$. Then $\tilde u$ solves the following problem

$$\begin{cases} L\tilde u=\widetilde F,\\ \tilde u|_{\partial\mathcal{K}}=0,\\ \tilde u(t,x)=0, & t\le 0,\end{cases}$$

where

$$\widetilde F=-c_2^2(\triangle\eta)u_0-2c_2^2\nabla_x\eta\cdot\nabla_x u_0-(c_1^2-c_2^2)\nabla(\nabla\eta\cdot u_0)-(c_1^2-c_2^2)\nabla\eta\,\mathrm{div}u_0.$$

Noting $\tilde u=u$ when $|x|<2$, and $\widetilde F(s,y)\equiv 0$ when $|y|>4$, it follows from estimates (4.7), (2.10) and (2.11) that

$$\begin{aligned}\|u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<2)} \le\ & C\|u_0'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<4)}+C\|u_0\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<4)}\\ \le\ & C\int_0^t \|Lu\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds, \qquad \forall\, t\ge 0.\end{aligned}$$

Repeating the proof of Proposition 4.1 and using (3.7), we obtain the following

Proposition 4.2. *Suppose that $u = u(t,x) \in C^\infty$ solves problem (3.1). Then*

$$\begin{aligned}&\sum_{\substack{|\alpha|+m\le N\\ m\le 1}} \|S^m\partial^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<2)}\\ &\le C\int_0^t \sum_{\substack{|\alpha|+m\le N\\ m\le 1}} \|LS^m\partial^\alpha u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds\\ &\quad + C\sum_{\substack{|\alpha|+m\le N-1\\ m\le 1}} \|LS^m\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}, \qquad \forall\, t\ge 0,\end{aligned}\tag{4.8}$$

$$\begin{aligned}&\sum_{\substack{|\alpha|+|\gamma|+m\le N\\ m\le 1}} \|S^m\widetilde{\Omega}^\gamma\partial^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<2)}\\ &\le C\int_0^t \sum_{\substack{|\alpha|+|\gamma|+m\le N\\ m\le 1}} \|LS^m\widetilde{\Omega}^\gamma\partial^\alpha u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds\\ &\quad + C\sum_{\substack{|\alpha|+|\gamma|+m\le N-1\\ m\le 1}} \|LS^m\widetilde{\Omega}^\gamma\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}, \qquad \forall\, t\ge 0.\end{aligned}\tag{4.9}$$

Proof of Theorem 4.1. First we prove estimate (4.1). Noting estimate (4.5), (4.1) holds if, instead of $\mathbb{R}^3\backslash\mathcal{K}$, the domain is in $\{|x|<2\}$.

Let $\beta \in C^\infty(\mathbb{R}^3)$ be a cut function satisfying

$$\beta(x) = \begin{cases} 1, & |x|\ge 2,\\ 0, & |x|\le 1.\end{cases}$$

Then $w = \beta u$ solves

$$\begin{cases} Lw = \beta Lu - c_2^2(\triangle\beta)u - 2c_2^2\nabla\beta\cdot\nabla u - (c_1^2-c_2^2)\nabla(\nabla\beta\cdot u) - (c_1^2-c_2^2)(\nabla\beta)\mathrm{div}u,\\ w(t,x) = 0, \qquad t\le 0.\end{cases}$$

Set $w = w_1 + w_2$, where w_1 and w_2 satisfy

$$\begin{cases} Lw_1 = \beta Lu,\\ w_1(t,x) = 0, \qquad t\le 0\end{cases}$$

and

$$\begin{cases} Lw_2 = -c_2^2(\triangle\beta)u - 2c_2^2\nabla\beta\cdot\nabla u - (c_1^2-c_2^2)\nabla(\nabla\beta\cdot u) - (c_1^2-c_2^2)(\nabla\beta)\mathrm{div}u = H,\\ w_2(t,x) = 0, \qquad t\le 0,\end{cases}$$

respectively. By Lemma 2.3, we have

$$\begin{aligned}&(\ln(2+t))^{-\frac12}\sum_{|\alpha|\le N} \|\langle x\rangle^{-\frac12}\partial^\alpha w_1'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|>2)}\\ &\le C\sum_{|\alpha|\le N}\int_0^t \|\partial^\alpha(\beta Lu)\|_{L^2(\mathbb{R}^3)}ds \le C\sum_{|\alpha|\le N}\int_0^t \|L\partial^\alpha u\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds, \qquad \forall\, t\ge 0.\end{aligned}\tag{4.10}$$

Decomposing w_2 and H as u and F in Lemma 2.1, we have

$$w_2 = v_1 + v_2, \qquad H = H_1 + H_2,$$

and

$$\|H_1\|_{L^2(\mathbb{R}^3)} \le C\|H\|_{L^2(\mathbb{R}^3)}, \qquad \|H_2\|_{L^2(\mathbb{R}^3)} \le C\|H\|_{L^2(\mathbb{R}^3)}.$$

Noting the fact that the support of H is contained in $\{1 < |x| < 2\}$, by the proof of Theorem 6.3 in [10] we conclude that

$$\begin{aligned}&\Big((\ln(2+t))^{-\frac{1}{2}} \sum_{|\alpha|\le N} \|\langle x\rangle^{-\frac{1}{2}}\partial^\alpha v_1'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|>2)}\Big)^2 \\ &\le C\sum_{|\alpha|\le N}\|\partial^\alpha H_1\|^2_{L^2([0,t]\times\mathbb{R}^3)} \le C\sum_{|\alpha|\le N}\|\partial^\alpha H\|^2_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:1\le|x|<2)}, \qquad \forall\, t\ge 0.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\Big((\ln(2+t))^{-\frac{1}{2}} \sum_{|\alpha|\le N} \|\langle x\rangle^{-\frac{1}{2}}\partial^\alpha v_2'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|>2)}\Big)^2 \\ &\le C\sum_{|\alpha|\le N}\|\partial^\alpha H\|^2_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:1\le|x|<2)}, \qquad \forall\, t\ge 0.\end{aligned}$$

Thus, noting (4.5) and Lemma 2.2, we get

$$\begin{aligned}&\ln(2+t))^{-\frac{1}{2}} \sum_{|\alpha|\le N} \|\langle x\rangle^{-\frac{1}{2}}\partial^\alpha w_2'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|>2)} \\ &\le C\int_0^t \sum_{|\alpha|\le N}\|L\partial^\alpha u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds + C\sum_{|\alpha|\le N-1}\|L\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}, \qquad \forall\, t\ge 0. \quad (4.11)\end{aligned}$$

The estimates (4.10) and (4.11) finish the proof of (4.1).

In a similar way, we can obtain estimates (4.2) and (4.3).

§ 5. L^2_x Estimates Outside an Obstacle

Suppose that v is a sufficiently smooth function such that

$$\|\nabla v\|_{L^\infty([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \le \delta, \tag{5.1}$$

$$\|\partial\nabla v\|_{L^1_tL^\infty_x([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \le C_0, \tag{5.2}$$

where C_0 is a positive constant, $\delta > 0$ is a sufficiently small constant. Let $\mathcal{L}$ be the following linear differential operator

$$\mathcal{L} = L - \sum_{l,m} C^{lm}(\nabla v)\partial_l\partial_m. \tag{5.3}$$

In this section we consider the mixed initial-boundary value problem of Dirichlet type

$$\begin{cases}\mathcal{L}w = G, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^3\backslash\mathcal{K}, \\ w|_{\partial\mathcal{K}} = 0, & \\ w(t,x) = 0, & t\le 0.\end{cases} \tag{5.4}$$

Set

$$Q_0 = |\partial_0 w|^2 + c_2^2|\nabla w|^2 + (c_1^2 - c_2^2)(\mathrm{div} w)^2 + \sum_{l,m=1}^{3} (\partial_l w)^T C^{lm}(\nabla v)\partial_m w,$$

$$Q_j = -2c_2^2(\partial_0 w)^T(\partial_j w) - 2(c_1^2 - c_2^2)\mathrm{div} w\partial_0 w_j - 2\sum_{k=1}^{3}(\partial_0 w)^T C^{jk}(\nabla v)\partial_k w \qquad (j = 1, 2, 3),$$

$$q = \sum_{l,m=1}^{3} \{(\partial_l w)^T \partial_0 C^{lm}(\nabla v)\partial_m w - 2(\partial_0 w)^T \partial_l C^{lm}(\nabla v)\partial_m w\}.$$

By the symmetry of $C^{lm}(\nabla v)$, we have

$$\sum_{\alpha=0}^{3} \partial_\alpha Q_\alpha(\partial w) = 2(\partial_0 w)^T \mathcal{L} w + q. \tag{5.5}$$

It is easy to see from (5.1) that there exist positive constants $\nu,\ \mu$ depending only on $c_1,\ c_2$ and λ, such that

$$\nu|w'|^2 \le Q_0 \le \mu|w'|^2. \tag{5.6}$$

Integrating (5.5) over $[0,T] \times \mathbb{R}^3 \setminus \mathcal{K}$ and applying Gronwall's inequality, we get

$$\|w'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le C\int_0^t \|G\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds, \qquad 0 \le t \le T. \tag{5.7}$$

In general, we have the following

Theorem 5.1. *Suppose that v satisfies* (5.1) *and* (5.2)*, and w solves problem* (5.4)*. Then for any given nonnegative integer N, there is a positive constant C such that*

$$\sum_{|\alpha|\le N} \|\partial^\alpha w'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le C\int_0^t \sum_{m\le N} \|\mathcal{L}\partial_s^m w(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds + C\sum_{|\alpha|\le N-1} \|L\partial^\alpha w(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}, \qquad 0 \le t \le T. \tag{5.8}$$

Proof. We can prove (5.8) by induction. The details are omitted.

§ 6. L_x^2 Estimates Involving Operators $S^j\widetilde{\Omega}\partial^\alpha$ Outside an Obstacle

Suppose that w solves problem (5.4). Let $P = P(t,x,D)$ be a differential operator. Suppose furthermore that Pw is not necessary to vanish on $\partial\mathcal{K}$. In this section we will give some rough L^2 estimates for Pw.

Proposition 6.1. *Suppose that $Pw(0,\cdot) = \partial_t Pw(0,\cdot) = 0$. Suppose furthermore that there is an integer M and a positive constant C_0 such that*

$$|(Pw)'(t,x)| \le C_0 t\sum_{|\alpha|\le M-1} |\partial_t\partial^\alpha w'(t,x)| + C_0\sum_{|\alpha|\le M} |\partial^\alpha w'(t,x)|, \qquad x \in \partial\mathcal{K},\ \forall\, t \ge 0. \tag{6.1}$$

Then

$$\begin{aligned}\|(Pw)'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le C\int_0^t &\|\mathcal{L}Pw(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds\\ &+C\int_0^t \sum_{\substack{|\alpha|+j\le M+1\\ j\le 1}} \|LS^j\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds\\ &+C\sum_{\substack{|\alpha|+j\le M\\ j\le 1}}\|LS^j\partial^\alpha w\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}, \qquad \forall\, t\ge 0. \end{aligned} \tag{6.2}$$

Proof. It is easy to see that

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus\mathcal{K}} Q_0(t,x)dx - \int_{\mathbb{R}^3\setminus\mathcal{K}} Q_0(0,x)dx - \int_{[0,t]\times\partial\mathcal{K}}\sum_{j=1}^3 Q_j n_j d\sigma\\ &= 2\int_{[0,t]\times\mathbb{R}^3\setminus\mathcal{K}}(\partial_0 Pw)^T\mathcal{L}Pwdsdx + \int_{[0,t]\times\mathbb{R}^3\setminus\mathcal{K}} qdsdx,\end{aligned} \tag{6.3}$$

where

$$Q_0 = |\partial_0 Pw|^2 + c_2^2|\nabla Pw|^2 + (c_1^2-c_2^2)(\mathrm{div}Pw)^2 + \sum_{l,m=1}^3(\partial_l Pw)^T C^{lm}(\nabla v)\partial_m Pw,$$

$$Q_j = -2c_2^2(\partial_0 Pw)^T(\partial_j Pw) - 2(c_1^2-c_2^2)\mathrm{div}Pw\partial_0 Pw_j - 2\sum_{k=1}^3(\partial_0 Pw)^T C^{jk}(\nabla v)\partial_k Pw,$$

$$q = \sum_{l,m=1}^3\{(\partial_l Pw)^T\partial_0 C^{lm}(\nabla u)\partial_m Pw - 2(\partial_0 Pw)^T\partial_l C^{lm}(\nabla v)\partial_m Pw\},$$

in which v satisfies (5.1) and (5.2). Then by applying Gronwall's inequality and the trace theorem, we can get (6.2) from (6.3).

Obviously, $P = S^j\widetilde{\Omega}^\mu\partial^\alpha$ $(j\le 1)$ satisfies (6.1). As an immediate corollary of Proposition 6.1 we have the following

Theorem 6.1. *Assume that w solves problem* (5.4). *Then, for $M=1,2,\cdots$,*

$$\begin{aligned}&\sum_{\substack{|\mu|+|\alpha|+j\le M\\ j\le 1}}\|(S^j\widetilde{\Omega}^\mu\partial^\alpha w)'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\ &\le C\int_0^t\sum_{\substack{|\mu|+|\alpha|+j\le M\\ j\le 1}}\|\mathcal{L}S^j\widetilde{\Omega}^\mu\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds\\ &\quad + C\int_0^t\sum_{\substack{|\alpha|+j\le M+1\\ j\le 1}}\|LS^j\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds\\ &\quad + C\sum_{\substack{|\alpha|+j\le M\\ j\le 1}}\|LS^j\partial^\alpha w(s,\cdot)\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}, \qquad \forall\, t\ge 0.\end{aligned} \tag{6.4}$$

§ 7. L^2_x Estimates Involving $S^m\partial^\alpha$ Outside a Star-shaped Obstacle

In this section we assume furthermore that

$$\|\nabla v\|_{L^\infty(\mathbb{R}^3\setminus\mathcal{K})} \le \frac{\delta}{1+t}, \tag{7.1}$$

where δ is a sufficiently small positive constant. Suppose that w solves (5.4). Noting that $\mathcal{K}$ is a star-shaped domain, we can get better estimates for Sw.

Proposition 7.1. *Suppose that* (7.1) *holds and* w *solves problem* (5.4). *Then*

$$\begin{aligned}\|(Sw)'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le{}& C\int_0^t \|\mathcal{L}Sw(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds + C\int_0^t \sum_{|\alpha|\le 2}\|L\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds\\ &+ C\sum_{|\alpha|\le 1}\|L\partial^\alpha w\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}, \qquad \forall\ t\ge 0.\end{aligned} \tag{7.2}$$

Proof. Similarly to (6.3), we have

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus\mathcal{K}} Q_0(t,x)dx - \int_{\mathbb{R}^3\setminus\mathcal{K}} Q_0(0,x)dx - \int_{[0,t]\times\partial\mathcal{K}}\sum_{j=1}^3 Q_j n_j d\sigma\\ &= 2\int_{[0,t]\times\mathbb{R}^3\setminus\mathcal{K}} (\partial_0 Sw)^T\mathcal{L}Sw dsdx + \int_{[0,t]\times\mathbb{R}^3\setminus\mathcal{K}} qdsdx,\end{aligned} \tag{7.3}$$

where

$$Q_0 = |\partial_0 Sw|^2 + c_2^2|\nabla Sw|^2 + (c_1^2-c_2^2)(\mathrm{div}Sw)^2 + \sum_{l,m=1}^3(\partial_l Sw)^T C^{lm}(\nabla u)\partial_m Sw,$$

$$Q_j = -2c_2^2(\partial_0 Sw)^T(\partial_j Sw) - 2(c_1^2-c_2^2)\mathrm{div}Sw\partial_0 Sw_j - 2\sum_{k=1}^3(\partial_0 Sw)^T C^{jk}(\nabla u)\partial_k Sw,$$

$$q = \sum_{l,m=1}^3\{(\partial_l Sw)^T\partial_0 C^{lm}(\nabla u)\partial_m Sw - 2(\partial_0 Sw)^T\partial_l C^{lm}(\nabla u)\partial_m Sw\}.$$

Now we deal with the last term on the left hand side of (7.3). Noticing that when $(s,x)\in\mathbb{R}_+\times\partial\mathcal{K}$, w satisfies the Dirichlet boundary condition, we have

$$\partial_s Sw = \langle x,\vec{n}\rangle\partial_n\partial_s w,$$

where $\partial_n w = \langle\vec{n},\nabla_x\rangle w$. Similarly

$$\sum_{j=1}^3 n_j\partial_j Sw = s\partial_n\partial_s w + \partial_n(\langle x,\nabla_x\rangle w) \qquad \text{on}\ \ \mathbb{R}_+\times\partial\mathcal{K}.$$

Therefore we have

$$\begin{aligned}-\sum_{j=1}^3 Q_j n_j ={}& 2c_2^2\sum_{k=1}^3\langle x,\vec{n}\rangle s(\partial_n\partial_s w_k)^2 + 2c_2^2\sum_{k=1}^3\langle x,\vec{n}\rangle\partial_n\partial_s w_k\partial_n(\langle x,\nabla\rangle w_k)\\ &+ 2(c_1^2-c_2^2)\mathrm{div}(s\partial_s w)\sum_{j=1}^3\langle x,\vec{n}\rangle\partial_n\partial_s w_j n_j\end{aligned}$$

$$
\begin{aligned}
&+ 2(c_1^2 - c_2^2)\mathrm{div}(\langle x, \nabla\rangle w)\sum_{j=1}^{3}\langle x, \vec{n}\rangle \partial_n \partial_s w_j n_j \\
&+ 2\sum_{k=1}^{3}(\langle x, \vec{n}\rangle \partial_n \partial_s w)^T \sum_{j=1}^{3} C^{jk}(\nabla v)(s\partial_k \partial_s w + \partial_k(\langle x, \nabla\rangle w))n_j \\
= {}& N_1 + N_2 + N_3 + N_4 + N_5.
\end{aligned}
$$

Obviously

$$|N_2| + |N_4| \le C \sum_{1\le|\alpha|\le 2} |\partial^\alpha w|^2 \qquad \text{on } \ \mathbb{R}_+ \times \partial\mathcal{K}.$$

Moreover, noting (7.1), we have

$$|N_5| \le C \sum_{1\le|\alpha|\le 2} |\partial^\alpha w|^2 \qquad \text{on } \ \mathbb{R}_+ \times \partial\mathcal{K}.$$

To estimate N_3 we need the following

Lemma 7.1. *Suppose that w is a vector field vanishing on $\partial\mathcal{K}$. Then*

$$\mathrm{div} w = \lambda \partial_n \langle w, \vec{n}\rangle \qquad \text{on } \ \partial\mathcal{K},$$

where λ is a positive function.

Proof. Introduce orthogonal curvilinear coordinate system $x_i = x_i(q_1, q_2, q_3)$ such that

$$\vec{e}_3 = \vec{n} \qquad \text{on } \ \partial\mathcal{K},$$

where $\vec{e}_1,\ \vec{e}_2,\ \vec{e}_3$ denote unit coordinate vectors of this system. Set

$$w = \bar{w}_1\vec{e}_1 + \bar{w}_2\vec{e}_2 + \bar{w}_3\vec{e}_3.$$

Then

$$\mathrm{div} w = \frac{1}{H_1H_2H_3}(\partial_{q_1}(H_2H_3\bar{w}_1) + \partial_{q_2}(H_3H_1\bar{w}_2) + \partial_{q_3}(H_1H_2\bar{w}_3)),$$

where $H_1,\ H_2$ and H_3 are Lamé coefficients of the curvilinear coordinate system with

$$H_i = \Big|\frac{\partial x}{\partial q_i}\Big|, \qquad i = 1, 2, 3.$$

Since ∂_{q_1} and ∂_{q_2} are tangential derivatives on $\partial\mathcal{K}$, we have

$$\partial_{q_1}(H_2H_3\bar{w}_1) = \partial_{q_2}(H_3H_1\bar{w}_2) = 0.$$

Therefore

$$\mathrm{div} w = \frac{1}{H_3}\frac{\partial \bar{w}_3}{\partial q_3} \qquad \text{on } \ \partial\mathcal{K}.$$

Since

$$\frac{\partial \bar{w}_3}{\partial q_3} = \partial_n\langle w, \vec{n}\rangle \qquad \text{on } \ \partial\mathcal{K},$$

Lemma 7.1 is valid with $\lambda = \frac{1}{H_3}$.

Continuation of the Proof of Proposition 7.1. By Lemma 7.1, we have

$$N_3 = 2(c_1^2 - c_2^2)\lambda s\langle x, \vec{n}\rangle(\partial_s\partial_n\langle w, \vec{n}\rangle)^2.$$

Noting that $\mathcal{K}$ is star-shaped, namely, $\langle x, \vec{n}\rangle > 0,\ \forall\, x \in \partial\mathcal{K}$, it follows from (7.3) that

$$\int_{\mathbb{R}^3\setminus\mathcal{K}} Q_0(t,x)dx \le 2\int_{[0,t]\times\mathbb{R}^3\setminus\mathcal{K}} (\partial_0 Sw)^T \mathcal{L}Swdsdx + \int_{[0,t]\times\mathbb{R}^3\setminus\mathcal{K}} qdsdx + C\int_{[0,t]\times\partial\mathcal{K}} \sum_{1\le|\alpha|\le 2} |\partial^\alpha w|^2 d\sigma. \tag{7.4}$$

So we conclude that

$$\|(Sw)'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le C\int_0^t \|\mathcal{L}Sw(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds + C\Big(\int_{[0,t]\times\partial\mathcal{K}} \sum_{1\le|\alpha|\le 2} |\partial^\alpha w|^2 d\sigma\Big)^{\frac12}. \tag{7.5}$$

By (4.5) and a trace argument, we have

$$\begin{aligned}&\Big(\int_{[0,t]\times\partial\mathcal{K}} \sum_{1\le|\alpha|\le 2} |\partial^\alpha w|^2 d\sigma\Big)^{\frac12}\\ &\le C\int_0^t \sum_{|\alpha|\le 2} \|L\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds + C\sum_{|\alpha|\le 1} \|L\partial^\alpha w\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}.\end{aligned} \tag{7.6}$$

Then (7.2) follows immediately from (7.5) and (7.6).

Repeating the procedure in the proof of Theorem 5.1 and using Proposition 7.1, we get the following

Theorem 7.1. *Suppose that* (7.1) *holds and* w *solves* (5.4). *Then for any given nonnegative integer* N*, we have*

$$\begin{aligned}&\sum_{\substack{|\alpha|+m\le N\\ m\le 1}} \|S^m\partial^\alpha w'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\ &\le C\int_0^t \sum_{\substack{|\alpha|+m\le N\\ m\le 1}} \|\mathcal{L}S^m\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds + C\sum_{\substack{|\alpha|+m\le N-1\\ m\le 1}} \|LS^m\partial^\alpha w(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\ &\quad + C\int_0^t \sum_{|\alpha|\le N+1} \|L\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds + C\sum_{|\alpha|\le N} \|L\partial^\alpha w\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}, \qquad \forall\, t \ge 0.\end{aligned} \tag{7.7}$$

§ 8. Main L^2 Estimates Outside a Star-shaped Obstacle

Assuming that v satisfies (5.2) and (7.1), we have the following

Proposition 8.1. *Let* $w \in C^\infty$ *solve* (5.4). *Then, for any given integer* $N \ge 0$*, we have*

$$\begin{aligned}&\sum_{|\alpha|\le N+4} \|\partial^\alpha w'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + \sum_{\substack{|\alpha|+m\le N+2\\ m\le 1}} \|S^m\partial^\alpha w'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\ &+ \sum_{\substack{|\alpha|+|\mu|+m\le N\\ m\le 1}} \|S^m\widetilde{\Omega}^\mu\partial^\alpha w'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^t \Big(\sum_{|\alpha|\le N+4}\|\mathcal{L}\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}+\sum_{\substack{|\alpha|+m\le N+2\\ m\le 1}}\|\mathcal{L}S^m\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\\
&\quad+\sum_{\substack{|\alpha|+|\mu|+m\le N\\ m\le 1}}\|\mathcal{L}S^m\widetilde{\Omega}^\mu\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\Big)ds\\
&\quad+C\sum_{|\alpha|\le N+3}\|\mathcal{L}\partial^\alpha w(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}+C\sum_{\substack{|\alpha|+m\le N+1\\ m\le 1}}\|\mathcal{L}S^m\partial^\alpha w(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\\
&\quad+C\sum_{|\alpha|\le N+2}\|L\partial^\alpha w\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}+C\sum_{\substack{|\alpha|+m\le N\\ m\le 1}}\|LS^m\partial^\alpha w\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})},\quad\forall\, t\ge 0.
\end{aligned}\tag{8.1}
$$

Proof. We denote the left hand side of (8.1) by $\mathrm{I}+\mathrm{II}+\mathrm{III}$, and the right hand side by RHS. Since $L=\mathcal{L}+\sum\limits_{l,m=1}^{3}C^{lm}(\nabla v)\partial_l\partial_m$, by Theorem 5.1 we have

$$
\mathrm{I}\le \mathrm{RHS}+C\sum_{l,m=1}^{3}\sum_{|\alpha|\le N+3}\|C^{lm}(\nabla v)\partial_l\partial_m\partial^\alpha w(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}. \tag{8.2}
$$

Similarly, by Theorem 7.1 and Theorem 6.1 we get the estimates of II and III respectively.

Noting (7.1), if δ is small enough, then an application of Gronwall's inequality implies that

$$
\mathrm{I}+\mathrm{II}+\mathrm{III}\le \mathrm{RHS}+C\int_0^t\Big(\sup_{\substack{x\in\mathbb{R}^3\\ j,k}}|C^{jk}(\nabla v)|\Big)(\mathrm{I}+\mathrm{II})ds.
$$

This completes the proof of Proposition 8.1.

Repeating the proof of Proposition 8.1 and applying the weighted L^2 estimates of linear elastodynamic operator (Theorem 4.1), we get the following

Theorem 8.1. *Suppose that w solves problem (5.4). Then for any given integer $N\ge 0$, we have*

$$
\begin{aligned}
&\sum_{|\alpha|\le N+4}\|\partial^\alpha w'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}+\sum_{\substack{|\alpha|+m\le N+2\\ m\le 1}}\|S^m\partial^\alpha w'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\\
&+\sum_{\substack{|\alpha|+|\mu|+m\le N\\ m\le 1}}\|S^m\widetilde{\Omega}^\mu\partial^\alpha w'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\\
&+(\ln(2+t))^{-\frac12}\Big(\sum_{|\alpha|\le N+3}\|\langle x\rangle^{-\frac12}\partial^\alpha w'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}\\
&+\sum_{\substack{|\alpha|+m\le N+1\\ m\le 1}}\|\langle x\rangle^{-\frac12}S^m\partial^\alpha w'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}\\
&+\sum_{\substack{|\alpha|+|\mu|+m\le N-1\\ m\le 1}}\|\langle x\rangle^{-\frac12}S^m\widetilde{\Omega}^\mu\partial^\alpha w'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\int_0^t \Big(\sum_{|\alpha|\leq N+4} \|\mathcal{L}\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + \sum_{\substack{|\alpha|+m\leq N+2\\ m\leq 1}} \|\mathcal{L}S^m\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \\
&\quad + \sum_{\substack{|\alpha|+|\mu|+m\leq N\\ m\leq 1}} \|\mathcal{L}S^m\widetilde{\Omega}^\mu\partial^\alpha w(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\Big) ds \\
&\quad + C\sum_{|\alpha|\leq N+3} \|\mathcal{L}\partial^\alpha w(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + C\sum_{\substack{|\alpha|+m\leq N+1\\ m\leq 1}} \|\mathcal{L}S^m\partial^\alpha w(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \\
&\quad + C\sum_{|\alpha|\leq N+2} \|L\partial^\alpha w\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})} + C\sum_{\substack{|\alpha|+m\leq N\\ m\leq 1}} \|LS^m\partial^\alpha w\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})} \\
&\quad + C\sum_{\substack{|\alpha|+|\mu|+m\leq N-2\\ m\leq 1}} \|LS^m\widetilde{\Omega}^\mu\partial^\alpha w\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}, \qquad \forall\, t\geq 0. \qquad (8.3)
\end{aligned}
$$

§ 9. Almost Global Existence for Nonlinear Elastodynamic System Outside a Star-shaped Domain

In this section, we give the proof of the main result (Theorem 2.1). We will apply the above estimates and the local existence theorem to getting the almost global existence of solution and the lower bound of the lifespan of solution by an iterative procedure.

Suppose that the integer $N \geq 14$ and we shall take $N = 14$ in what follows. Thus, we assume that the initial data satisfies (2.4) for $N = 14$. Then Theorem 2.3 (also see [18]) implies that if $\varepsilon > 0$ is small enough, problem (2.3) admits a local solution in $0 < t < 1$, such that

$$
\begin{aligned}
&\sup_{0\leq t\leq 1} \Big(\sum_{|\alpha|\leq 14} \|\partial^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + \sum_{\substack{|\alpha|+m\leq 12\\ m\leq 1}} \|S^m\partial^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \\
&+ \sum_{\substack{|\alpha|+m+|\mu|\leq 10\\ m\leq 1}} \|S^m\widetilde{\Omega}^\mu\partial^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\Big) \\
&+ \sum_{|\alpha|\leq 13} \|\langle x\rangle^{-\frac{1}{2}}\partial^\alpha u'(t,\cdot)\|_{L^2([0,1]\times\mathbb{R}^3\setminus\mathcal{K})} \\
&+ \sum_{\substack{|\alpha|+m\leq 11\\ m\leq 1}} \|\langle x\rangle^{-\frac{1}{2}}S^m\partial^\alpha u'(t,\cdot)\|_{L^2([0,1]\times\mathbb{R}^3\setminus\mathcal{K})} \\
&+ \sum_{\substack{|\alpha|+m+|\mu|\leq 9\\ m\leq 1}} \|\langle x\rangle^{-\frac{1}{2}}S^m\widetilde{\Omega}^\mu\partial^\alpha u'(t,\cdot)\|_{L^2([0,1]\times\mathbb{R}^3\setminus\mathcal{K})} \leq C\varepsilon. \qquad (9.1)
\end{aligned}
$$

In fact, by Theorem 2.3, estimate (9.1) holds when x is in a bounded region. By the finiteness of the propagation speed and the estimates for the Cauchy problem we can derive the above estimate outside a bounded region.

Let $\eta(t) \in C^\infty(\mathbb{R})$ be a cut function satisfying

$$\eta(t)=\begin{cases}1, & t\le \frac{1}{2},\\ 0, & t>1.\end{cases}$$

Set $u_0=\eta u,\ w=u-u_0=(1-\eta)u$, where u is the local solution. Because of $w=0$ when $t\le\frac{1}{2}$, we shall prove the almost global existence of w instead of u by iteration. Thus, problem (2.3) is transformed into the following problem on w:

$$\begin{cases}Lw=(1-\eta)F(\nabla(u_0+w),\nabla^2(u_0+w))-[L,\eta](u_0+w),\\ w|_{\partial\mathcal{K}}=0,\\ w(t,x)=0, \qquad t\le 0.\end{cases}\tag{9.2}$$

Set $w_0=0$ and define $w_k,\ k=1,2,\cdots$, inductively by

$$\begin{cases}Lw_k=(1-\eta)F(\nabla(u_0+w_{k-1}),\nabla^2(u_0+w_k))-[L,\eta](u_0+w_k)=F_k,\\ w_k|_{\partial\mathcal{K}}=0,\\ w_k(t,x)=0, \qquad t\le 0.\end{cases}\tag{9.3}$$

Let

$$\begin{aligned}M_k(T)=\sup_{0\le t\le T}\Bigg(&\sum_{|\alpha|\le 14}\|\partial^\alpha w_k'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}+\sum_{\substack{|\alpha|+m\le 12\\ m\le 1}}\|S^m\partial^\alpha w_k'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\\ &+\sum_{\substack{|\alpha|+|\mu|+m\le 10\\ m\le 1}}\|S^m\widetilde{\Omega}^\mu\partial^\alpha w_k'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}+(1+t)\sum_{|\alpha|\le 2}\|\partial^\alpha w_k(t,\cdot)\|_{L^\infty(\mathbb{R}^3\backslash\mathcal{K})}\Bigg)\\ &+(\ln(2+T))^{-\frac{1}{2}}\Bigg(\sum_{|\alpha|\le 13}\|\langle x\rangle^{-\frac{1}{2}}\partial^\alpha w_k'\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}\\ &+\sum_{\substack{|\alpha|+m\le 11\\ m\le 1}}\|\langle x\rangle^{-\frac{1}{2}}S^m\partial^\alpha w_k'\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}\\ &+\sum_{\substack{|\alpha|+|\mu|+m\le 9\\ m\le 1}}\|\langle x\rangle^{-\frac{1}{2}}S^m\widetilde{\Omega}^\mu\partial^\alpha w_k'\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}\Bigg).\end{aligned}\tag{9.4}$$

Now we show inductively that these exists a positive constant C_1 such that for sufficiently small $\varepsilon>0$,

$$M_k(T_\varepsilon)\le C_1\varepsilon, \qquad k=0,1,2,\cdots\tag{9.5}$$

provided that the constant c occurring in $T_\varepsilon=\mathrm{e}^{\frac{c}{\varepsilon}}$ is sufficiently small. Obviously, (9.5) holds when $k=0$. Suppose that $M_{k-1}(T_\varepsilon)\le C_1\varepsilon$. We shall show that (9.5) holds for k. To do this, we first prove that

$$\begin{aligned}M_k(T_\varepsilon)\le{}& C\varepsilon^2+CC_1\varepsilon^2+C\varepsilon M_k(T_\varepsilon)+CC_1\varepsilon M_k(T_\varepsilon)\\ &+CC_1cM_k(T_\varepsilon)+C\int_0^{T_\varepsilon}\chi_{[0,1]}M_k(s)ds.\end{aligned}\tag{9.6}$$

By Theorem 3.1, (9.1), the inductive hypothesis and a straightforward application of the argument in [19], the fourth term in the expression to $M_k(T_\varepsilon)$ can be controlled by the right hand side of (9.6). The other terms in $M_k(T_\varepsilon)$ can be dominated by the right hand side of (8.3) with $N = 10$ and $w = w_k$. Write the right hand side of (8.3) as

$$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8.$$

$A_1 + A_2 + A_3$ can be dominated by the right hand side of (9.6) by using the argument in [19]. Moreover, it is easy to see that $A_4 + A_5$ can be also dominated by the right hand side of (9.6). Now, we deal with $A_6 + A_7 + A_8$.

When $t > 1$, we have

$$\begin{aligned}\int_1^{T_\varepsilon} \sum_{|\alpha|\le 12} |L\partial^\alpha w_k|^2 ds \le{}& C\int_1^{T_\varepsilon} \Big(\sum_{|\beta|\le 6} \|\partial^\beta w'_{k-1}\|_{L^\infty(\mathbb{R}^3\backslash\mathcal{K})}\Big)^2 \Big(\sum_{|\alpha|\le 13} \|\partial^\alpha w'_k\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\Big)^2 ds\\ &+ C\int_1^{T_\varepsilon} \Big(\sum_{|\beta|\le 6} \|\partial^\beta w'_{k}\|_{L^\infty(\mathbb{R}^3\backslash\mathcal{K})}\Big)^2 \Big(\sum_{|\alpha|\le 13} \|\partial^\alpha w'_{k-1}\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\Big)^2 ds\\ ={}& B_1 + B_2.\end{aligned}$$

By Lemma 2.6, we conclude that

$$B_1 \le CC_1\varepsilon \ln(T_\varepsilon) M_k(T_\varepsilon) \le CC_1 c M_k(T_\varepsilon).$$

Similarly, we have $B_2 \le CC_1 c M_k(T_\varepsilon)$. When $t < 1$, applying the estimate on the local solution (then on u_0) and the inductive hypothesis, we can show that A_6 can be dominated by the right hand side of (9.6). Similarly, we have

$$A_7 + A_8 \le \text{ the right hand side of (9.6)}.$$

Thus, estimate (9.6) holds. By Gronwall's inequality, from (9.6) we get

$$M_k(T_\varepsilon) \le C\varepsilon^2 + CC_1\varepsilon^2 + C\varepsilon M_k(T_\varepsilon) + CC_1\varepsilon M_k(T_\varepsilon) + CC_1 c M_k(T_\varepsilon). \tag{9.7}$$

Hence, if c is sufficiently small (independent of ε), there exists a positive constant C_1 such that

$$M_k(T_\varepsilon) \le C_1\varepsilon$$

for sufficiently small $\varepsilon > 0$. Then we get (9.5). Applying the argument in [19], we can show that

$$\begin{aligned}&\sup_{0\le t\le T_\varepsilon} \|w'_k(t,x) - w'_{k-1}(t,x)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\\ &\le \frac{1}{2} \sup_{0\le t\le T_\varepsilon} \|w'_{k-1}(t,x) - w'_{k-2}(t,x)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}, \qquad k = 2, 3, \cdots.\end{aligned}$$

Hence, $\{w_k(t,x)\}$ converges in the energy norm. Suppose that its limit is $w(t,x)$. Then $u = u_0 + w$ is a solution to problem (2.3). If the initial data $f,\ g \in C^\infty$ satisfy the compatibility conditions of infinite order, then from Theorem 2.3 we know that $u \in C^\infty([0,T_\varepsilon] \times \mathbb{R}^3\backslash\mathcal{K})$. This proves Theorem 2.1.

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