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GLOBAL EXISTENCE FOR A CLASS OF SYSTEMS OF NONLINEAR WAVE EQUATIONS IN THREE SPACE DIMENSIONS

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Abstract

Consider a system of nonlinear wave equations

$$(\partial_t^2 - c_i^2\Delta_x)u_i = F_i(u, \partial u, \partial_x\partial u) \qquad \text{in } \ (0,\infty)\times\mathbb{R}^3$$

for $i = 1, \cdots, m$, where F_i $(i = 1, \cdots, m)$ are smooth functions of degree 2 near the origin of their arguments, and $u = (u_1, \cdots, u_m)$, while ∂u and $\partial_x\partial u$ represent the first and second derivatives of u, respectively. In this paper, the author presents a new class of nonlinearity for which the global existence of small solutions is ensured. For example, global existence of small solutions for

$$\begin{cases}(\partial_t^2 - c_1^2\Delta_x)u_1 = u_2(\partial_t u_2) + \text{arbitrary cubic terms},\\(\partial_t^2 - c_2^2\Delta_x)u_2 = u_1(\partial_t u_2) + (\partial_t u_1)u_2 + \text{arbitrary cubic terms}\end{cases}$$

will be established, provided that $c_1^2 \neq c_2^2$.

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§ 1. Introduction

Throughout this paper, we use the notation $\partial_0 = \partial_t$ and $\partial_j = \partial_{x_j}$ for $j = 1, 2, 3$. For multi-indices $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$, ∂^α and ∂_x^β denote $\partial_0^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ and $\partial_1^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3}$, respectively. All the functions which will appear below are supposed to be real-valued.

This paper is devoted to the study of the Cauchy problem for systems of nonlinear wave equations of the type

$$\Box_i u_i = F_i(u, \partial u, \partial_x\partial u) \qquad \text{in } \ (0,\infty)\times\mathbb{R}^3 \quad (i = 1, \cdots, m) \tag{1.1}$$

with initial data

$$u(0,x) = \varepsilon f(x), \qquad u_t(0,x) = \varepsilon g(x), \tag{1.2}$$

where $\Box_i = \partial_t^2 - c_i^2\Delta_x$ $(i = 1, \cdots, m)$, $u = (u_j)_{j=1,\cdots,m}$, $\partial u = (\partial_a u_j)_{\substack{j=1,\cdots,m\\ a=0,\cdots,3}}$, $\partial_x\partial u = (\partial_k\partial_a u_j)_{\substack{j=1,\cdots,m\\ k=1,\cdots,3\\ a=0,\cdots,3}}$. We assume that all c_i's in the definition of $\Box_i$ are positive constants, and

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that $F = (F_i)_{i=1,\cdots,m}$ is a smooth function of degree 2 around the origin. More precisely, we assume

$$\begin{aligned} F(u,v,w) &= F\Big((u_j)_{j=1,\cdots,m},\ (v_{j,a})_{\substack{j=1,\cdots,m\\ a=0,\cdots,3}},\ (w_{j,ka})_{\substack{j=1,\cdots,m\\ k=1,\cdots,3\\ a=0,\cdots,3}} \Big) \\ &= O(|u|^2+|v|^2+|w|^2) \end{aligned} \tag{1.3}$$

around the origin in $\mathbb{R}^m \times \mathbb{R}^{4m} \times \mathbb{R}^{12m}$. Here the variables $v_{j,a}$ and $w_{j,ka}$ correspond to $\partial_a u_j$ and $\partial_k \partial_a u_j$, respectively. We also suppose that $f,\, g \in C_0^\infty(\mathbb{R}^3)$. ε is a sufficiently small and positive parameter. To ensure the hyperbolicity of the system (1.1), we always assume

$$c_{ka}^{ij}(u,v,w) = c_{ka}^{ji}(u,v,w) \qquad (i,j \in \{1,\cdots,m\},\ k \in \{1,2,3\},\ a \in \{0,\cdots,3\}) \tag{1.4}$$

for any (u,v,w) in some neighbourhood of the origin in $\mathbb{R}^m \times \mathbb{R}^{4m} \times \mathbb{R}^{12m}$, where

$$c_{ka}^{ij}(u,v,w) = \frac{\partial F_i}{\partial w_{j,ka}}(u,v,w). \tag{1.5}$$

Because we only consider classical solutions, we may also assume, without loss of generality, that

$$c_{kl}^{ij} = c_{lk}^{ij} \qquad (i,j \in \{1,\cdots,m\},\ k,l \in \{1,2,3\}). \tag{1.6}$$

We are concerned with the condition to ensure the global existence of solutions to (1.1) for small data. In the following, we say that (GE) holds when for any $f,\, g \in C_0^\infty(\mathbb{R}^3)$, there exists a positive constant ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$, the Cauchy problem (1.1)–(1.2) admits a unique solution $u \in C^\infty([0,\infty) \times \mathbb{R}^3; \mathbb{R}^m)$.

We want to recall some known results briefly. It is known that (GE) does not hold for some quadratic nonlinearity. Hence we need some condition on F to ensure (GE). Such a condition is called the Null Condition. The case where $c_1 = \cdots = c_m$ and $F = F(u, \partial u, \partial_x \partial u)$ was studied by Klainerman [10] and Christodoulou [2]. The case where the speeds c_i do not necessarily coincide with each other and F depends only on derivatives of u, namely $F = F(\partial u, \partial_x \partial u)$, was studied by Kovalyov [12], Agemi-Yokoyama [1], Yokoyama [19] and Sideris-Tu [17]. To state their results precisely, we introduce the Null Condition for the case of multiple speeds. Set

$$I(j) = \{k \in \{1,\cdots,m\};\ c_k = c_j\} \qquad \text{for } j \in \{1,2,\cdots,m\} \tag{1.7}$$

and

$$Y_i^m = \{y = (y_1,\cdots,y_m) \in \mathbb{R}^m; y_j = 0 \text{ if } j \notin I(i)\}. \tag{1.8}$$

For a smooth function $G(u,v,w)$ and a positive integer p, we define

$$G^{(p)}(u,v,w) = \sum_{|\alpha|+|\beta|+|\gamma|=p} (\partial_u^\alpha \partial_v^\beta \partial_w^\gamma G)(0,0,0) \frac{u^\alpha v^\beta w^\gamma}{\alpha!\beta!\gamma!},$$

where we have used the standard multi-index notation.

Definition 1.1. (The Null Condition) *We say $F(u,v,w) = (F_i(u,v,w))_{i=1,\cdots,m}$ satisfies the Null Condition if the following two conditions hold for each $i \in \{1,\cdots,m\}$:*

(i) *For any U, μ, $\nu \in Y_i^m$ and any $X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4$ with $X_0^2 - X_1^2 - X_2^2 - X_3^2 = 0$, we have*

$$F_i^{(2)}(U, V(\mu, X), W(\nu, X)) = 0,$$

where $V \in \mathbb{R}^{4m}$ and $W \in \mathbb{R}^{12m}$ are given by

$$V(\mu, X) = (V_{j,a}(\mu, X))_{j,a} = (\mu_j X_a)_{j,a} \tag{1.9}$$

and

$$W(\nu, X) = (W_{j,ka}(\nu, X))_{j,k,a} = (\nu_j X_k X_a)_{j,k,a}, \tag{1.10}$$

respectively with $1 \le j \le m$, $0 \le a \le 3$ *and* $1 \le k \le 3$.

(ii) *For any* $(u, v, w) \in \mathbb{R}^m \times \mathbb{R}^{4m} \times \mathbb{R}^{12m}$ *and any* $j, k \in \{1, \cdots, m\}$, *we have*

$$\frac{\partial^2 F_i^{(2)}}{\partial u_j \partial u_k}(u, v, w) = 0.$$

Note that the above Null Condition coincides with Klainerman's Null Condition when $c_1 = \cdots = c_m$, and with Agemi-Yokoyama's when F does not depend on u explicitly, namely $F = F(v, w)$.

Using this Null Condition, known results mentioned in the above can be summarized as the following: If F satisfies the Null Condition and furthermore if we have either $c_1 = \cdots = c_m$ or $F = F(v, w)$, then (GE) holds (see [3–7] for the correponding results of two space dimensional systems with cubic nonlinearity). Therefore we are interested in the general case where $F = F(u, v, w)$ and the speeds may be different from each other. Such a case was treated by Kubota–Yokoyama [13], and quite recently the author [9] extended their result and showed that if F satisfies the Null Condition and

(H1) For each $i \in \{1, \cdots, m\}$, $F_i^{(2)}$ does not depend on u, namely

$$\frac{\partial F_i^{(2)}}{\partial u_j}(u, v, w) = 0 \quad \text{for all} \;\; j \in \{1, \cdots, m\} \;\; \text{and any} \;\; (u, v, w) \in \mathbb{R}^m \times \mathbb{R}^{4m} \times \mathbb{R}^{12m},$$

then (GE) holds. Notice that (H1) is a stronger version of the condition (ii) in the definition of the Null Condition. Note also that, in all the results mentioned above, $F_i^{(2)}$ is always independent of u itself (it follows from the Null Condition if $c_1 = \cdots = c_m$, and it is apparent when either $F = F(v, w)$ or (H1) holds).

Our aim in this paper is to establish a global existence result for equations with nonlinearity F whose quadratic part $F_i^{(2)}$ may depend also on u. In other words, we would like to find a condition, an alternative to (H1), which allows quadratic terms including u. Our main result is the following

Theorem 1.1. *Suppose that* (1.4) *holds and that* F *satisfies the Null Condition. Furthermore, assume*

(H2) *For each* $i \in \{1, \cdots, m\}$, *there exist some expressions* $G_{i,a}$ $(a = 0, 1, 2, 3)$ *such that*

$$F_i^{(2)}(u, \partial u, \partial_x \partial u) = \sum_{a=0}^{3} \partial_a \{G_{i,a}(u, \partial u)\} \qquad (i = 1, \cdots, m)$$

holds for any "small" functions $u \in C^2$.
Then we have (GE) *for the Cauchy problem* (1.1)–(1.2).

Remark 1.1. We have assumed that the functions f and g in initial data (1.2) are compactly supported for simplicity of exposition of the known results, but it is not used essentially in the proof of Theorem 1.1. The support condition on data is not essential in most of the known results in the above, but is used essentially in the proof of some results including [6, 7, 13].

The first point of our theorem is that we can treat $F_i^{(2)}$ including terms depending on both of u and derivatives of u. For example, $u_j(\partial_a u_j)$ with $j \notin I(i)$ can be contained in $F_i(u, \partial u, \partial_x \partial u)$, because it satisfies the Null Condition and $u_j(\partial_a u_j) = \partial_a(u_j^2/2)$. Similarly, terms like $u_j(\partial_a u_k) + u_k(\partial_a u_j)$ with $(j,k) \notin I(i) \times I(i)$ also can be included in F_i.

The second point is that there is no restriction on higher nonlinearity. It may sound obvious that nonlinearity of higher power than the critical power does not affect the existence of the solution. However, sometimes it is not so easy to prove such a result, especially when the nonlinearity depends on both of u and its derivatives. For example, in two space dimensions, we have "almost global" existence for $\Box u = u(u_t)^2$, and global existence for $\Box u = u^4$. Therefore, it is natural to expect we have at least "almost global" existence also for $\Box u = u(u_t)^2 + u^4$, but the known frameworks to prove each results are completely different from each other, and this problem still remains open as far as the author knows (see [14, 15, 8]).

Before concluding this section, we will give an explicit representation of F satisfying both our Null Condition and the assumption (H2).

First we introduce some notations. Throughout this paper, for a set of functions $\{f_\lambda\}_{\lambda\in\Lambda}$ and a function g, we write $g = \sum'_{\lambda\in\Lambda} f_\lambda$ if there exist some constants C_λ $(\lambda \in \Lambda)$ such that $g = \sum_{\lambda\in\Lambda} C_\lambda f_\lambda$.

For a positive constant c, and smooth functions ϕ and ψ, we define the null forms:

$$Q_0(\phi, \psi; c) = (\partial_t\phi)(\partial_t\psi) - c^2 \sum_{j=1}^{3} (\partial_j\phi)(\partial_j\psi), \tag{1.11}$$

$$Q_{ab}(\phi, \psi) = (\partial_a\phi)(\partial_b\psi) - (\partial_b\phi)(\partial_a\psi) \qquad (0 \le a, b \le 3). \tag{1.12}$$

For all $i \in \{1, \cdots, m\}$, we set $H_i = F_i - F_i^{(2)}$. Then, from the definition of $F_i^{(2)}$, we have

$$H_i(u, v, w) = O(|u|^3 + |v|^3 + |w|^3). \tag{1.13}$$

Using these notations, if F satisfies the Null Condition, we have

$$F_i(u, \partial u, \partial_x\partial u) = N_i(\partial u, \partial_x\partial u) + R_i(u, \partial u, \partial_x\partial u) + H_i(u, \partial u, \partial_x\partial u)$$

for all $i = 1, \cdots, m$, where

$$N_i(\partial u, \partial_x\partial u) = {\sum_{(j,k)\in I(i)\times I(i)}}' \Bigg({\sum_{|\alpha|,|\beta|=0,1}}' Q_0(\partial^\alpha u_j, \partial^\beta u_k; c_i) + {\sum_{\substack{|\alpha|,|\beta|=0,1\\ a,b}}}' Q_{ab}(\partial^\alpha u_j, \partial^\beta u_k) \Bigg),$$

$$R_i(u, \partial u, \partial_x\partial u) = {\sum_{(j,k)\notin I(i)\times I(i)}}' \Bigg({\sum_{\substack{|\alpha|,|\beta|=0,1,2\\ \text{with } |\alpha|+|\beta|\ge 1}}}' (\partial^\alpha u_j)(\partial^\beta u_k) \Bigg)$$

(refer to [1, 2, 10, 19]). Moreover, if we further assume (H2) in addition to the Null Condition, we easily get the expression

$$F_i(u, \partial u, \partial_x\partial u) = \Big(\sum_{a=0}^{3} \partial_a \{G_{i,a}(u, \partial u)\} \Big) + H_i(u, \partial u, \partial_x\partial u) \tag{1.14}$$

for $i \in \{1, \cdots, m\}$, where $G_{i,a}$ has the form

$$G_{i,a}(u, \partial u) = N_{i,a}(\partial u) + R_{i,a}(u, \partial u) \tag{1.15}$$

with

$$N_{i,a}(\partial u) = {\sum_{(j,k)\in I(i)\times I(i)}}' \Big(Q_0(u_j, u_k; c_i) + {\sum_{a,b}}' Q_{ab}(u_j, u_k) \Big), \tag{1.16}$$

$$R_{i,a}(u, \partial u) = {\sum_{(j,k)\notin I(i)\times I(i)}}' \Big({\sum_{|\alpha|,|\beta|=0,1}}' (\partial^\alpha u_j)(\partial^\beta u_k) \Big). \tag{1.17}$$

§ 2. Notations and Preliminary Results

First we introduce some notations which will be used throughout this paper.

2.1. Notations. Let f and g be functions in $\mathbf{S}$, where $\mathbf{S}$ denotes the class of rapidly decreasing functions, and $c_1, \cdots, c_m$ be positive constants given in (1.1). For each $i \in \{1, \cdots, m\}$, we define a mapping $U_i^*[f,g]$ by

$$U_i^*[f,g](t,x) = u(t,x) \qquad \text{for} \ \ (t,x) \in [0,\infty) \times \mathbb{R}^3, \tag{2.1a}$$

where u is the unique classical solution to

$$\begin{cases} (\partial_t^2 - c_i^2 \Delta_x) u(t,x) = 0 & \text{for} \ \ (t,x) \in (0,\infty) \times \mathbb{R}^3, \\ u(0,x) = f(x), \ (\partial_t u)(0,x) = g(x) & \text{for} \ \ x \in \mathbb{R}^3. \end{cases} \tag{2.1b}$$

Also, for a given function $\phi = \phi(t,x)$, we define another mapping $U_i[\phi]$ by

$$U_i[\phi](t,x) = v(t,x) \qquad \text{for} \ \ (t,x) \in [0,\infty) \times \mathbb{R}^3, \tag{2.2a}$$

where v is the unique classical solution to

$$\begin{cases} (\partial_t^2 - c_i^2 \Delta_x) v(t,x) = \phi(t,x) & \text{for} \ \ (t,x) \in (0,\infty) \times \mathbb{R}^3, \\ v(0,x) = (\partial_t v)(0,x) = 0 & \text{for} \ \ x \in \mathbb{R}^3. \end{cases} \tag{2.2b}$$

Now, we introduce some vector fields:

$$\begin{gathered} \Gamma_0 = t\partial_t + \sum_{j=1}^{3} x_j \partial_j, \ \Gamma_1 = \partial_t, \ \Gamma_2 = \partial_1, \ \Gamma_3 = \partial_2, \ \Gamma_4 = \partial_3, \\ \Gamma_5 = \Omega_{12}, \ \Gamma_6 = \Omega_{13}, \ \Gamma_7 = \Omega_{23}, \end{gathered} \tag{2.3}$$

where $\Omega_{ij} = x_i \partial_j - x_j \partial_i$. We write Γ^α for $\Gamma_0^{\alpha_0} \Gamma_1^{\alpha_1} \cdots \Gamma_7^{\alpha_7}$ using a multi-index α. It is easy to check that for any smooth function ϕ, we have

$$\Gamma^\alpha \Gamma^\beta \phi(t,x) = \Gamma^{\alpha+\beta} \phi(t,x) + \sideset{}{'}\sum_{|\gamma| \le |\alpha| + |\beta| - 1} \Gamma^\gamma \phi(t,x), \tag{2.4}$$

$$\Gamma^\alpha \partial_a \phi(t,x) = \partial_a \Gamma^\alpha \phi(t,x) + \sideset{}{'}\sum_{\substack{b \in \{0,1,2,3\} \\ |\beta| \le |\alpha| - 1}} \partial_b \Gamma^\beta \phi(t,x) \qquad \text{for any } a \in \{0,1,2,3\}, \tag{2.5}$$

$$\partial_a \Gamma^\alpha \phi(t,x) = \Gamma^\alpha \partial_a \phi(t,x) + \sideset{}{'}\sum_{\substack{b \in \{0,1,2,3\} \\ |\beta| \le |\alpha| - 1}} \Gamma^\beta \partial_b \phi(t,x) \qquad \text{for any } a \in \{0,1,2,3\}. \tag{2.6}$$

We also have

$$\Box_i \Gamma^\alpha \phi(t,x) = \Gamma^\alpha(\Box_i \phi(t,x)) + \sideset{}{'}\sum_{|\beta| \le |\alpha| - 1} \Gamma^\beta (\Box_i \phi(t,x)), \tag{2.7}$$

where $\Box_i = \partial_t^2 - c_i^2 \Delta_x$.

For a non-negative integer s and a function v, we define

$$|v(t,x)|_s = \sum_{0 \le |\alpha| \le s} |\Gamma^\alpha v(t,x)|, \tag{2.8}$$

$$\|v(t,\cdot)\|_{s,p} = \| |v(t,\cdot)|_s \|_{L^p(\mathbb{R}^3)} \qquad (1 \le p \le \infty). \tag{2.9}$$

2.2. Preliminary Results. We start this subsection with the well-known energy inequality for hyperbolic systems.

Lemma 2.1. (The Energy Inequality) *Let $v=(v_1,\cdots,v_m)$ be a solution to*

$$\partial_t^2 v_i(t,x)-\sum_{j=1}^{m}\sum_{k=1}^{3}\sum_{a=0}^{3}S_{k,a}^{i,j}(t,x)(\partial_k\partial_a v_j)(t,x)=\phi_i(t,x)\qquad in\ \ (0,\infty)\times\mathbb{R}^3$$

for $i=1,\cdots,m$, with initial data $v=f$ and $v_t=g$ at $t=0$. Assume that we have $S_{k,a}^{i,j}=S_{k,a}^{j,i}$ and $S_{k,l}^{i,j}=S_{l,k}^{i,j}$ for any $i,j\in\{1,\cdots,m\}$, $k,l\in\{1,2,3\}$ and $a\in\{0,1,2,3\}$. We also suppose that there exists some positive constant M such that

$$M^{-1}|\xi|^2\le\sum_{i,j=1}^{m}\sum_{k,l=1}^{3}S_{k,l}^{i,j}(t,x)\xi_{i,k}\xi_{j,l}\le M|\xi|^2$$

holds for any $(t,x)\in[0,\infty)\times\mathbb{R}^3$ and any $\xi=(\xi_{i,k})_{\substack{i=1,\cdots,m\\ k=1,2,3}}\in\mathbb{R}^{3m}$. Then we have

$$\|\partial v(t,\cdot)\|_{L^2}\le C(\|f\|_{H^1}+\|g\|_{L^2})+C\int_0^t(\|\partial S(\tau,\cdot)\|_{L^\infty}\|\partial v(\tau,\cdot)\|_{L^2}+\|\phi(\tau,\cdot)\|_{L^2})d\tau,\tag{2.10}$$

where $S=(S_{k,a}^{i,j})_{\substack{i,j=1,\cdots,m\\ k=1,\cdots,3\\ a=0,\cdots,3}}$ and $\phi=(\phi_i)_{i=1,\cdots,m}$.

Since this energy inequality can be shown by the classical and standard argument, we omit the proof here.

Let $\phi=\phi(t,x)$ be a sufficiently smooth function on $(0,\infty)\times\mathbb{R}^3$. We will use notations in Subsection 2.1. The following lemma is classical (see [18]).

Lemma 2.2. *Let $i\in\{1,\cdots,m\}$. Then we have*

$$\|U_i[\phi](t,\cdot)\|_{L^2(\mathbb{R}^3)}\le C\int_0^t\|\phi(\tau,\cdot)\|_{L^{6/5}(\mathbb{R}^3)}d\tau\qquad for\ \ t>0,\tag{2.11}$$

$$\|U_i^*[f,g](t,\cdot)\|_{L^2(\mathbb{R}^3)}\le C(\|f\|_{L^2(\mathbb{R}^3)}+\|g\|_{L^{6/5}(\mathbb{R}^3)})\qquad for\ \ t>0.\tag{2.12}$$

Since we have

$$U_i[\partial_a\phi]=\partial_a U_i[\phi]-\delta_{0,a}U_i^*[0,\phi(0,\cdot)],\qquad a=0,1,2,3,\tag{2.13}$$

by using Kronecker's delta $\delta_{i,j}$, the standard energy inequality for wave equations, together with (2.12), implies the following

Lemma 2.3. *Let $i\in\{1,\cdots,m\}$ and $a\in\{0,1,2,3\}$. Then we have*

$$\|U_i[\partial_a\phi](t,\cdot)\|_{L^2}\le C\Big(\int_0^t\|\phi(\tau,\cdot)\|_{L^2}d\tau+\delta_{0,a}\|\phi(0,\cdot)\|_{L^{6/5}}\Big)\qquad for\ \ t>0.\tag{2.14}$$

For the detail of the proof, see [16] for example.

§ 3. L^∞ Decay Estimates for Linear Wave Equations

In this section we will derive L^∞ decay estimates for solutions of linear wave equations.

We introduce some weight functions which are concerned with decay of solutions to wave equations. For $(t,x)\in\mathbb{R}_+\times\mathbb{R}^3$, we define

$$w_+(t,x)=1+t+|x|,\tag{3.1}$$

$$w_i(t,x)=1+|c_it-|x||\qquad \text{for}\ \ i=1,\cdots,m.\tag{3.2}$$

For convenience of exposition, we set $c_0 = 0$, and define

$$w_0(t,x) = 1 + |c_0 t - |x|| = 1 + |x|. \tag{3.3}$$

First we state a known decay estimate for homogeneous wave equations (see [13] for instance).

Lemma 3.1. *Let $i \in \{1, \cdots m\}$. Suppose that f, $g \in \mathbf{S}(\mathbb{R}^3)$. Then there exists some constant C, depending on f and g, such that*

$$(w_+ w_i)(t,x)|U_i^*[f,g](t,x)| \le C \qquad \text{for } \ t > 0 \text{ and } x \in \mathbb{R}^3. \tag{3.4}$$

Now we turn our attention to inhomogeneous wave equations. The following weighted $L^\infty - L^\infty$ estimate played an essential role in [19].

Lemma 3.2. *Let $c_1, \cdots, c_m$ be positive constants appeared in* (1.1)*, and $c_0 = 0$. Assume $i = 1, \cdots, m$ and $J = 0, 1, \cdots, m$. We define*

$$\Phi_\theta(t) = \begin{cases} \log(2+t), & \text{when } \ \theta = 0, \\ 1, & \text{when } \ \theta > 0. \end{cases} \tag{3.5}$$

(i) *We have*

$$\begin{aligned} |\partial_a U_i[\phi](t,x)| \le{} & C w_0^{-1}(t,x)\{w_i^{-\mu}(t,x)\Phi_{\nu-1}(t) + w_i^{-\nu}(t,x)\Phi_{\mu-1}(t)\} \\ & \times \sup_{\substack{0\le\tau\le t \\ y\in\mathbb{R}^3}} |y|(w_+^\mu w_J^\nu)(\tau,y)|\phi(\tau,y)|_1 \end{aligned} \tag{3.6}$$

for $\mu \ge 1$ and $\nu \ge 1$.

(ii) *Moreover, if $c_i \ne c_J$, we have*

$$|\partial_a U_i[\phi](t,x)| \le C(w_0^{-1} w_i^{-\mu})(t,x)\Phi_{\nu-1}(t) \sup_{\substack{0\le\tau\le t \\ y\in\mathbb{R}^3}} |y|(w_+^\mu w_J^\nu)(\tau,y)|\phi(\tau,y)|_1 \tag{3.7}$$

for $\mu > 0$ and $\nu \ge 1$.
Here the constant C may depend on i, J, μ and ν, but is independent of the other quantities.

The proof of Lemma 3.2 is rather complicated, but can be done in a similar spirit to the proof of Proposition 3.4 below. Refer to [19] for the details.

Remark 3.1. More precisely, we do not need Γ_0 in Lemma 3.1, i.e., (3.6) and (3.7) stay valid if $|\phi|_1$ is replaced by $|\phi| + |\partial\phi| + |\Omega\phi|$.

From Lemma 3.2, we get the following corollary.

Corollary 3.1. *Let $i, k \in \{1, \cdots, m\}$. Then for $\mu \ge 1$, we have*

$$(w_0 w_i^\mu)(t,x)|\partial_a U_i[\phi](t,x)| \le C\Phi_{\mu-1}(t) \sup_{\substack{0\le\tau\le t \\ y\in\mathbb{R}^3}} |y|(w_0^\mu w_k^\mu)(\tau,y)|\phi(\tau,y)|_1, \tag{3.8}$$

where Φ_θ is given by (3.5).

Proof. First, suppose that $\phi = \phi(\tau, y)$ is supported on the set $\{(\tau, y); c_k\tau \le 3|y|\}$. Then there exists a constant C such that $w_+ \le Cw_0$ in the support of ϕ. Hence (3.8) for such ϕ follows from Lemma 3.2(i) with the choice $(J, \nu) = (k, \mu)$. Similarly, if ϕ is supported on the set $\{(\tau, y); c_k\tau \ge 2|y| \text{ or } |\tau|^2 + |y|^2 \le 2\}$, we get (3.8) from Lemma 3.2(ii) with the choice $(J, \nu) = (0, \mu)$, because we have $w_+ \le Cw_k$ on the support of ϕ. Now an appropriate partition of unity implies the assertion for general ϕ.

To control the contribution of the higher nonlinearity to the solution, we have to estimate L^∞ norm of the solution itself in addition to its derivatives.

Proposition 3.1. *Let $i, j \in \{1, \cdots, m\}$. Then we have*

$$(w_+ w_i)(t,x)|U_i[\phi](t,x)| \le C \sup_{\substack{0\le\tau\le t\\ y\in\mathbb{R}^3}} |y|(w_0^2 w_j^2)(\tau,y)|\phi(\tau,y)|. \tag{3.9}$$

Proof. It suffices to prove (3.9) assuming $c_i = 1$ and $c_j = c\ (>0)$.
Let $T > 0$, and set $M = \sup_{\substack{0\le\tau\le T\\ y\in\mathbb{R}^3}} |y|(w_0^2 w_j^2)(\tau,y)|\phi(\tau,y)|$. Let v be a solution to

$$(\partial_t^2 - \Delta_x)v(t,x) = r^{-1}(1+r)^{-2}(1+|ct-r|)^{-2}M \tag{3.10}$$

with $v = v_t = 0$ at $t = 0$, where $r = |x|$. Since $|\phi(t,x)|$ is dominated by the right hand side of (3.10) for any $(t,x) \in [0,T]\times\mathbb{R}^3$, the positivity of the fundamental solution to the wave equation in three space dimensions implies

$$|U_i[\phi](t,x)| \le v(t,x) \qquad \text{for } \ 0 \le t \le T \ \text{ and } \ x \in \mathbb{R}^3. \tag{3.11}$$

Since v is spherically symmetric, and

$$(\partial_t^2 - \partial_r^2)(rv) = (1+r)^{-2}(1+|ct-r|)^{-2}M,$$

v can be expressed explicitly as

$$v(t,x) = MI(t,r), \tag{3.12}$$

where

$$I(t,r) = \frac{1}{r}\int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{d\rho}{(1+\rho)^2(1+|cs-\rho|)^2}. \tag{3.13}$$

Hence the assertion of the proposition is an immediate consequence of (3.11), if we can show that

$$I(t,r) \le C(1+t+r)^{-1}(1+|t-r|)^{-1}. \tag{3.14}$$

Now we are going to prove (3.14). By changing variables in the integral by $p = \rho + s$ and $q = \rho - cs$, we get

$$I(t,r) = \frac{1}{(c+1)r}\int_{|r-t|}^{t+r} dp \int_{p_1}^{p} \frac{dq}{\left(1+\frac{cp+q}{c+1}\right)^2(1+|q|)^2}, \tag{3.15}$$

where $p_1 = p_1(p,r,t) = \frac{(1-c)p+(1+c)(r-t)}{2}$. Set

$$R_{r,t} = \{(p,q);\ p_1(p,r,t) \le q \le p,\ |r-t| \le p \le r+t\}.$$

We define

$$I_1(p,t,r) = \int_0^p \frac{dq}{\left(1+\frac{cp+q}{c+1}\right)^2(1+|q|)^2}, \tag{3.16}$$

$$I_2(p,t,r) = \int_{p_1}^0 \frac{dq}{\left(1+\frac{cp+q}{c+1}\right)^2(1+|q|)^2}. \tag{3.17}$$

Then we have

$$I(t,r) = \frac{1}{(c+1)r}\int_{|t-r|}^{t+r} (I_1(p,t,r) + I_2(p,t,r))dp. \tag{3.18}$$

Since
$$1+\frac{cp+q}{c+1}\geq \frac{c}{c+1}(1+p) \qquad \text{when} \quad q\geq 0,$$
we get
$$I_1(p,t,r)\leq \frac{C}{(1+p)^2}\int_{-\infty}^{\infty}\frac{dq}{(1+|q|)^2}=\frac{2C}{(1+p)^2}, \tag{3.19}$$
where C is a constant independent of t and r.

If $p_1\geq 0$, it is clear that $I_2(t,r)\leq 0$, and I_2 is negligible. Therefore we may assume $p_1<0$ in the following estimate of I_2. Let δ be a constant satisfying $0<\delta<1$. We split I_2 into two parts:
$$\begin{aligned} I_2(p,t,r) &= \int_{p_1}^{\delta p_1}\frac{dq}{\left(1+\frac{cp+q}{c+1}\right)^2(1+|q|)^2}+\int_{\delta p_1}^{0}\frac{dq}{\left(1+\frac{cp+q}{c+1}\right)^2(1+|q|)^2} \\ &\equiv I_3(p,t,r)+I_4(p,t,r). \end{aligned} \tag{3.20}$$
We have
$$1+\frac{cp+q}{c+1}\geq 1+\frac{cp+\delta p_1}{c+1}\geq 1+\frac{c}{c+1}(1-\delta)p$$
for any (p,q) satisfying $\delta p_1\leq q\leq 0$ and $|t-r|\leq p$, because we have $p_1\geq -cp$ for such p and q. Hence, similarly to (3.19), we obtain
$$I_4(p,t,r)\leq \frac{C_\delta}{(1+p)^2}. \tag{3.21}$$

From now on, we concentrate on I_3. Let ε be a positive constant. First we consider the case $p\geq(1+\varepsilon)|t-r|$. We have
$$1+\frac{cp+q}{c+1}\geq 1+\frac{p-|t-r|}{2}\geq 1+\frac{\varepsilon}{2(1+\varepsilon)}p\geq C_\varepsilon(1+p) \qquad \text{for} \quad q\geq p_1.$$
Therefore, similarly to (3.19), we obtain
$$I_3(p,t,r)\leq \frac{C_\varepsilon}{(1+p)^2} \qquad \text{for} \quad p\geq (1+\varepsilon)|t-r|. \tag{3.22}$$
Secondly we consider the case $|t-r|\leq p\leq (1+\varepsilon)|t-r|$. We may assume $r<t$ because $r\geq t$ implies $p_1\geq 0$ for such p, provided that ε is small enough. This time we have
$$\begin{aligned} 1+|q|=1-q &\geq 1+\delta\Big(\frac{(1+c)(t-r)-(1-c)p}{2}\Big) \\ &\geq 1+\frac{(1+c)-|1-c|(1+\varepsilon)}{2(1+\varepsilon)}\delta p \end{aligned} \tag{3.23}$$
for any (p,q) satisfying $p_1<q<\delta p_1(<0)$ and $(0<)t-r\leq p\leq(1+\varepsilon)(t-r)$. Hence there exists a positive constant $C_{\varepsilon,\delta}$ such that
$$1+|q|\geq C_{\varepsilon,\delta}(1+p) \tag{3.24}$$
for (p,q) satisfying $p_1<q<\delta p_1$ and $0<t-r\leq p\leq(1+\varepsilon)(t-r)$, provided that ε satisfies $1+c-|1-c|(1+\varepsilon)>0$. Note that we can find such ε (>0), because $1+c-|1-c|>0$ for $c>0$. By (3.24), we obtain
$$I_3(p,t,r)\leq \frac{C}{(1+p)^2}\int_{p_1}^{\delta p_1}\frac{dq}{\left(1+\frac{cp+q}{c+1}\right)^2}\leq \frac{C}{(1+p)^2}\frac{1}{1+\frac{p-(t-r)}{2}}\leq \frac{C}{(1+p)^2} \tag{3.25}$$

for $t-r \le p \le (1+\varepsilon)(t-r)$, where C is a constant independent of t and r.

Finally (3.21), (3.22) and (3.25) imply

$$I_2(p,t,r) \le \frac{C}{(1+p)^2}, \tag{3.26}$$

which, together with (3.19), leads to

$$\begin{aligned} I(t,r) \le & \frac{C}{r}\int_{|t-r|}^{t+r} \frac{dp}{(1+p)^2} = \frac{C}{r}\Big(\frac{1}{1+|t-r|} - \frac{1}{1+t+r}\Big) \\ = & \frac{C(t+r-|t-r|)}{r(1+|t-r|)(1+t+r)} \le \frac{2C}{(1+|t-r|)(1+t+r)}, \end{aligned} \tag{3.27}$$

because $t+r-|t-r| \le 2r$. This completes the proof.

Now we are going to unify the weight functions for each speeds to one weight function. We set

$$w_-(t,x) = \min_{j=1,\cdots,m} w_j(t,x). \tag{3.28}$$

Then, making use of an appropriate partition of unity, we find the following inequalities from Corollary 3.1 and Proposition 3.1.

Corollary 3.2. *Let $i \in \{1,\cdots,m\}$. Then we have*

$$(w_0 w_i)(t,x)|\partial_a U_i[\phi](t,x)| \le C \sup_{\substack{0\le\tau\le t\\ y\in\mathbb{R}^3}} |y|(w_0^\mu w_-^\mu)(\tau,y)|\phi(\tau,y)|_1 \qquad \text{if } \mu > 1, \tag{3.29}$$

$$(w_0 w_i)(t,x)|\partial_a U_i[\phi](t,x)| \le C\log(2+t) \sup_{\substack{0\le\tau\le t\\ y\in\mathbb{R}^3}} |y|(w_0 w_-)(\tau,y)|\phi(\tau,y)|_1 \tag{3.30}$$

for any $a = 0,1,2,3$, and

$$(w_+ w_i)(t,x)|U_i[\phi](t,x)| \le C \sup_{\substack{0\le\tau\le t\\ y\in\mathbb{R}^3}} |y|(w_0^2 w_-^2)(\tau,y)|\phi(\tau,y)|. \tag{3.31}$$

§ 4. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. Because of the classical local existence theorem, it suffices to get some a priori estimate of the solution.

Let $T > 0$ and $u^{(\varepsilon)} = (u_1^{(\varepsilon)},\cdots,u_m^{(\varepsilon)})$ be the solution to (1.1)–(1.2) for $0 \le t < T$, where ε is the parameter appeared in (1.2).

We define

$$E_\varepsilon(t) = \sup_{0\le\tau<t} \sum_{i=1}^m e_{\varepsilon,i}(\tau), \tag{4.1}$$

where

$$\begin{aligned} e_{\varepsilon,i}(t) = & \|(w_0 w_i)(t,\cdot)\,|u_i^{(\varepsilon)}(t,\cdot)|_{K+2}\|_{L^\infty(\mathbb{R}^3)} \\ & + (1+t)^{-\lambda}(\|u_i^{(\varepsilon)}(t,\cdot)\|_{2K,2} + \|\partial u_i^{(\varepsilon)}(t,\cdot)\|_{2K,2}) \\ & + \|(w_0^{1-2\lambda} w_i^{1-2\lambda})(t,\cdot)\,|u_i^{(\varepsilon)}(t,\cdot)|_{2K-3}\|_{L^\infty(\mathbb{R}^3)} \\ & + \|(w_0 w_i^{1-2\lambda})(t,\cdot)\,|u_i^{(\varepsilon)}(t,\cdot)|_{2K-8}\|_{L^\infty(\mathbb{R}^3)} + \|u_i^{(\varepsilon)}(t,\cdot)\|_{2K-4,2}. \end{aligned} \tag{4.2}$$

In the above, λ is a positive and sufficiently small constant, and K is an integer satisfying $K \geq 12$.

Since $1+t \leq C(w_0 w_i)(t,x)$ for any $t \geq 0$ $(i=1,\cdots,m)$, we have

$$\sup_{x\in\mathbb{R}^3} |u_i^{(\varepsilon)}(t,x)|_{K+2} \leq C(1+t)^{-1}E_\varepsilon(T) \qquad \text{for} \ \ 0 \leq t < T. \tag{4.3}$$

By the Sobolev type inequality

$$\sup_{x\in\mathbb{R}^3} |x|\,|v(x)| \leq C \sum_{|\alpha|+|\beta|\leq 2} \|\partial_x^\alpha \Omega^\beta v\|_{L^2(\mathbb{R}^3)},$$

which holds for any $v \in \mathbf{S}(\mathbb{R}^3)$ (see [11] for the proof), we also have

$$\sup_{x\in\mathbb{R}^3} |x|\{(1+t)^{-\lambda}(|u_i^{(\varepsilon)}(t,x)|_{2K-2} + |\partial u_i^{(\varepsilon)}(t,x)|_{2K-2}) + |u_i^{(\varepsilon)}(t,x)|_{2K-6}\} \leq CE_\varepsilon(t) \tag{4.4}$$

for any $i \in \{1,\cdots,m\}$ and $t>0$.

Our aim in this section is to prove the following proposition.

Proposition 4.1. *Let $T>0$. There exist positive constants M_0, C_0 and ε_1, which are independent of T, such that $E_\varepsilon(T) \leq M_0$ implies*

$$E_\varepsilon(T) \leq C_0(\varepsilon + E_\varepsilon(T)^2), \tag{4.5}$$

provided that $\varepsilon \leq \varepsilon_1$.

If once Proposition 4.1 is established, the bootstrap argument implies that there exists a positive constant ε_0 $(\leq \varepsilon_1)$ such that for any $\varepsilon \leq \varepsilon_0$, $E_\varepsilon(t)$ stays bounded as far as the solution exists. This a priori estimate, together with the local existence theorem, implies the existence of global solutions. This completes the proof of Theorem 1.1. Hence our task is to prove Proposition 4.1.

Now we are in a position to prove Proposition 4.1. We always assume that $E_\varepsilon(T)$ and ε are sufficiently small in what follows. We also abbreviate $u^{(\varepsilon)}$ as u for simplicity of exposition.

Proof of Proposition 4.1. Operating Γ^α to (1.1), by (2.7) we have

$$\Box_i(\Gamma^\alpha u_i) = \sum_{|\beta|\leq|\alpha|}{}' \, \Gamma^\beta F_i(u,\partial u,\partial_x\partial u). \tag{4.6}$$

Therefore, remembering (2.5) and the expression (1.14), we find

$$\begin{aligned}\Gamma^\alpha u_i = U_i^*[f_{i,\alpha},g_{i,\alpha}] &+ \sum_{\substack{|\beta|\leq|\alpha| \\ a,b\in\{0,1,2,3\}}}{}' \, (U_i[\partial_b(\Gamma^\beta N_{i,a})] + U_i[\partial_b(\Gamma^\beta R_{i,a})]) \\ &+ \sum_{|\beta|\leq|\alpha|}{}' \, U_i[\Gamma^\beta H_i],\end{aligned} \tag{4.7}$$

where $f_{i,\alpha} = (\Gamma^\alpha u_i)(0,\cdot)$ and $g_{i,\alpha} = (\partial_t\Gamma^\alpha u_i)(0,\cdot)$.

4.1. Estimates for $|u_i(t,x)|_{2K-3}$. Let $|\alpha| \leq 2K-3$ in (4.7). Throughout this subsection, we assume that β is a multi-index with $|\beta| \leq 2K-3$.

Lemma 3.1 implies that

$$(w_+w_i)(t,x)|U_i^*[f_{i,\alpha},g_{i,\alpha}](t,x)|\le C\varepsilon, \tag{4.8}$$

$$(w_+w_i)(t,x)|U_i^*[0,(\Gamma^\beta R_{i,a})(0,\cdot)](t,x)|\le C\varepsilon^2. \tag{4.9}$$

Since $|R_{i,a}(u,\partial u)|\le C(|u|^2+|\partial u|^2)$, (3.30) in Corollary 3.2 leads to

$$\begin{aligned}
&(w_0w_i)(t,x)|\partial_b U_i[\Gamma^\beta R_{i,a}](t,x)|\\
&\le C\log(2+t)\sup_{(\tau,y)\in[0,t)\times\mathbb{R}^3}(w_0w_-)(\tau,y)|u(\tau,y)|_{K+2}\\
&\quad\times\{|y|(|u(\tau,y)|_{2K-2}+|\partial u(\tau,y)|_{2K-2})\}\\
&\le C\{\log(2+t)\}(1+t)^\lambda E_\varepsilon(t)^2.
\end{aligned}\tag{4.10a}$$

Here we have used (4.4) to get the last line. Therefore, noting that $\log(2+t)\le C_\lambda(1+t)^\lambda$, we obtain

$$(w_0^{1-2\lambda}w_i^{1-2\lambda})(t,x)|\partial_b U_i[\Gamma^\beta R_{i,a}](t,x)|\le CE_\varepsilon(T)^2 \qquad \text{for}\quad (t,x)\in[0,T)\times\mathbb{R}^3. \tag{4.10b}$$

Since $U_i[\partial_b(\Gamma^\beta R_{i,a})]=\partial_b U_i[\Gamma^\beta R_{i,a}]-\delta_{0,b}U_i^*[0,(\Gamma^\beta R_{i,a})(0,\cdot)]$, from (4.9) and (4.10b) we find

$$(w_0^{1-2\lambda}w_i^{1-2\lambda})(t,x)|U_i[\partial_b(\Gamma^\beta R_{i,a})](t,x)|\le C(\varepsilon^2+E_\varepsilon(T)^2) \tag{4.11}$$

for $(t,x)\in[0,T)\times\mathbb{R}^3$.

Just in the same manner, we can also show

$$(w_0^{1-2\lambda}w_i^{1-2\lambda})(t,x)|U_i[\partial_b(\Gamma^\beta N_{i,a})](t,x)|\le C(\varepsilon^2+E_\varepsilon(T)^2) \tag{4.12}$$

for $(t,x)\in[0,T)\times\mathbb{R}^3$.

Now we are going to estimate $U_i[\Gamma^\beta H_i]$. Since $|H_i(u,\partial u,\partial_x\partial u)|\le C(|u|^3+|\partial u|^3+|\partial_x\partial u|^3)$, (3.31) in Corollary 3.2 implies

$$\begin{aligned}
&(w_+w_i)(t,x)|U_i[\Gamma^\beta H_i](t,x)|\\
&\le C\sup_{(\tau,y)\in[0,t)\times\mathbb{R}^3}(w_0^2w_-^2)(\tau,y)|u(\tau,y)|_{K+2}^2\{|y|(|u(\tau,y)|_{2K-3}+|\partial u(\tau,y)|_{2K-2})\}\\
&\le C(1+t)^\lambda E_\varepsilon(T)^3 \qquad \text{for}\quad |\beta|\le 2K-3.
\end{aligned}\tag{4.13a}$$

Here we have used (4.4) again. (4.13a) implies

$$(w_0^{1-\lambda}w_i^{1-\lambda})(t,x)|U_i[\Gamma^\beta H_i](t,x)|\le CE_\varepsilon(T)^3\le CE_\varepsilon(T)^2 \tag{4.13b}$$

for $(t,x)\in[0,T)\times\mathbb{R}^3$.

Finally, from (4.7), (4.8), (4.11), (4.12) and (4.13b), we obtain

$$(w_0^{1-2\lambda}w_i^{1-2\lambda})(t,x)|u_i(t,x)|_{2K-3}\le C(\varepsilon+E_\varepsilon(T)^2) \qquad \text{for}\quad (t,x)\in[0,T)\times\mathbb{R}^3. \tag{4.14}$$

4.2. Estimates for $\|u(t,\cdot)\|_{2K-4,2}$. We are going to prove

$$\|u(t,\cdot)\|_{2K-4,2}\le C(\varepsilon+E_\varepsilon(T)^2) \qquad \text{for}\quad 0\le t<T. \tag{4.15}$$

From (4.7) and (2.12) in Lemma 2.2, we have

$$\|u(t)\|_{2K-4,2} \le C\varepsilon + C \sum_{\substack{|\beta|\le 2K-4\\ 0\le a,b\le 3}} (\|U_i[\partial_b\Gamma^\beta N_{i,a}](t)\|_{L^2} + \|U_i[\partial_b\Gamma^\beta R_{i,a}](t)\|_{L^2}) + \sum_{|\beta|\le 2K-4} \|U_i[\Gamma^\beta H_i](t)\|_{L^2}. \tag{4.16}$$

First, we will get a bound of $\|U_i[\partial_b(\Gamma^\beta N_{i,a})](t,\cdot)\|_{L^2}$. By Lemma 2.3, we have

$$\|U_i[\partial_b\Gamma^\beta N_{i,a}](t,\cdot)\|_{L^2} \le C\Big(\varepsilon^2 + \int_0^t \|N_{i,a}(\tau,\cdot)\|_{2K-4,2}d\tau\Big). \tag{4.17}$$

Because $N_{i,a}$ is a quadratic function of ∂u_j with $j\in I(i)$, we have

$$\begin{aligned}|N_{i,a}(\tau,y)|_{2K-4} &\le C\sum_{j,k\in I(i)} |u_j(\tau,y)|_{K-1}|u_k(\tau,y)|_{2K-3}\\ &\le C(w_0^{-2+2\lambda}w_i^{-2+2\lambda})(t,x)E_\varepsilon(T)^2\end{aligned} \tag{4.18}$$

for $(\tau,y)\in[0,T)\times\mathbb{R}^3$.

Since $N_{i,a}$ satisfies the Null Condition, we have a better estimate around the light cone. To explain it, we recall the estimates for the null forms.

Lemma 4.1. *Let c be a positive constant, and $a,b\in\{0,1,2,3\}$. Then we have*

$$\begin{aligned}|Q_0(\phi,\psi;c)(t,x)| \le& C|ct-r|(1+t+r)^{-1}|\partial\phi|\,|\partial\psi|\\ &+C(1+t+r)^{-1}(|\phi|_1\,|\partial\psi|+|\partial\phi|\,|\psi|_1),\end{aligned} \tag{4.19}$$

$$|Q_{ab}(\phi,\psi)(t,x)| \le C(1+t+r)^{-1}(|\phi|_1\,|\partial\psi|+|\partial\phi|\,|\psi|_1) \tag{4.20}$$

for any (t,x) satisfying $|ct-r|<\dfrac{ct}{2}$, where $r=|x|$.

Proof. See [19] for the proof.

Set

$$\Lambda_i(\tau) = \Big\{y\in\mathbb{R}^3;\, |\,c_i\tau-|y|\,|<\frac{c_i\tau}{2}\Big\} \qquad \text{for} \quad i=1,\cdots,m. \tag{4.21}$$

Note that $w_0\ge Cw_+$ in $\Lambda_i(\tau)$.

Since $N_{i,a}$ can be written as a linear combination of the null forms, by Lemma 4.1 we get

$$\begin{aligned}|N_{i,a}(\tau,y)|_{2K-4} &\le C(w_+^{-1}w_i)(\tau,y)\sum_{j,k\in I(i)}|u_j(\tau,y)|_{K-1}|u_k(\tau,y)|_{2K-3}\\ &\le C(w_+^{-1}w_0^{-2+2\lambda}w_i^{-1+2\lambda})(\tau,y)E_\varepsilon(T)^2\\ &\le C(w_+^{-3+2\lambda}w_i^{-1+2\lambda})(\tau,y)E_\varepsilon(T)^2\end{aligned} \tag{4.22a}$$

for any $y \in \Lambda_i(\tau)$ and any $\tau \in [0,T)$. Hence we obtain

$$\begin{aligned}
&\int_{y\in\Lambda_i(\tau)} |N_{i,a}(\tau,y)|_{2K-4}^2 dy \\
&\le C(1+\tau)^{-6+4\lambda} E_\varepsilon(T)^4 \int_{c_i\tau/2}^{3c_i\tau/2} (1+|c_i\tau - r|)^{-2+4\lambda} r^2 dr \\
&\le C(1+\tau)^{-4+4\lambda} E_\varepsilon(T)^4 \int_0^\infty (1+|c_i\tau - r|)^{-2+4\lambda} dr \\
&\le C(1+\tau)^{-4+4\lambda} E_\varepsilon(T)^4 \qquad (4.22\text{b})
\end{aligned}$$

for $\tau \in [0,T)$, provided that $-2+4\lambda < -1$.

On the other hand, observing that $1+\tau+|y| \le C w_i(\tau,y)$ for $y \notin \Lambda_i(\tau)$, by (4.18) we get

$$\begin{aligned}
&\int_{y\notin\Lambda_i(\tau)} |N_{i,a}(\tau,y)|_{2K-4}^2 dy \\
&\le C(1+\tau)^{-4+4\lambda} E_\varepsilon(T)^4 \int_0^\infty (1+r)^{-4+4\lambda} r^2 dr \le C(1+\tau)^{-4+4\lambda} E_\varepsilon(T)^4 \qquad (4.23)
\end{aligned}$$

for any $\tau \in [0,T)$, provided that $-2+4\lambda < -1$.

Since (4.22b) and (4.23) lead to

$$\|N_{i,a}(\tau,\cdot)\|_{2K-4,2} \le C(1+\tau)^{-2+2\lambda} E_\varepsilon(T)^2, \qquad (4.24)$$

from (4.17) we obtain

$$\|U_i[\partial_b \Gamma^\beta N_{i,a}](t,\cdot)\|_{L^2} \le C\Big(\varepsilon^2 + E_\varepsilon(T)^2 \int_0^\infty (1+\tau)^{-2+2\lambda} d\tau\Big) \le C(\varepsilon^2 + E_\varepsilon(T)^2) \qquad (4.25)$$

for $|\beta| \le 2K-4$ and $t \in [0,T)$, provided that $-2+2\lambda < -1$.

Secondly we will estimate $\|U_i[\partial_b \Gamma^\beta R_{i,a}]\|_{L^2}$ for $|\beta| \le 2K-4$. From the assumption on $R_{i,a}$, we have

$$\|\partial U_i[\Gamma^\beta R_{i,a}](t,\cdot)\|_{L^2} \le C \sum_{\substack{(j,k)\notin I(i)\times I(i)\\ |\gamma_1|+|\gamma_2|\le|\beta|\\ |p|,|q|=0,1}} \|\partial U_i[(\Gamma^{\gamma_1}\partial^p u_j)(\Gamma^{\gamma_2}\partial^q u_k)](t,\cdot)\|_{L^2}. \qquad (4.26)$$

Hence our task is to get a bound for $\|\partial U_i[(\Gamma^{\gamma_1}\partial^p u_j)(\Gamma^{\gamma_2}\partial^q u_k)](t,\cdot)\|_{L^2}$ with $(j,k) \notin I(i) \times I(i)$.

Set $v = U_i[(\Gamma^{\gamma_1}\partial^p u_j)(\Gamma^{\gamma_2}\partial^q u_k)]$ for a while. Multiplying the equation

$$\Box_i v = (\Gamma^{\gamma_1}\partial^p u_j)(\Gamma^{\gamma_2}\partial^q u_k)$$

by $\partial_t v$, and then following the standard derivation of the energy equality, we obtain

$$\|\partial v(t,\cdot)\|_{L^2}^2 \le 2\int_0^t \int_{y\in\mathbb{R}^3} \{(\Gamma^{\gamma_1}\partial^p u_j)(\Gamma^{\gamma_2}\partial^q u_k)(\partial_t v)\}(\tau,y) dy d\tau. \qquad (4.27\text{a})$$

Similarly to (4.10b), we can prove

$$(w_0^{1-2\lambda} w_i^{1-2\lambda})(t,x)\, |\partial v(t,x)| \le C E_\varepsilon(T)^2 \qquad \text{for } \ 0 \le t < T. \qquad (4.27\text{b})$$

Since $(j,k) \notin I(i) \times I(i)$, we have either $c_j \neq c_i$ or $c_k \neq c_i$. Without loss of generality, we may assume $c_j \neq c_i$. From (4.27a) and (4.27b), we get

$$\begin{aligned}
\|\partial v(t,\cdot)\|_{L^2}^2 &\leq C \int_0^T \|\{(\Gamma^{\gamma_1}\partial^p u_j)(\partial_t v)\}(\tau,\cdot)\|_{L^2} \|\Gamma^{\gamma_2}\partial^q u_k(\tau,\cdot)\|_{L^2} d\tau \\
&\leq C \int_0^T \||u_j|_{2K-3}|\partial v|\|_{L^2} \|u_k\|_{2K-3,2} d\tau \\
&\leq C E_\varepsilon(T)^4 \int_0^T \|w_{j,i}(\tau,\cdot)^{-1+2\lambda}\|_{L^2} (1+\tau)^\lambda d\tau, \qquad (4.27c)
\end{aligned}$$

where

$$w_{j,i}(\tau,y) = (w_0^2 w_j w_i)(\tau,y).$$

If $|y| \leq c_{j,i}\tau/2$ with $c_{j,i} = \min\{c_j, c_i\}$, we have

$$w_{j,i}^{-1+2\lambda}(\tau,y) \leq C(1+|y|)^{-2+4\lambda}(1+\tau)^{-2+4\lambda}. \qquad (4.27d)$$

Therefore we get

$$\begin{aligned}
&\int_{|y|\leq c_{j,i}\tau/2} |w_{j,i}(\tau,y)^{-1+2\lambda}|^2 dy \\
&\leq C(1+\tau)^{-4+8\lambda} \int_0^\infty (1+r)^{-4+8\lambda} r^2 dr \leq C(1+\tau)^{-4+8\lambda}, \qquad (4.27e)
\end{aligned}$$

provided that $-2+8\lambda < -1$.

On the other hand, since the assumption $c_j \neq c_i$ implies $(w_j w_i)(\tau,y) \geq C(1+\tau)$, we obtain

$$w_{j,i}(\tau,y)^{-1+2\lambda} \leq C(1+|y|)^{-2+4\lambda}(1+\tau)^{-1+2\lambda}, \qquad (4.27f)$$

which leads to

$$\begin{aligned}
\int_{|y|\geq c_{j,i}\tau/2} |w_{j,i}(\tau,y)^{-1+2\lambda}|^2 dy &\leq C(1+\tau)^{-2+4\lambda} \int_{c_{j,i}\tau/2}^\infty (1+r)^{-4+8\lambda} r^2 dr \\
&\leq C(1+\tau)^{-2+4\lambda} \Big(1 + \frac{c_{j,i}\tau}{2}\Big)^{-1+8\lambda} \\
&\leq C(1+\tau)^{-3+12\lambda}, \qquad (4.27g)
\end{aligned}$$

provided that $-1+8\lambda < 0$.

From (4.27e) and (4.27g), we find

$$\|w_{j,i}(\tau,\cdot)^{-1+2\lambda}\|_{L^2} \leq C(1+\tau)^{-3/2+6\lambda} \qquad (4.27h)$$

for sufficiently small λ, and from (4.27c) we obtain

$$\sup_{0\leq t<T} \|\partial v(t,\cdot)\|_{L^2} \leq C E_\varepsilon(T)^2 \Big(\int_0^\infty (1+\tau)^{-3/2+7\lambda} d\tau\Big)^{\frac{1}{2}} \leq C E_\varepsilon(T)^2, \qquad (4.27i)$$

provided that $-3/2+7\lambda < -1$.

As an immediate consequence of (4.27i), (2.13) and (2.12), we get

$$\sup_{0\leq t<T} \|U_i[\partial_b \Gamma^\beta R_{i,a}](t,\cdot)\|_{L^2} \leq C(\varepsilon^2 + E_\varepsilon(T)^2) \qquad (4.28)$$

for β with $|\beta| \leq 2K-4$.

Now, we will get cotrol of $\|U_i[\Gamma^\beta H_i](t,\cdot)\|_{L^2}$ with $|\beta| \leq 2K-4$. We note that

$$|H_i(\tau,y)|_{2K-4} \leq C|u(\tau,y)|_K^2(|u(\tau,y)|_{2K-4} + |\partial u(\tau,y)|_{2K-3}). \tag{4.29a}$$

From Lemma 2.2 and Hölder's inequality, we have

$$\begin{aligned}\|U_i[\Gamma^\beta H_i](t,\cdot)\|_{L^2} &\leq C\int_0^t \|H_i(\tau,\cdot)\|_{2K-4,6/5}d\tau \\ &\leq C\int_0^t \|u(\tau,\cdot)\|_{K,\infty}^{4/3}\|u(\tau,\cdot)\|_{K,2}^{2/3}(\|u(\tau,\cdot)\|_{2K-4,2} + \|\partial u(\tau,\cdot)\|_{2K-3,2})d\tau \\ &\leq CE_\varepsilon(T)^3\int_0^\infty (1+\tau)^{5\lambda/3-4/3}d\tau \leq CE_\varepsilon(T)^3, \end{aligned}\tag{4.29b}$$

provided that $5\lambda/3 - 4/3 < -1$.

Finally, (4.15) follows from (4.16), (4.25), (4.28) and (4.29b).

4.3. Estimates for $|u_i(t,x)|_{2K-8}$ and $|u_i(t,x)|_{K+2}$. Lemma 3.1 and Corollary 3.1 with $\mu = \nu = 1+\lambda$ imply

$$\begin{aligned}&(w_0w_i)(t,x)|U_i[\partial_b\Gamma^\beta N_{i,a}](t,x)| \\ &\leq C\Big(\varepsilon^2 + C\sup_{(\tau,y)\in(0,t)\times\mathbb{R}^3}|y|(w_0w_i)^{1+\lambda}(\tau,y)|N_{i,a}(\tau,y)|_{2K-7}\Big)\end{aligned}\tag{4.30a}$$

for $|\beta| \leq 2K-8$.

By Lemma 4.1, we have

$$\begin{aligned}|y|\,|N_{i,a}(\tau,y)|_{2K-7} &\leq C|y|(w_+^{-1}w_i)(\tau,y)\sum_{j,k\in I(i)}|u_j(\tau,y)|_{K-2}\,|u_k(\tau,y)|_{2K-6} \\ &\leq C(w_+^{-1}w_0^{-1+2\lambda}w_i^{-1+2\lambda})(\tau,y)E_\varepsilon(T)^2 \\ &\leq C(w_+^{-1+6\lambda}w_0^{-1-\lambda}w_i^{-1-\lambda})(\tau,y)E_\varepsilon(T)^2\end{aligned}\tag{4.30b}$$

for $y \in \Lambda_i(\tau)$. On the other hand, for $y \notin \Lambda_i(\tau)$, we get

$$\begin{aligned}|y|\,|N_{i,a}(\tau,y)|_{2K-7} &\leq C|y|(w_0^{-2+2\lambda}w_i^{-2+2\lambda})(\tau,y)E_\varepsilon(T)^2 \\ &\leq C(w_0^{-1+2\lambda}w_+^{-2+2\lambda})(\tau,y)E_\varepsilon(T)^2 \\ &\leq C(w_0^{-1-\lambda}w_i^{-1-\lambda}w_+^{-1+6\lambda})(\tau,y)E_\varepsilon(T)^2.\end{aligned}\tag{4.30c}$$

Consequently, if λ is small enough to satisfy $-1+6\lambda \leq 0$, we obtain

$$(w_0^{1+\lambda}w_i^{1+\lambda})(t,y)\,|y|\,|N_{i,a}(\tau,y)|_{2K-7} \leq CE_\varepsilon(T)^2 \qquad \text{for any } \ (\tau,y)\in[0,T)\times\mathbb{R}^3,$$

and (4.30a) leads to

$$(w_0w_i)(t,x)|U_i[\partial_b\Gamma^\beta N_{i,a}](t,x)| \leq C(\varepsilon^2 + E_\varepsilon(T)^2) \tag{4.30d}$$

for $|\beta| \leq 2K-8$ and $t \in [0,T)$.

Now we turn our attention to the estimate of $U_i[\partial_b\Gamma^\beta R_{i,a}]$.

First, suppose $j,k \in \{1,\cdots,m\}$ and $I(j) \neq I(k)$. Then, for $|\beta| \leq 2K-8$ and for $0 \leq |p|,|q| \leq 1$, Lemma 3.1 and (3.29) in Corollary 3.2 lead to

$$\begin{aligned}&(w_0w_i)(t,x)|U_i[\partial_b\Gamma^\beta(\partial^p u_j\,\partial^q u_k)](t,x)| \\ &\leq C\Big(\varepsilon^2 + \sup_{(\tau,y)\in(0,t)\times\mathbb{R}^3}|y|(w_0^{1+\lambda}w_-^{1+\lambda})(\tau,y)|(\partial^p u_j)(\partial^q u_k)(\tau,y)|_{2K-7}\Big).\end{aligned}\tag{4.31a}$$

Since we have $(w_j w_k)(\tau,y) \ge (w_+ w_-)(\tau,y)$, we obtain

$$\begin{aligned}|y|\,|(\partial^p u_j)(\partial^q u_k)(\tau,y)|_{2K-7} &\le C|y||u_j(\tau,y)|_{2K-6}|u_k(\tau,y)|_{2K-6}\\ &\le C|y|(w_0^{-2+4\lambda}w_+^{-1+2\lambda}w_-^{-1+2\lambda})(\tau,y)E_\varepsilon(T)^2\\ &\le C(w_0^{-1-\lambda}w_-^{-1-\lambda}w_+^{-1+10\lambda})(\tau,y)E_\varepsilon(T)^2. \end{aligned}\tag{4.31b}$$

Therefore, if λ is so small that $-1+10\lambda \le 0$ holds, we get

$$|y|(w_0^{1+\lambda}w_-^{1+\lambda})(\tau,y)|(\partial^p u_j)(\partial^q u_k)(\tau,y)|_{2K-7} \le CE_\varepsilon(T)^2 \tag{4.31c}$$

for any $(\tau,y)\in[0,T)\times\mathbb{R}^3$, and consequently (4.31a) implies

$$(w_0w_i)(t,x)|U_i[\partial_b\Gamma^\beta(\partial^p u_j\partial^q u_k)](t,x)| \le C(\varepsilon^2+E_\varepsilon(T)^2) \qquad \text{for} \ \ t\in[0,T), \tag{4.31d}$$

provided $|\beta|\le 2K-8$.

Next, let $j,k\in\{1,\cdots,m\}$ with $I(j)=I(k)\ne I(i)$. We can choose a cutoff function $\chi\in C^\infty(\mathbb{R}_+\times\mathbb{R}^3)$ which satisfies

$$\begin{cases}\operatorname{supp}\chi\subset\{(\tau,y)\in\mathbb{R}_+\times\mathbb{R}^3; \tau^2+|y|^2\ge 1 \text{ and } |c_j\tau-|y|\,|\le c_j\tau/2\},\\ \chi\equiv 1 \text{ on the set } \{(\tau,y);\tau^2+|y|^2\ge 2 \text{ and } |c_j\tau-|y|\,|\le c_j\tau/4\},\\ 0\le\chi(\tau,y)\le 1 \text{ and } \displaystyle\sup_{(\tau,y)\in\mathbb{R}_+\times\mathbb{R}^3}|\chi(\tau,y)|_1\le C.\end{cases}\tag{4.32}$$

Since $c_j=c_k$, we have

$$\begin{aligned}&|y|\,|(\partial^p u_j\partial^q u_k)(\tau,y)|_{2K-7}\\ &\le C|y|(|u_j(\tau,y)|_{K-3}|u_k(\tau,y)|_{2K-6}+|u_k(\tau,y)|_{K-3}|u_j(\tau,y)|_{2K-6})\\ &\le C|y|(w_0^{-2+2\lambda}w_j^{-2+2\lambda})(\tau,y)E_\varepsilon(T)^2.\end{aligned}\tag{4.33a}$$

For (τ,y) satisfying $|\,c_j\tau-|y|\,|\ge c_j\tau/4$, we get

$$|y|(w_0^{-2+2\lambda}w_j^{-2+2\lambda})(\tau,y) \le C(w_0^{-1-\lambda}w_+^{-2+5\lambda})(\tau,y). \tag{4.33b}$$

It is easy to see that (4.33b) is valid also for (τ,y) satisfying $\tau^2+|y|^2\le 2$. Therefore, if $2-5\lambda>1$, Lemma 3.2(i) with the choice

$$J=0, \qquad \mu=2-5\lambda \qquad \text{and} \qquad \nu=1+\lambda$$

gives us

$$(w_0w_i)(t,x)|\partial U_i[(1-\chi)\Gamma^\beta(\partial^p u_j\,\partial^q u_k)](t,x)| \le CE_\varepsilon(T)^2 \tag{4.33c}$$

for $|\beta|\le 2K-8$.

On the other hand, for (τ,y) with $|\,c_j\tau-|y|\,|\le c_j\tau/2$, we get

$$|y|(w_0^{-2+2\lambda}w_j^{-2+2\lambda})(\tau,y) \le C(w_+^{-1+2\lambda}w_j^{-2+2\lambda})(\tau,y). \tag{4.34a}$$

Since we may assume $2-2\lambda>1$ for small λ, by (4.33a) and Lemma 3.2(ii) with the choice

$$J=j, \qquad \mu=1-2\lambda \qquad \text{and} \qquad \nu=2-2\lambda,$$

we obtain

$$(w_0w_i^{1-2\lambda})(t,x)|\partial U_i[\chi\Gamma^\beta(\partial^p u_j\,\partial^q u_k)](t,x)| \le CE_\varepsilon(T)^2 \tag{4.34b}$$

for $|\beta| \le 2K-8$.

Now we are going to get a similar estimate for $|\beta| \le K+2$. Since we have $(K+3)/2+1 \le K+2$ and $K+4 \le 2K-8$ for $K \ge 12$, we get

$$\begin{aligned} &|y|\,|(\partial^p u_j\, \partial^q u_k)(\tau,y)|_{K+3} \\ &\le C|y|(|u_j(\tau,y)|_{K+2}|u_k(\tau,y)|_{2K-8} + |u_k(\tau,y)|_{K+2}|u_j(\tau,y)|_{2K-8}) \\ &\le C(w_0^{-1}w_j^{-2+2\lambda})(\tau,y)E_\varepsilon(T)^2 \end{aligned} \tag{4.35a}$$

in place of (4.33a). For (τ,y) with $|\,c_j\tau - |y|\,| \le c_j\tau/2$, instead of (4.34a), we have

$$(w_0^{-1}w_j^{-2+2\lambda})(\tau,y) \le C(w_+^{-1}w_j^{-2+2\lambda})(\tau,y). \tag{4.35b}$$

Now we can apply Lemma 3.2(ii) with the choice

$$J=j, \qquad \mu=1, \qquad \nu=2-2\lambda,$$

and we obtain

$$(w_0w_i)(t,x)|\partial U_i[\chi\Gamma^\beta(\partial^p u_j\,\partial^q u_k)](t,x)| \le CE_\varepsilon(T)^2 \qquad \text{for} \quad |\beta| \le K+2. \tag{4.35c}$$

From (4.33c) and (4.34b), we get

$$(w_0w_i^{1-2\lambda})(t,x)|\partial U_i[\Gamma^\beta(\partial^p u_j\,\partial^q u_k)](t,x)| \le CE_\varepsilon(T)^2 \qquad \text{for} \quad |\beta| \le 2K-8, \tag{4.36}$$

provided that $I(j)=I(k)\ne I(i)$.

Similarly, (4.33c) and (4.35c) lead to

$$(w_0w_i)(t,x)|\partial U_i[\Gamma^\beta(\partial^p u_j\,\partial^q u_k)](t,x)| \le CE_\varepsilon(T)^2 \qquad \text{for} \quad |\beta| \le K+2, \tag{4.37}$$

provided that $I(j)=I(k)\ne I(i)$.

By (4.31d) and (4.36) together with (4.9), we see that

$$(w_0w_i^{1-2\lambda})(t,x)|U_i[\partial_b\Gamma^\beta R_{i,a}](t,x)| \le C(\varepsilon^2 + E_\varepsilon(T)^2) \qquad \text{for} \quad |\beta| \le 2K-8, \tag{4.38}$$

and, replacing (4.36) by (4.37), we also obtain

$$(w_0w_i)(t,x)|U_i[\partial_b\Gamma^\beta R_{i,a}](t,x)| \le C(\varepsilon^2 + E_\varepsilon(T)^2) \qquad \text{for} \quad |\beta| \le K+2. \tag{4.39}$$

Finally, since H_i is a function of degree 3, by (4.4) we have

$$\begin{aligned} &|y|(w_0^2w_-^2)(\tau,y)|H_i(\tau,y)|_{2K-8} \\ &\le C\{(w_0w_-)(\tau,y)|u(\tau,y)|_{K-2}\}^2(|y|\,|u(\tau,y)|_{2K-6}) \le CE_\varepsilon(T)^3 \end{aligned} \tag{4.40a}$$

for all $(\tau,y)\in[0,T)\times\mathbb{R}^3$. Therefore (3.31) in Corollary 3.2 implies

$$(w_0w_i)(t,x)|U_i[\Gamma^\beta H_{i,a}](t,x)| \le CE_\varepsilon(T)^3 \qquad \text{for} \quad |\beta| \le 2K-8. \tag{4.40b}$$

Summing up, from (4.8), (4.30d), (4.39) and (4.40b) we get

$$(w_0w_i)(t,x)|u_i(t,x)|_{K+2} \le C(\varepsilon + E_\varepsilon(T)^2), \tag{4.41}$$

while, using (4.38) instead of (4.39), we obtain

$$(w_0w_i^{1-2\lambda})(t,x)|u_i(t,x)|_{2K-8} \le C(\varepsilon + E_\varepsilon(T)^2). \tag{4.42}$$

4.4. Estimates for $\|\partial u(t,\cdot)\|_{2K,2}$. Let $|\alpha| \le 2K$ in (4.7), which can be rewritten as

$$\Box_i(\Gamma^\alpha u_i) - \sum_{j,k,a} c_{ka}^{ij}\partial_k\partial_a(\Gamma^\alpha u_j) = \Gamma^\alpha F_i - \sum_{j,k,a} c_{ka}^{ij}\partial_k\partial_a(\Gamma^\alpha u_j) + {\sum_{|\beta|\le|\alpha|-1}}' \Gamma^\beta F_i, \tag{4.43}$$

where the coefficients $c_{ka}^{ij}(u,v,w)$ are given by (1.5).

Since $F_i(u,v,w) = O(|u|^2+|v|^2+|w|^2)$ for small (u,v,w), the Leibniz formula, together with (2.4) and (2.5), implies that the right hand side of (4.43) is dominated by

$$C|u(t,x)|_{K+2}(|u(t,x)|_{2K} + |\partial u(t,x)|_{2K}).$$

We also have $c_{ka}^{ij}(u,\partial u) \le C|u(t,x)|_1 (\ll 1)$. Hence, remembering the assumptions (1.4) and (1.6), we can apply Lemma 2.1 to (4.43), and we get

$$\begin{aligned}
\|\partial u_i(t,\cdot)\|_{2K,2} &\le C\Big(\varepsilon + \int_0^t \|u(\tau)\|_{K+2,\infty}(\|u(\tau)\|_{2K,2} + \|\partial u(\tau)\|_{2K,2})d\tau\Big)\\
&\le C\Big(\varepsilon + E_\varepsilon(t)^2\int_0^t (1+\tau)^{\lambda-1}d\tau\Big)\\
&\le C(1+t)^\lambda(\varepsilon + E_\varepsilon(T)^2) \qquad \text{for } \ 0\le t<T.
\end{aligned} \tag{4.44}$$

4.5. Estimates for $\|u(t,\cdot)\|_{2K,2}$. Let $|\alpha| \le 2K$ in (4.7), again.

Remembering that $F_i = \sum_a \partial_a G_{i,a} + H_i$, and using (2.4)–(2.7), we find

$$\Box_i(\Gamma^\alpha u) = {\sum_{\substack{|\beta|\le|\alpha|\\ 0\le a,b\le 3}}}' \partial_b(\Gamma^\beta G_{i,a}) + \sum_{j,k,a}\partial_k\Big(\frac{\partial H_i}{\partial w_{j,ka}}\Gamma^\alpha\partial_a u_j\Big) + H_{i,\alpha}^*, \tag{4.45}$$

where

$$H_{i,\alpha}^* = \Gamma^\alpha H_i - \sum_{j,k,a}\partial_k\Big(\frac{\partial H_i}{\partial w_{j,ka}}\Gamma^\alpha\partial_a u_j\Big) + {\sum_{|\beta|\le|\alpha|-1}}' \Gamma^\beta H_i.$$

Since $|H_{i,\alpha}^*| \le C|u|_{K+2}^2(|u|_{2K} + |\partial u|_{2K})$, Lemma 2.2 implies that

$$\begin{aligned}
\|U_i[H_{i,\alpha}^*](t,\cdot)\|_{L^2} &\le C\int_0^t \||u(\tau)|_{K+2}^2(|u(\tau)|_{2K} + |\partial u(\tau)|_{2K})\|_{L^{6/5}}d\tau\\
&\le C\int_0^t \|u\|_{K+2,\infty}^{\frac43}\|u\|_{K+2,2}^{\frac23}(\|u\|_{2K,2} + \|\partial u\|_{2K,2})d\tau\\
&\le CE_\varepsilon(t)^3\int_0^\infty (1+\tau)^{\frac53\lambda-\frac43}d\tau \le CE_\varepsilon(T)^3 \qquad \text{for } \ 0\le t<T,
\end{aligned} \tag{4.46}$$

provided that λ is small enough to satisfy $\frac53\lambda - \frac43 < -1$.

On the other hand, by virtue of Lemma 2.3, we have

$$\begin{aligned}
&\Big\|U_i\Big[{\sum_{\substack{|\beta|\le|\alpha|\\ a,b\in\{0,1,2,3\}}}}' \partial_b(\Gamma^\alpha G_{i,a}) + \sum_{j,k,a}\partial_k\Big(\frac{\partial H_i}{\partial w_{j,ka}}\Gamma^\alpha\partial_a u_j\Big)\Big](t,\cdot)\Big\|_{L^2}\\
&\le C\Big(\varepsilon^2 + \int_0^t \|u(\tau)\|_{K+2,\infty}(\|u(\tau)\|_{2K,2} + \|\partial u(\tau)\|_{2K,2})d\tau\Big)\\
&\le C\Big(\varepsilon^2 + E_\varepsilon(T)^2\int_0^t (1+\tau)^{\lambda-1}d\tau\Big) \le C(1+t)^\lambda(\varepsilon^2 + E_\varepsilon(T)^2).
\end{aligned} \tag{4.47}$$

Summing up, we obtain

$$(1+t)^{-\lambda}\|u_i(t,\cdot)\|_{2K,2} \le C(\varepsilon + E_\varepsilon(T)^2) \qquad \text{for} \ \ 0 \le t < T. \tag{4.48}$$

4.6. Conclusion. Finally Proposition 4.1 follows from (4.14), (4.15), (4.41), (4.42), (4.44) and (4.48). This completes the proof.

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