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ESTIMATES OF LOWER CRITICAL MAGNETIC FIELD AND VORTEX PINNING BY INHOMOGENEITIES IN TYPE II SUPERCONDUCTORS***

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Abstract

The effect of an applied magnetic field on an inhomogeneous superconductor is studied and the value of the lower critical magnetic field H_{c_1} at which superconducting vortices appear is estimated. In addition, the authors locate the vortices of local minimizers, which depends on the inhomogeneous term $a(x)$.

Keywords Superconductor, Vortices, Pinning mechanism, Critical magnetic field
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§ 1 . Introduction

Consider the following functional:

$$J(u,A)=\frac{1}{2}\int_{\Omega}|\nabla_A u|^2+\frac{1}{2\varepsilon^2}(a(x)-|u|^2)^2+|h-h_{ex}|^2-h_{ex}^2, \tag{1.1}$$

which corresponds to the free energy of a superconductor in a prescribed constant magnetic field h_{ex}. Here, $\Omega\subseteq\mathbb{R}^2$ is the smooth, bounded, simply connected section of the superconductor and $a(x):\Omega\to\mathbb{R}^2$ is a given function satisfying $0<\min a(x)\le a(x)$ in Ω. The unknowns are the complex-valued order parameter $u\in H^1(\Omega,\mathbb{C})$ and the $U(1)$ connection $A\in H^1(\Omega,\mathbb{R}^2)$. $h=\mathrm{curl}A$ is the induced magnetic field, $\nabla_A=\nabla-iA$. The order parameter u indicates the local state of the material, viz., $|u|$ is the density of superconducting electron pairs so that, when $|u|\simeq 1$, the material is in its superconducting state, whereas when $|u|\simeq 0$, it is in its normal state. $\kappa=\frac{1}{\varepsilon}$ is the Ginzburg-Landau parameter depending on the material. The modified Ginzburg-Landau functional (1.1) was first written down by Likharev [12]. Then, this model has been used and developed by Aftalion, Sandier, Serfaty, Chapman and Richardson [3, 9]. The minima of $a(x)$ corresponds to the impurities in the material.

It is well known that a superconductor placed in an applied magnetic field may change its phase when the field varies. There are two critical fields H_{c_1} and H_{c_2} for which a phase transition occurs. Above H_{c_2}, superconductivity is destroyed and material is in the normal

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phase. Below H_{c_1}, the material is superconducting everywhere. When $H_{c_1} \leq h_{ex} \leq H_{c_2}$, the material is in mixed state. In this state, since the works of Abrikosov [1], the minimizers of (1.1) exhibit vortices, i.e. points where the order parameter vanishes, if the Ginzburg-Landau parameter kappa is large enough. Actually these vortices are observed in nature experimentally. It is of importance to determine where these points are located and estimate the values of H_{c_1} and H_{c_2}. There have been many works toward to dealing with such problems [2, 4–11, 13–22]. We do not attempt to give an exhaustive list of reference, but briefly summarize the advances concerning this problem. In the case $a(x) \equiv 1$ in Ω, the value of the first critical field (as a function of kappa) was predicted by Abrikosov when the domain is the plane. This computation was extended to arbitrary domain by others. However, these were only formal works. Very recently, S. Serfaty [18] was able to derive a completely rigorous proof of the previous predictions. In the case $a(x) \neq 1$ in Ω, Kim and Liu [14, 15, 16] studied the global minimizers of the functional (1.1).

In this paper, we want to find the first critical magnetic field H_{c_1} and study the local minimiers of (1.1). Inspired by [10, 18], we will choose a suitable M and construct the local minimizer of functional (1.1) in

$$D_M^a = \{(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2) : F_a(u) < M|\ln \varepsilon|\}, \tag{1.2}$$

where

$$F_a(u) = \frac{1}{2} \int_\Omega \Big[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(a(x) - |u|^2)^2\Big]. \tag{1.3}$$

Recall that the Ginzburg-Landau equation of any critical point $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ associated to the functional (1.1) is

$$\text{(G.L.)} \qquad \begin{cases} -\nabla_A^2 u = \dfrac{1}{\varepsilon^2}(a - |u|^2)u, \\ -\nabla^\perp h = (iu, \nabla_A u) \end{cases}$$

with the boundary conditions

$$\begin{cases} h = h_{ex} & \text{on } \partial\Omega, \\ (\nabla u - iAu) \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Here $\nabla^\perp$ denotes $(-\partial_{x_2}, \partial_{x_1})$, ν is the unit outer normal vector to $\partial\Omega$ and $(z, w) = \mathrm{Re}(z\bar{w})$ where z and w are in $\mathbb{C}$.

Let us now state our main results and hypothesis. Notice that $J(u, A)$ is invariant under $U(1)$-gauge transformation, i.e. of the type

$$v = e^{i\phi} u, \quad B = A + \nabla\phi \qquad \text{for any } \phi \in H^2(\Omega, \mathbb{R}),$$

which makes the problem non-compact. Therefore, throughout this paper, we shall impose the gauge condition

$$\begin{cases} \operatorname{div}(a(x)A) = 0 & \text{on } \Omega, \\ A \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Moreover, since we assume that Ω is simply connected, we can say that there exists $\xi \in H^2(\Omega, \mathbb{R})$ such that

$$\begin{cases} a(x)A = (-\xi_{x_2}, \xi_{x_1}) & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Hence

$$h = \mathrm{curl} A = \mathrm{div}\left(\frac{1}{a}\nabla\xi\right). \tag{1.7}$$

Without loss of generality, we make the following hypothesis on the function $a(x)$:

$$0 < b_0 = \min_{\overline{\Omega}} a(x) \leq a(x) \leq 1. \tag{1.8}$$

Now let ξ_0 be a smooth function satisfying

$$\begin{cases} -\mathrm{div}\left(\frac{1}{a}\nabla\xi_0\right) + \xi_0 = -1 & \text{in} \quad \Omega, \\ \xi_0 = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.9}$$

and define

$$\Lambda = \left\{x \in \Omega : \left|\frac{\xi_0(x)}{a(x)}\right| = \max_{\overline{\Omega}}\left|\frac{\xi_0}{a}\right|\right\}. \tag{1.10}$$

From Lemma 4.2 in [16], the set Λ is a finite set under the assumption that $a(x)$ is analytic. Our main theorem is the following

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, smooth, simply connected domain. Assume that $a(x)$ is a analytic function and* (1.8) *holds. Assume that $M > \pi(\mathrm{Card}\Lambda)\max_{\Lambda} a(x)$. Then, there exist $k_1 = \frac{1}{2\max_{\overline{\Omega}}|\frac{\xi_0(x)}{a(x)}|}$, $k_2^\varepsilon = \mathrm{O}_\varepsilon(1)$, $k_3^\varepsilon = \mathrm{o}_\varepsilon(1)$ and $\varepsilon_0 = \varepsilon_0(M) > 0$ such that*

$$H_{c_1} = k_1|\ln\varepsilon| + k_2^\varepsilon, \tag{1.11}$$

and the following holds for $\varepsilon < \varepsilon_0$:

(i) *If $h_{ex} \leq H_{c_1}$, a solution of* (G.L.)*, which minimizes $J(u, A)$ in D_M^a, exists and satisfies $\frac{b_0}{2} \leq |u| \leq 1$.*

(ii) *If $H_{c_1} + k_3^\varepsilon \leq h_{ex} \leq H_{c_1} + \mathrm{O}_\varepsilon(1)$, there exists a solution of* (G.L.) *that minimizes $J(u, A)$ in D_M^a. It has a bounded positive number of vortices a_i^ε of degree one such that*

$$\mathrm{dist}\,(a_i^\varepsilon, \Lambda) \to 0 \qquad \text{as} \ \ \varepsilon \to 0, \tag{1.12}$$

and there exists a constant $\alpha > 0$ such that

$$\mathrm{dist}\,(a_i^\varepsilon, a_j^\varepsilon) \geq \alpha \qquad \text{for} \ \ i \neq j. \tag{1.13}$$

Remark 1.1. It follows from (1.10) and (1.12) that the distribution of the location of vortices are governed both by the term $a(x)$ and by the function ξ_0 which is related to the magnetic field. This is called the pinning mechanism in superconductivity.

Remark 1.2. In Theorem 1.1, we only assume the smooth property of the inhomogeneous term $a(x)$. This results can be extended to the model of variable thickness superconducting thin film in [10, 18].

This paper is organized as follows. In the next section we shall give some basic estimate for $J(u, A)$. In Section 3, we pay attention to the splitting of the energy $J(u, A)$. In Section 4, we shall split the magnetic field. In Section 5, we give some estimates for F_a. In Section 6, we shall obtain the critical magnetic field. In Section 7, we prove Theorem 1.1.

In the discussion of the following sections, we always assume that the Abrikosov estimate $H_{c_1} \leq C|\ln\varepsilon|$ holds and $1 \leq h_{ex} \leq C|\ln\varepsilon|$. Also, for simplicity of the notation, we denote D_M^a by D hereafter.

§ 2 . Basic Estimates

In this section, we give the following lemmas and omit their proofs here.

Lemma 2.1. *For $(\tilde{u}, \widetilde{A})$ minimizing $J(u,A)$ in $\overline{D}$ with gauge condition* (1.5)*, we have*

$$J(\tilde{u}, \widetilde{A}) \leq Ch_{ex}^2, \tag{2.1}$$

$$\frac{1}{2}\int_\Omega |\nabla_{\widetilde{A}}\tilde{u}|^2 \leq Ch_{ex}^2, \tag{2.2}$$

$$\frac{1}{4\varepsilon^2}\int_\Omega (a(x)-|\tilde{u}|^2)^2 \leq Ch_{ex}^2, \tag{2.3}$$

$$\|\tilde{\xi}\|_{H^2(\Omega,\mathbf{R})} \leq Ch_{ex}, \tag{2.4}$$

where $\tilde{\xi}$ is defined by (1.6)*.*

Lemma 2.2. *For any $(\tilde{u}, \widetilde{A})$ in $\overline{D}$ satisfying gauge condition* (1.5) *and $J(\tilde{u}, \widetilde{A}) \leq Ch_{ex}^2$, there exists $(u,A) \in \overline{D}$ satisfying* (1.5) *such that*

$$|u| \leq 1, \tag{2.5}$$

$$F_a(u) \leq F_a(\tilde{u}), \tag{2.6}$$

$$J(u,A) \leq J(\tilde{u}, \widetilde{A}) + \mathrm{o}(1), \tag{2.7}$$

$$\|\nabla\xi\|_{L^\infty(\Omega,R)} \leq Ch_{ex}. \tag{2.8}$$

In addition, if $(\tilde{u}, \widetilde{A})$ minimizes J in $\overline{D}$, then there holds

$$F_a(u) = F_a(\tilde{u}) + \mathrm{o}(1) \qquad \text{as } \varepsilon \to 0, \tag{2.9}$$

$$J(u,A) = J(\tilde{u}, \widetilde{A}) + \mathrm{o}(1) = J(u, \widetilde{A}) + \mathrm{o}(1) \qquad \text{as } \varepsilon \to 0. \tag{2.10}$$

Lemma 2.3. *For $(u,A) \in \overline{D}$ and $|u| \leq 1$, let $u^\gamma \in H^1(\Omega, \mathbb{C})$ be a minimizer of the following minimization problem*

$$\min_{v\in H^1(\Omega,\mathbb{C})}\Big\{\int_\Omega\Big[\frac{1}{2}|\nabla v|^2 + \frac{1}{4\varepsilon^2}(a-|v|^2)^2\Big] + \int_\Omega \frac{|u-v|^2}{2\varepsilon^{2\gamma}}\Big\}, \tag{2.11}$$

where $0<\gamma<1$, $0 \leq b_0 \leq a(x) \leq 1$ in Ω and $a \in C^2(\overline{\Omega})$. Then, we have u^γ in $H^3(\Omega,\mathbb{C})$ and u^γ satisfies

$$-\Delta u^\gamma = \frac{1}{\varepsilon^2}u^\gamma(a-|u^\gamma|^2) + \frac{u-u^\gamma}{\varepsilon^{2\gamma}}, \tag{2.12}$$

$$F_a(u^\gamma) \leq F_a(u) \leq M|\ln\varepsilon|, \tag{2.13}$$

$$|u^\gamma| \leq 1, \tag{2.14}$$

$$|\nabla u^\gamma| \leq \frac{C}{\varepsilon}. \tag{2.15}$$

Since $|\nabla u^\gamma| \leq \frac{C}{\varepsilon}$, the vortices are well defined in the following lemma (see [5]).

Lemma 2.4. *There exists $\lambda > 0$ and vortex points $\{a_i^\varepsilon\}_{i\in\mathcal{T}}$ in Ω such that* $\mathrm{Card}\mathcal{T} \leq Ch_{ex}^2$*, and*

$$|u^\gamma(x)| \geq \frac{1}{2}b_0 \qquad \text{in } \Omega \setminus \bigcup_{i\in\mathcal{T}} B(a_i^\varepsilon, \lambda\varepsilon). \tag{2.16}$$

In the remaining part of this section, we shall pay our attentions to the discussions on the minimizer u^γ. Although we have put a function $a(x)$ in the functional (2.11) on u^γ, we found that the proofs of the following four lemmas on the properties of u^γ can be obtained directly from adjusting the corresponding ones in [2, 18] and [19] by replacing the energy density with $\frac{1}{2}[|\nabla v|^2+\frac{1}{\varepsilon^2}a(1-|v|^2)^2]$ where $v=\frac{u^\gamma}{\sqrt{a}}$. Thus, we omit the proof of the following lemmas in this paper.

Lemma 2.5. *For any $0<\gamma<\beta<1$, u^γ has no vortex $\Big($i.e. $|u^\gamma|\ge \dfrac{1}{2}b_0\Big)$ on the domain $\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)\le\varepsilon^\beta\}$.*

Lemma 2.6. Card $\mathcal{T}$ *is uniformly bounded by a constant N which is independent of ε. Let $0<\gamma<\beta<\mu<1$ be such that $\bar\mu=\mu^{N+1}>\beta$. For ε small enough we may choose a subset $\mathcal{T}_1\subset\mathcal{T}$ and a radius $\rho>0$ with $\lambda\varepsilon\le\varepsilon^\mu\le\rho\le\varepsilon^{\bar\mu}<\varepsilon^\beta$ such that*

$$|u^\gamma|\ge\frac{1}{2}b_0 \qquad \textit{in}\quad \Omega\setminus\bigcup_{i\in\mathcal{T}_1}B(a_i^\varepsilon,\rho), \tag{2.17}$$

$$|u^\gamma|\ge\sqrt{a}-\frac{C}{|\ln\varepsilon|^2} \qquad \textit{on}\quad \partial B(a_i^\varepsilon,\rho)\quad \textit{for}\quad i\in\mathcal{T}_1, \tag{2.18}$$

$$\int_{\partial B(a_i^\varepsilon,\rho)} e_\varepsilon(u^\gamma)\le\frac{C(\beta,\mu)}{\rho} \qquad \textit{for}\quad i\in\mathcal{T}_1, \tag{2.19}$$

where $e_\varepsilon(u^\gamma)=\dfrac{1}{2}\Big[|\nabla u^\gamma|^2+\dfrac{1}{2\varepsilon^2}(a-|u^\gamma|^2)^2\Big]$ *and*

$$|a_i^\varepsilon-a_j^\varepsilon|\ge 8\rho \qquad \textit{for}\quad i\ne j\in\mathcal{T}_1. \tag{2.20}$$

Lemma 2.7. *For any $u\in\overline{D}$, denote $d_i=\deg(u^\gamma,\partial B(a_i^\varepsilon,\rho))$. Then we have*

$$|d_i|=\mathrm{O}(1),\qquad \forall\, i\in\mathcal{T}_1. \tag{2.21}$$

If the fact $|\nabla u|\le\frac{C}{\varepsilon}$ is proved (if u is a solution of (G.L.), then this is true), then in the sense of [5] the vortices of u are well defined and has the same uniformly bound on the number as the vortices of u^γ has. For u, we may also have bigger vortices (b_i^ε,q_i) of size ρ, such that u satisfies the same conclusions as in Lemma 2.6 for u^γ. We may compare (a_i^ε,d_i) (the vortices of u^γ) with (b_i^ε,q_i) (the vortices of u) by the minimal connection between the vortices as in [18]. Now define positive points p_i and negative points n_i as follows:

(i) The set of p_i's consists of all the b_i's such that $q_i>0$ repeated $|q_i|$ times, with the a_i's such that $d_i<0$ repeated $|d_i|$ times.

(ii) The set of n_i's consists of all the b_i's such that $q_i<0$ repeated $|q_i|$ times, with the a_i's such that $d_i>0$ repeated $|d_i|$ times.

Also define the positive holes by $P_i=B(p_i,\rho)$ and negative holes by $N_i=B(n_i,\rho)$ and add $\mathbb{R}^2\setminus\Omega$ as a hole of multiplicity $\sum d_i-\sum q_i$. Then there are k positive holes and k negative holes, and the distance between the configurations $\bar a=(a_i,d_i)$ and $\bar b=(b_i,q_i)$ is defined by the minimal connection [7],

$$\operatorname{dist}(\bar a,\bar b)=L(p_i,n_i)=\inf_{\sigma\in S_k}\sum_{i=1}^k|p_i-n_{\sigma(i)}|.$$

Lemma 2.8. *For small ε, there holds*

$$\operatorname{dist}(\bar a,\bar b)\le C\varepsilon^\gamma|\ln\varepsilon|. \tag{2.22}$$

§ 3. Splitting of the Energy $J(u, A)$

The results in this section are closely related to those in Section IV.1 in [18]. As in [18], we shall decompose the energy $J(u, A)$ with the following lemma.

Lemma 3.1. *For any* $(\tilde{u}, \widetilde{A}) \in \widetilde{D}$ *satisfying* (1.5), (2.1)–(2.4), *let* (u, A) *be associated with* $(\tilde{u}, \widetilde{A})$ *as in Lemma* 2.2 *and let* u *be associated with* u^γ, *solving the minimization problem* (2.11), *which has vortices* (a_i, d_i) *satisfying Lemma* 2.6. *Then we have*

$$J(u, A) = F_a(u) + V(\xi) + \mathrm{o}(1), \tag{3.1}$$

where

$$V(\xi) = \frac{1}{2}\int_\Omega \frac{1}{a}|\nabla\xi|^2 + \frac{1}{2}\int_\Omega \Big|\mathrm{div}\left(\frac{1}{a}\nabla\xi\right)\Big|^2 - h_{ex}\int_\Omega \mathrm{div}\left(\frac{1}{a}\nabla\xi\right) + 2\pi\sum_{i\in\mathcal{T}_1} d_i\xi(a_i), \tag{3.2}$$

and ξ *is defined by* (1.6).

§ 4. Splitting of the Magnetic Field $V(\xi)$

In this section, we will split the magnetic field under the assumptions of Lemma 3.1. Let

$$\zeta = \xi - h_{ex}\xi_0, \tag{4.1}$$

where ξ_0 is the unique solution of the equation

$$-\mathrm{div}\left(\frac{1}{a}\nabla\Big(\mathrm{div}\left(\frac{1}{a}\nabla\xi_0\right)\Big)\right) + \mathrm{div}\left(\frac{1}{a}\nabla\xi_0\right) = 0 \qquad \text{in } \ \Omega, \tag{4.2}$$

$$\mathrm{div}\left(\frac{1}{a}\nabla\xi_0\right) = 1 \qquad \text{on } \ \partial\Omega, \tag{4.3}$$

$$\xi_0 = 0 \qquad \text{on } \ \partial\Omega. \tag{4.4}$$

By computation, we have the following lemma.

Lemma 4.1. *Under the assumptions of Lemma* 3.1, *we have*

$$V(\xi) = V(h_{ex}\xi_0) + \widetilde{V}(\zeta), \tag{4.5}$$

where

$$\widetilde{V}(\zeta) = \frac{1}{2}\int_\Omega \frac{1}{a}\Big|\nabla\zeta\Big|^2 + \Big|\mathrm{div}\left(\frac{1}{a}\nabla\zeta\right)\Big|^2 + 2\pi\sum_{i\in\mathcal{T}_1} d_i\zeta(a_i). \tag{4.6}$$

Now, from Lemmas 3.1 and 4.1, we can write $J(u, A)$ as

$$J(u, A) = J_0 + F_a(u) + 2\pi h_{ex}\sum_{i\in\mathcal{T}_1} d_i\xi_0(a_i) + \widetilde{V}(\zeta) + \mathrm{o}(1), \tag{4.7}$$

where

$$J_0 = -\frac{1}{2}h_{ex}^2\int_\Omega \Big|\mathrm{div}\left(\frac{1}{a}\nabla\xi_0\right)\Big|^2 + \frac{1}{a}|\nabla\xi_0|^2. \tag{4.8}$$

By a simple computation we also have the following estimate.

Lemma 4.2. *J_0 satisfies*

$$\inf_{\{(u,A)\in\overline{D}:\mathcal{T}_1=\emptyset\}} J(u,A) = J_0 + \frac{1}{2}\int_\Omega |\nabla\sqrt{a}|^2 + \mathrm{o}(1), \tag{4.9}$$

i.e. $J_0 + \frac{1}{2}\int_\Omega |\nabla\sqrt{a}|^2$ *is (almost) the minimal energy for vortex-less configuration.*

Lemma 4.3. *ξ_0 is the unique solution of the equations (4.2)–(4.4). Then ξ_0 satisfies*

$$-\mathrm{div}\left(\frac{1}{a}\nabla\xi_0\right) + \xi_0 = -1 \qquad in \quad \Omega, \tag{4.10}$$

$$\xi_0 = 0 \qquad on \quad \partial\Omega, \tag{4.11}$$

$$-1 < \xi_0 < 0 \qquad in \quad \Omega. \tag{4.12}$$

Proof. By the uniqueness, (4.10) follows directly from (4.2)–(4.4). (4.12) follows from the maximum principle.

Lemma 4.4.

$$J_0 + \frac{1}{2}\int_\Omega h_{ex}^2 = \frac{1}{2}h_{ex}^2\int_\Omega |\xi_0|. \tag{4.13}$$

Proof. By the definition of J_0 and (4.10), the conclusion follows.

In the last part of this section, we will pay our attention to the discussions on $\widetilde{V}(\zeta)$. Since we found that the proof of the following two lemmas on the properties of ζ can be obtained directly from adjusting the corresponding proof of lemmas in [18], we give the statements of the lemmas.

Lemma 4.5. *The configurations (a_i, d_i) being set, $\widetilde{V}(\zeta)$ is minimal for ζ satisfying the following equation*

$$-\mathrm{div}\left(\frac{1}{a}\nabla\Big(\mathrm{div}\left(\frac{1}{a}\nabla\zeta\right)\Big)\right) + \mathrm{div}\left(\frac{1}{a}\nabla\zeta\right) = 2\pi\sum_{i\in\mathcal{T}_1} d_i\delta_{a_i} \qquad in \quad \Omega, \tag{4.14}$$

$$\mathrm{div}\left(\frac{1}{a}\nabla\zeta\right) = 0 \qquad on \quad \partial\Omega, \tag{4.15}$$

$$\zeta = 0 \qquad on \quad \partial\Omega, \tag{4.16}$$

and

$$\widetilde{V}(\zeta) = \pi\sum_{i\in\mathcal{T}_1} d_i\zeta(a_i). \tag{4.17}$$

Lemma 4.6.

$$\widetilde{V}(\zeta) = \pi\sum_{i,j} d_i d_j \zeta^{a_i}(a_j) \leq C\Big(\sum_i |d_i|\Big)^2 \tag{4.18}$$

remains bounded as ε tends to 0. Furthermore

$$\|\zeta^p\|_{L^\infty(\Omega)} \leq C\mathrm{dist}\,(p,\partial\Omega)^k, \qquad \forall k < 1, \tag{4.19}$$

$$\|\zeta^p - \zeta^q\|_{H^2(\Omega)} \leq C|p-q|^k, \qquad \forall k < 1. \tag{4.20}$$

Here ζ^p is the solution of

$$-\mathrm{div}\left(\frac{1}{a}\nabla\Big(\mathrm{div}\left(\frac{1}{a}\nabla\zeta^p\right)\Big)\right)+\mathrm{div}\left(\frac{1}{a}\nabla\zeta^p\right)=2\pi\delta_p \qquad in \quad \Omega, \tag{4.21}$$

$$\mathrm{div}\left(\frac{1}{a}\nabla\zeta^p\right)=0 \qquad on \quad \partial\Omega, \tag{4.22}$$

$$\zeta^p=0 \qquad on \quad \partial\Omega. \tag{4.23}$$

§ 5. Estimates for F_a

Denote $e_\varepsilon(u)=\frac{1}{2}[|\nabla u|^2+\frac{1}{2\varepsilon^2}(a-|u|^2)^2]$ and Ω_ρ as before. The same argument of [10] gives the following lemma.

Lemma 5.1.

$$\frac{1}{2}\int_{\Omega_\rho}|\nabla u^\gamma|^2\geq\pi\sum_{i\in\mathcal{T}_1}a(a_i)d_i^2|\ln\rho|+W((a_1,d_1),\cdots,(a_k,d_k))+\mathrm{O}(1), \tag{5.1}$$

where we have assumed $\mathcal{T}_1=\{1,2,\cdots,k\}$ and

$$W((a_1,d_1),\cdots,(a_k,d_k))=-\pi\sum_{i\neq j\in\mathcal{T}_1}a(a_i)d_id_j\ln|a_i-a_j|-\pi\sum_{i\in\mathcal{T}_1}d_iR_0(a_i),$$

$$R_0(x)=\Phi_0(x)-\sum_{i\in\mathcal{T}_1}a(a_i)d_i\ln|x-a_i|,$$

where $\Phi_0(x)$ solves

$$-\mathrm{div}\left(\frac{1}{a}\nabla\Phi_0\right)=2\pi\sum_{i\in\mathcal{T}_1}d_i\,\delta_{a_i} \qquad in \quad \Omega, \tag{5.2}$$

$$\Phi_0=0 \qquad on \quad \partial\Omega. \tag{5.3}$$

Lemma 5.2.

$$F_a(u^\gamma)\geq\pi\sum_{i\in\mathcal{T}_1}a(a_i)d_i^2|\ln\rho|+\pi\sum_{i\in\mathcal{T}_1}a(a_i)|d_i|\ln\frac{\rho}{\varepsilon}$$
$$+W((a_1,d_1),\cdots,(a_k,d_k))+\mathrm{O}(1), \tag{5.4}$$

$$F_a(u^\gamma)\geq\pi\sum_{i\in\mathcal{T}_1}a(a_i)|d_i|\Big|\ln\frac{\rho}{\varepsilon}\Big|+\mathrm{O}(1). \tag{5.5}$$

Proof. First, we have

$$F_a(u^\gamma)\geq\frac{1}{2}\int_{\Omega_\rho}|\nabla u^\gamma|^2+\sum_{i\in\mathcal{T}_1}\int_{B(a_i,\rho)}e_\varepsilon(u^\gamma). \tag{5.6}$$

But we know from [11] that

$$\int_{B(a_i,\rho)}e_\varepsilon(u^\gamma)=\int_{B(a_i,\rho)}\frac{1}{2}\Big[|\nabla u^\gamma|^2+\frac{1}{2\varepsilon^2}(a-|u^\gamma|^2)^2\Big]\geq\pi a(a_i)|d_i|\ln\frac{\rho}{\varepsilon}+\mathrm{O}(1).$$

Hence

$$\sum_{i\in\mathcal{T}_1}\int_{B(a_i,\rho)}e_\varepsilon(u^\gamma)\geq\sum_{i\in\mathcal{T}_1}\pi a(a_i)|d_i|\ln\frac{\rho}{\varepsilon}+\mathrm{O}(1).$$

Now combining the above result with (5.1), we have the conclusions of this lemma.

§ 6. Estimates of the Critical Magnetic Field H_{c_1}

In this section, we shall obtain the critical magnetic field H_{c_1}.

Lemma 6.1. *Suppose $M > \pi \max\limits_{\overline{\Omega}} a(x)$. Then there exist $k_2^\varepsilon = \mathrm{O}(1)$, $k_3^\varepsilon = \mathrm{o}(1)$ as $\varepsilon \to 0$ such that for*

$$h_{ex} = \frac{|\ln \varepsilon|}{2 \max\limits_{\Omega} \left|\frac{\xi_0}{a}\right|} + t, \tag{6.1}$$

there holds

(i) *If $t < k_2^\varepsilon$, considering $(\tilde{u}, \widetilde{A})$ a minimizing configuration in $\overline{D}$, then $\mathcal{T}_1 = \emptyset$ and*

$$J(\tilde{u}, \widetilde{A}) = \inf_{\{(u,A) \in \overline{D}: \mathcal{T}_1 = \emptyset\}} J = J_0 + \frac{1}{2} \int_\Omega |\nabla \sqrt{a}|^2 + \mathrm{o}(1). \tag{6.2}$$

(ii) *If $t = k_2^\varepsilon$, then there is $(u, A) \in \overline{D}$ with one vortex of degree one such that*

$$J(u, A) \le \inf_{\{(u,A) \in \overline{D}: \mathcal{T}_1 = \emptyset\}} J. \tag{6.3}$$

(iii) *If $t \ge k_2^\varepsilon + k_3^\varepsilon$, there exists (u, A) in $\overline{D}$ having one vortex of degree one such that*

$$J(u, A) < \inf_{\{(u,A) \in \overline{D}: \mathcal{T}_1 = \emptyset\}} J.$$

Proof. Recall that, for (u, A) obtained from $(\tilde{u}, \widetilde{A})$ with the aid of Lemma 2.2, (4.7) yields

$$J(u, A) = F_a(u) + J_0 + 2\pi h_{ex} \sum_{i \in \mathcal{T}_1} d_i \xi_0(a_i) + \widetilde{V}(\zeta) + \mathrm{o}(1), \tag{6.4}$$

and if $\mathcal{T}_1 \neq \emptyset$, using Lemma 5.2, we have

$$F_a(u) \ge F_a(u^\gamma) \ge \pi \sum_{i \in \mathcal{T}_1} a(a_i) |d_i| |\ln \varepsilon| + \mathrm{O}(1). \tag{6.5}$$

Now substituting (6.5) into (6.4), we get

$$J(u, A) \ge J_0 + \pi \sum_{i \in \mathcal{T}_1} a(a_i) |d_i| |\ln \varepsilon| - 2\pi h_{ex} \sum_{i \in \mathcal{T}_1} a(a_i) |d_i| \left| \frac{\xi_0(a_i)}{a(a_i)} \right| + \widetilde{V}(\zeta) + \mathrm{O}(1).$$

Hence

$$J(u, A) > \inf_{\{(u,A) \in \overline{D}: \mathcal{T}_1 = \emptyset\}} J(u, A)$$

as long as

$$\pi h_{ex} \sum_{i \in \mathcal{T}_1} a(a_i) |d_i| \left| \frac{\xi_0(a_i)}{a(a_i)} \right| \le \pi \sum_{i \in \mathcal{T}_1} a(a_i) |d_i| |\ln \varepsilon| + \mathrm{O}(1),$$

which is valid if

$$h_{ex} \le \frac{|\ln \varepsilon|}{2 \max\limits_{\overline{\Omega}} \left| \frac{\xi_0(x)}{a(x)} \right|} + \mathrm{O}(1). \tag{6.6}$$

As in [18], set

$$Z^\varepsilon = \Big\{ t \in \mathbb{R} : \text{there exist } (u, A) \in \overline{D} \text{ with at least one vortex and}$$
$$J(u, A) < \inf_{\{(u,A)\in\overline{D}:\mathcal{T}_1=\emptyset\}} J \quad \text{for } h_{ex} = k_1|\ln\varepsilon| + t \Big\},$$

where $k_1 = \frac{1}{2\max_{\overline{\Omega}}|\xi_0(x)/a(x)|}$. In the following, we want to show $Z^\varepsilon \neq \emptyset$ and then define $k_2^\varepsilon = \inf Z^\varepsilon$ and prove that there exists $k_3^\varepsilon = \mathrm{o}(1)$ such that $[k_2^\varepsilon + k_3^\varepsilon, +\infty] \subset Z^\varepsilon$.

To show $Z^\varepsilon \neq \emptyset$, let $c \in \overline{\Omega}$ such that $|\xi_0(c)/a(c)| = \max_{\overline{\Omega}} |\xi_0(x)/a(x)|$. Then, since $\xi_0 = 0$ on $\partial\Omega$, we have $c \in \Omega$. Now consider the following minimization problem

$$\nu_\varepsilon(c) = \min_W \frac{1}{2} \int_{\Omega\setminus B(c,\varepsilon)} |\nabla u|^2, \tag{6.7}$$

where $W = \{u : u = \sqrt{a}v,\ v \in H^1(\Omega \setminus B(c,\varepsilon), S^1),\ \deg(v, \partial B(c,\varepsilon)) = 1\}$.

Then, it follows from discussions in [11] that

$$\nu_\varepsilon(c) = \frac{1}{2}\int_{\Omega\setminus B(c,\varepsilon)} \frac{1}{a(x)}|\nabla\Phi_\varepsilon|^2 + \frac{1}{2}\int_\Omega \left|\nabla\sqrt{a}\right|^2 = \pi a(c)|\ln\varepsilon| + \mathrm{O}(1), \tag{6.8}$$

where Φ_ε satisfies

$$\begin{cases} -\mathrm{div}\left(\dfrac{1}{a}\nabla\Phi_\varepsilon\right) = 0 & \text{in } \ \Omega\setminus B(c,\varepsilon), \\ \Phi_\varepsilon = 0 & \text{on } \ \partial\Omega, \\ \Phi_\varepsilon = \text{const.} & \text{on } \ \partial B(c,\varepsilon), \\ \displaystyle\int_{\partial B(c,\varepsilon)} \frac{1}{a}\frac{\partial\Phi_\varepsilon}{\partial n} = 2\pi. \end{cases} \tag{6.9}$$

Let u be a minimizer of the problem (6.7) which is well defined on $\Omega\setminus B(c,\varepsilon)$. Now we extend it to the whole domain Ω by defining it on $B(c,\varepsilon)$ as taking polar system centered at c such that $\bar{u}(r,\theta) = f(r)w(\theta)$ with $w(\theta)|_{\partial B(c,\varepsilon)} = u|_{\partial B(c,\varepsilon)}$, $f \equiv 1$ in $\Omega\setminus B(c, \frac{\varepsilon}{2})$, $f \equiv 0$ in $B(c, \frac{\varepsilon}{4})$ and $|f'(r)| \le \frac{C}{\varepsilon}$. Then, since $\bar{u}$ has one vortex at c, $\bar{u}^\gamma$ has one vortex in $B(c, C\varepsilon^\gamma|\ln\varepsilon|)$. Thus, we have

$$\begin{aligned} F_a(\bar{u}, B(c,\varepsilon)) &= \frac{1}{2}\int_{B(c,\varepsilon)} \left[|\nabla\bar{u}|^2 + \frac{1}{2\varepsilon}(a - |\bar{u}|^2)^2\right] \\ &\le \frac{1}{2}\int_{B(c,\varepsilon)} \|f'\|^2_{L^\infty} + \frac{C}{\varepsilon^2}\|w'\|^2_{L^\infty} + \frac{1}{2\varepsilon^2} \le C, \end{aligned} \tag{6.10}$$

and hence

$$F_a(\bar{u}, \Omega) = F_a(\bar{u}, B(c,\varepsilon)) + \frac{1}{2}\int_{\Omega\setminus B(c,\varepsilon)} |\nabla\bar{u}|^2 \le K + a(c)\pi|\ln\varepsilon|. \tag{6.11}$$

Then, taking $\xi = h_{ex}\xi_0 + \zeta^c$, where ζ^c is defined by (4.21)–(4.23), we know from Section 4 that

$$V(\xi) = V(h_{ex}\xi_0) + \widetilde{V}(\zeta^c) = J_0 + 2\pi h_{ex}\xi_0(c) + \pi\zeta^c(c) + \mathrm{o}(1).$$

Now we derive that for

$$h_{ex} = \frac{1}{2\max|\xi_0(x)/a(x)|}|\ln\varepsilon| + t = -\frac{1}{2\xi_0(c)/a(c)}|\ln\varepsilon| + t,$$

$$\begin{aligned} J(\bar{u}, A) &\le K + a(c)\pi|\ln\varepsilon| + J_0 + 2\pi h_{ex}\xi_0(c) + \mathrm{o}(1) \\ &\le K + a(c)\pi|\ln\varepsilon| + J_0 + 2\pi\xi_0(c)\Big[t - \frac{a(c)}{\xi_0(c)}|\ln\varepsilon|\Big] + \mathrm{o}(1) \\ &= K + 2\pi\xi_0(c)t + J_0 + \mathrm{o}(1). \end{aligned}$$

This in turn implies $t \in Z^\varepsilon$ when

$$t \ge \frac{K - \frac{1}{2}\int_\Omega \left|\nabla\sqrt{a}\right|^2}{2\pi|\xi_0(c)|} + \mathrm{o}(1).$$

Therefore $Z^\varepsilon \neq \emptyset$ and

$$k_2^\varepsilon = \inf Z^\varepsilon \le \frac{K - \frac{1}{2}\int_\Omega \left|\nabla\sqrt{a}\right|^2}{2\pi|\xi_0(c)|} + \mathrm{o}(1).$$

On the other hand, by (6.6), we have $h_{ex} \le k_1|\ln\varepsilon| + \mathrm{O}(1)$. So, we know that $k_2^\varepsilon \ge \mathrm{O}(1)$ and thus $k_2^\varepsilon = \mathrm{O}(1)$. Now we prove that there exists $k_3^\varepsilon = \mathrm{o}(1)$ such that $[k_2^\varepsilon + k_3^\varepsilon, +\infty] \subset Z^\varepsilon$. In fact, let $t \in Z^\varepsilon$ and let (u, A) satisfy

$$J(u, A) < \inf_{\{(u,A)\in\overline{D}:\mathcal{T}_1=\emptyset\}} J \tag{6.12}$$

for $h_{ex,1} = k_1|\ln\varepsilon| + t$ and u^γ has vortices (a_i, d_i). Assume $t' > t$ and $h_{ex,2} = k_1|\ln\varepsilon| + t'$. Then we have

$$\begin{aligned} J(u, A) &= F_a(u) + V(h_{ex,2}\xi_0) + \widetilde{V}(\zeta) + \mathrm{o}(1) \\ &= F_a(u) + J_0 + h_{ex,2}\sum_{i\in\mathcal{T}_1} 2\pi d_i\xi_0(a_i) + \widetilde{V}(\zeta) + \mathrm{o}(1) \\ &\le F_a(u) + J_0 + h_{ex,1}\sum_{i\in\mathcal{T}_1} 2\pi d_i\xi_0(a_i) + \widetilde{V}(\zeta) + \mathrm{o}(1) + (t'-t)\sum_{i\in\mathcal{T}_1} 2\pi d_i\xi_0(a_i). \end{aligned}$$

This means that if $t' - t \ge k_3^\varepsilon = \mathrm{o}(1)$, then by (6.12) $J(u, A) < \inf_{\{(u,A)\in\overline{D}:\mathcal{T}_1=\emptyset\}} J(u, A)$. This in turn implies $[k_2^\varepsilon + k_3^\varepsilon, +\infty] \subset Z^\varepsilon$ and the proof of this lemma is complete.

In summary, we have deduced that $H_{c_1} = k_1|\ln\varepsilon| + k_2^\varepsilon$ where H_{c_1} is the first critical field defined as the value of h_{ex} for which the minimal energy among vortex-less configurations is equal to the minimal energy among single-vortex configurations.

§ 7. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 7.1. *Let $h_{ex} = k_1|\ln\varepsilon| + \mathrm{o}(|\ln\varepsilon|)$ and $(\tilde{u}, \widetilde{A})$ be a minimizer of $J(u, A)$ in $\overline{D}$. Let $\{(a_i, d_i)\}_{i=1}^k$ be vortices of u^γ. Then $d_i = 1$ for $i \in \mathcal{T}_1 = \{1, 2, \cdots, k\}$ and there are $\varepsilon_0 > 0$ and $\alpha > 0$ such that* $\operatorname{dist}(a_i, \partial\Omega) \ge \alpha > 0$.

Proof. The conclusion follows by the same argument in [10, 18].

Now to prove next lemmas we need the following assumption

$$H_{c_1} + k_3^\varepsilon \le h_{ex} \le k_1|\ln\varepsilon| + \mathrm{O}(1), \tag{7.1}$$

where

$$H_{c_1} = k_1|\ln\varepsilon| + k_2^\varepsilon \tag{7.2}$$

with

$$k_1 = \frac{1}{2\max\limits_{\Omega}|\xi_0(x)/a(x)|}, \qquad k_2^\varepsilon = \mathrm{O}(1), \qquad k_3^\varepsilon = \mathrm{o}(1). \tag{7.3}$$

Lemma 7.2. *Let (u, A) be obtained from a minimizer $(\tilde{u}, \widetilde{A})$ of J in $\overline{D}$, and $(a_i^\varepsilon, d_i^\varepsilon)$ be the vortices of u^γ. Denote*

$$\Lambda = \Big\{x \in \Omega : \Big|\frac{\xi_0(x)}{a(x)}\Big| = \max_{y\in\Omega}\Big|\frac{\xi_0(y)}{a(y)}\Big|\Big\}. \tag{7.4}$$

Then we have

$$\operatorname{dist}(a_i^\varepsilon, \Lambda) \to 0 \qquad \textit{as} \ \ \varepsilon \to 0, \quad \forall\, i \in \mathcal{T}_1, \tag{7.5}$$

$$\operatorname{dist}(a_i^\varepsilon, a_j^\varepsilon) \geq \alpha > 0 \qquad \textit{for} \ \ i \neq j \in \mathcal{T}_1. \tag{7.6}$$

The first result (7.5) *remains true if we only assume $h_{ex} \leq k_1|\ln\varepsilon| + \mathrm{o}(|\ln\varepsilon|)$.*

Proof. If $\mathcal{T}_1 \neq \emptyset$, we have from Lemma 7.1 that $d_i^\varepsilon = 1$ for all $i \in \mathcal{T}_1$. Denote $d = \sum\limits_{i\in\mathcal{T}_1} d_i^\varepsilon = \mathrm{Card}\mathcal{T}_1 = \deg(u^\gamma, \partial\Omega)$. Then it follows from Lemma 5.2, as in [10, 18], that

$$W(a_1, \cdots, a_k) + \pi\sum_{i\in\mathcal{T}_1} a(a_i^\varepsilon)|\ln\varepsilon| + \mathrm{O}(1) \leq F_a(u^\gamma) \leq -2\pi h_{ex}\sum_{i\in\mathcal{T}_1}\xi_0(a_i^\varepsilon) + \mathrm{o}(|\ln\varepsilon|),$$

where $W(a_1, \cdots, a_k) = W((a_1, 1), \cdots, (a_k, 1))$. Then

$$\begin{aligned}\pi\sum_{i\in\mathcal{T}_1} a(a_i^\varepsilon)|\ln\varepsilon| &\leq 2\pi h_{ex}\sum_{i\in\mathcal{T}_1} a(a_i^\varepsilon)\max_{\overline{\Omega}}\Big|\frac{\xi_0(x)}{a(x)}\Big| \\ &\quad - 2\pi h_{ex}\sum_{i\in\mathcal{T}_1} a(a_i^\varepsilon)\Big(\frac{\xi_0(a_i^\varepsilon)}{a(a_i^\varepsilon)} + \max_{\overline{\Omega}}\Big|\frac{\xi_0(x)}{a(x)}\Big|\Big) + \mathrm{o}(|\ln\varepsilon|),\end{aligned}$$

which in turn gives

$$\sum_{i\in\mathcal{T}_1}\Big(\frac{\xi_0(a_i^\varepsilon)}{a(a_i^\varepsilon)} + \max_{\overline{\Omega}}\Big|\frac{\xi_0(x)}{a(x)}\Big|\Big) \leq \frac{\mathrm{o}(|\ln\varepsilon|)}{h_{ex}} \to 0,$$

hence

$$\operatorname{dist}(a_i^\varepsilon, \Lambda) \to 0 \qquad \text{as} \ \ \varepsilon \to 0, \quad \forall\, i \in \mathcal{T}_1.$$

Moreover, since $W(a_1^\varepsilon, \cdots, a_k^\varepsilon) \geq \mathrm{O}(1)$ and

$$W(a_1^\varepsilon, \cdots, a_k^\varepsilon) + \pi\sum_{i\in\mathcal{T}_1} a(a_i^\varepsilon)|\ln\varepsilon| + \mathrm{O}(1) \leq -2\pi h_{ex}\sum_{i\in\mathcal{T}_1}\xi_0(a_i^\varepsilon) + \mathrm{O}(1),$$

we have $W(a_1^\varepsilon, \cdots, a_k^\varepsilon) \leq \mathrm{O}(1)$ if $h_{ex} \leq k_1|\ln\varepsilon| + \mathrm{O}(1)$. Therefore we conclude that $|a_i^\varepsilon - a_j^\varepsilon|$ remains bounded from below uniformly since as in [5] we could prove that $W \to +\infty$ if $|a_i^\varepsilon - a_j^\varepsilon| \to 0$ for some $i \neq j$. Lemma 7.2 is proved.

Lemma 7.3. *Assume that $M > \pi(\mathrm{Card}\Lambda)\max\limits_{\Lambda} a(x)$. Then, for ε sufficiently small, a minimizer $(\tilde{u}, \widetilde{A})$ of J in $\overline{D}$ is a solution of* (G.L.), *and $(u, A) = (\tilde{u}, \widetilde{A})$, where (u, A) comes from $(\tilde{u}, \widetilde{A})$ in Lemma* 2.2.

Proof. It suffices to show that $(\tilde{u}, \widetilde{A})$ is not on the boundary of $\overline{D}$. Since we have proved that W is a bounded quantity in Lemma 7.2 and dist $(a_i^\varepsilon, a_j^\varepsilon) \geq \alpha > 0$, we have

$$\pi \sum_{i\in\mathcal{T}_1} a(a_i)|\ln\varepsilon| + \mathrm{O}(1) \leq \pi \sum_{i\in\mathcal{T}_1} a(a_i)|\ln\varepsilon| + \mathrm{O}(1).$$

Hence, this inequality together with Lemma 2.2 yields that

$$F_a(u) = \pi \sum_{i\in\mathcal{T}_1} a(a_i)|\ln\varepsilon| + \mathrm{O}(1). \tag{7.7}$$

Now it follows from Lemma 7.2 that as $\varepsilon \to 0$, $a_i^\varepsilon \to c_i \in \Lambda$ and thus we get from (7.6) that $c_i \neq c_j$ for $i \neq j$. Hence

$$d = \mathrm{Card}\mathcal{T}_1 \leq \mathrm{Card}\Lambda. \tag{7.8}$$

Thus

$$\pi \sum_{i\in\mathcal{T}_1} a(a_i) \leq \pi \mathrm{Card}\Lambda \max_\Lambda a(x) + \delta < M \qquad \text{for } \varepsilon \leq \varepsilon_0. \tag{7.9}$$

Here, $\delta \leq \frac{1}{2}\Big(M - \pi\mathrm{Card}\Lambda \max_\Lambda a(x)\Big)$. Thus

$$F_a(\tilde{u}) \leq \pi \sum_{i\in\mathcal{T}_1} a(a_i)|\ln\varepsilon| + \mathrm{O}(1) \leq (M-\delta)|\ln\varepsilon|, \tag{7.10}$$

which implies that $(\tilde{u}, \widetilde{A})$ is not on ∂D. Thus Lemma 7.3 is proved.

Now since (u, A) is a solution of (G.L.), we can show that $|\nabla u| \leq C/\varepsilon$. Then u has bigger vortices of size ρ : $\{(b_i, q_i)\}_{i\in\mathcal{T}_1}$. The following lemma compares the vortices of u^γ with the vortices of u.

Lemma 7.4. *For sufficiently small ε, we have*

(i) *If (u, A) is a solution of* (G.L.) *such that $J(u, A) \leq Ch_{ex}^2$, then $|u| \leq 1$, and $|\nabla u| \leq \dfrac{C}{\varepsilon}$.*

(ii) *If (u, A) is a solution of* (G.L.) *such that u^γ has no vortex $\Big($i.e. $|u^\gamma| \geq \dfrac{b_0}{2}\Big)$ and $J(u, A) \leq J_0 + \dfrac{1}{2}\displaystyle\int_\Omega |\nabla\sqrt{a}|^2 + \mathrm{o}(1)$, then u has not vortex on Ω $\Big(|u| \geq \dfrac{b_0}{2}\Big)$.*

(iii) *If (u, A) is a solution of* (G.L.) *given by Theorem* 1.1, *then its vortices* (*of size ρ*) *satisfy the same conclusions as those of u^γ.*

(iv) *In addition, if $(a_i)_{i\in\mathcal{T}_1}$ are the vortices of u^γ of degree one, then the vortices $\{b_i\}_{i\in\mathcal{T}_1}$ of u are of degree one and Lemma* 2.6 (*on u^γ*) *is satisfied by u.*

Proof. The conclusions follow by the same argument in [19].

Proof of Theorem 1.1. Let $(\tilde{u}, \widetilde{A})$ be a minimizer of $J(u, A)$ in $\overline{D}$. Then, from Lemma 7.3, we have that $(\tilde{u}, \widetilde{A})$ is a solution of (G.L.), and $(u, A) = (\tilde{u}, \widetilde{A})$, where (u, A) comes from $(\tilde{u}, \widetilde{A})$ by Lemma 2.2.

Now, in the case that $h_{ex} \leq H_{c_1}$, we know that u^γ has not a vortex from Lemma 6.1. Combining this with the conclusion (ii) of Lemma 7.4, we obtain that u has no vortex and $|u| \geq \frac{b_0}{2}$ in $\overline{\Omega}$. Moreover, from the conclusion (i) of Lemma 7.4, we have $|u| \leq 1$ in $\overline{\Omega}$. Thus the conclusion (i) of Theorem 1.1 is proved.

In the case that $H_{c_1} + k_3^2 \leq h_{ex} \leq H_{c_1} + O(1)$, we know that u^γ has vortices from Lemma 6.1. Then by the conclusions of Lemma 7.1, Lemma 7.2, and the conclusions (iii) and (iv) in Lemma 7.4, we obtain the conclusion (ii) of Theorem 1.1. The proof of Theorem 1.1 is complete.

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