Chin. Ann. Math.
25B:4(2004),507–522.

ON THE RECOVERY OF A CURVE ISOMETRICALLY IMMERSED IN $\mathbb{E}^n$

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Abstract

It is known from classical differential geometry that one can reconstruct a curve with $(n-1)$ prescribed curvature functions, if these functions can be differentiated a certain number of times in the usual sense and if the first $(n-2)$ functions are strictly positive. It is established here that this result still holds under the assumption that the curvature functions belong to some Sobolev spaces, by using the notion of derivative in the distributional sense. It is also shown that the mapping which associates with such prescribed curvature functions the reconstructed curve is of class $\mathcal{C}^\infty$.

Keywords Differential geometry, Nonlinear elasticity, Curves in Euclidean space, Frenet equations, Weak derivatives
2000 MR Subject Classification 53A04, 34G05, 74K05

§ 1 . Introduction

Physically, we can think of a curve as being obtained from a straight line by bending and twisting. After reflecting on this construction, we are led to conjecture that, roughly speaking, the curvature functions describe completely the behavior of the curve. This statement is true: we can find a proof of this classical result of differential geometry for example in [4], where the case of a curve isometrically immersed in the n-dimensional Euclidean space is treated, without specifying the regularity requirements for the initial data.

The same reconstruction problem can be posed for a surface or for an open set of $\mathbb{R}^n$ and there exist different methods to prove this result of differential geometry. Recently, motivated by some problems encountered in nonlinear elasticity, some extensions have been obtained for the case of Sobolev-type functions. More specifically, the classical result for an open set states that if the metric tensor is of class $\mathcal{C}^2$ and satisfies the Riemann compatibility conditions, then it is induced by an immersion (see for instance [10] for a “local” version, or [2] for the proof of the existence of a global immersion if in addition the open set is simply-connected). Then, it has been proved by C. Mardare in [7] that the same result holds under the assumption that the metric is of class $\mathcal{C}^1$; moreover, we can find in S. Mardare [8] an even stronger result, for the case when the metric is only of class $W^{1,\infty}_{\mathrm{loc}}$.

Another interesting question concerns the regularity of the mapping that can be defined by associating with the prescribed data (metric and curvature) the reconstructed manifold in $\mathbb{R}^n$. In this direction, it has been established that this mapping is continuous for certain natural metrizable topologies, in the case of an open set of $\mathbb{R}^n$ by Ciarlet and Laurent in [3] and in the case of a surface (using a different method) by Ciarlet in [1].

Manuscript received December 5, 2003.
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The purpose of this paper is twofold. First we provide a proof of the existence and uniqueness of a curve immersed in $\mathbb{R}^n$, whose curvature functions are $(n-1)$ prescribed functions in $H^{n-2}(I;\mathbb{R})\times H^{n-3}(I;\mathbb{R})\times\cdots\times H^1(I;\mathbb{R})\times L^2(I;\mathbb{R})$; we emphasize that, instead of the classical framework of differential geometry, where all functions are considered to be indefinitely derivable, our setting will be that of distributions and we will always use the notion of derivative in the general sense. Second, we show that the mapping constructed in this fashion is of class $\mathcal{C}^\infty$. As corollaries, we derive the same results in the classical setting, where derivatives are considered in the usual sense.

The paper is organized as follows. In Section 2, we present some technical results which will be used in the sequel. In Section 3, we prove the existence and uniqueness (or uniqueness up to rigid motions, if the assumptions are weakened) of a curve with prescribed curvatures and in Section 4 we show that the mapping constructed in this manner is of class $\mathcal{C}^\infty$. Finally, in Section 5 we gather some additional commentaries about our problem.

The results of this paper have been announced in [11].

§ 2. Preliminaries

To begin with, we introduce some conventions and notations that will be used throughout the article. For any $n\geq 1$, the n-dimensional Euclidean space $\mathbb{E}^n$ will be identified with $\mathbb{R}^n$ and will be endowed with the Euclidean norm defined by $|a|=\sqrt{\langle a,a\rangle}$, where $\langle a,b\rangle$ denotes the Euclidean inner product of $a,b\in\mathbb{R}^n$. The notations $\mathbb{M}^{n\times n}$ and $\mathbb{O}^n_+$, respectively designate the set of all real square matrices and of all proper orthogonal matrices of order n (a matrix Q is proper orthogonal if Q is orthogonal and $\det Q=1$). A mapping $\varphi\colon\mathbb{R}^n\to\mathbb{R}^n$ defined by $\varphi(x)=x_0+Qx$, where $x_0\in\mathbb{R}^n$ and $Q\in\mathbb{O}^n_+$ is called a proper isometry or rigid motion. We denote by

$$|A|:=\sup_{v\in\mathbb{R}^n\setminus\{0\}}\frac{|Av|}{|v|}$$

the operator norm of a matrix $A\in\mathbb{M}^{n\times n}$ and by I_n the identity matrix of order n.

If X is a Hilbert space, we denote by $|\cdot|_X$ its norm induced by the inner product and by $\mathcal{D}'((0,T);X)$ the space of X-valued distributions. For all integer $m\geq 0$, the m-th derivative of $f\in\mathcal{D}'((0,T);X)$ is denoted as $f^{(m)}$ and the first derivative is denoted as f' or $f^{(1)}$. The function spaces used in this paper are denoted as follows: $L^p((0,T);X)$ for $1\leq p<+\infty$ is the space of all measurable functions $f\colon(0,T)\to X$ such that

$$|f|_{L^p((0,T);X)}:=\Big(\int_0^T|f(t)|_X^p\,dt\Big)^{\frac{1}{p}}<+\infty,$$
$$H^m((0,T);X):=\{v\in L^2((0,T);X);\ v^{(k)}\in L^2((0,T);X),\ \forall\,k\leq m\}.$$

Let $H^0((0,T);X):=L^2((0,T);X)$. For all integer $m\geq 0$, the space $H^m((0,T);X)$ endowed with the inner product

$$\langle u,v\rangle_{H^m((0,T);X)}:=\sum_{k=0}^m\int_0^T\langle u^{(k)}(t),v^{(k)}(t)\rangle_X\,dt$$

is a Hilbert space.

In the sequel, we present three lemmas that will be key ingredients in the proof of our main results (Theorems 3.1 and 4.1). The first lemma establishes the existence and uniqueness of the solution of a differential system and the second and third lemmas are about mappings between Sobolev spaces. In what follows, the derivatives are to be understood in the distributional sense and classes of functions in $H^1((0,T);\mathbb{M}^{n\times n})$ are identified with their continuous representative, as allowed by the Sobolev imbedding theorem. In particular, it makes sense to consider $Y(0)$ in the system below, since $Y\in\mathcal{C}^0([0,T];\mathbb{M}^{n\times n})$.

Lemma 2.1. *Consider the system of differential equations:*

$$\begin{aligned} &Y'(t)=A(t)Y(t)+B(t) \qquad \text{for almost all } \; t \; \text{ in } \; (0,T),\\ &Y(0)=Y_0, \end{aligned} \tag{2.1}$$

where A and B belong to the space $L^2((0,T);\mathbb{M}^{n\times n})$ and Y_0 is a matrix of $\mathbb{M}^{n\times n}$.

Then there exists a unique solution $Y\in H^1((0,T);\mathbb{M}^{n\times n})$ to this system.

Proof. It is a direct consequence of Theorem 4.1 and Remark 4.3 of [5].

Lemma 2.2. *Let $k\geq 0$ and $m\geq 1$ be two integers such that $k\leq m$. Then the mapping*

$$(f,g)\in H^k((0,T);\mathbb{R})\times H^m((0,T);\mathbb{R}^n)\to(fg)\in H^k((0,T);\mathbb{R}^n)$$

is of class $\mathcal{C}^\infty$.

Proof. The proof is straightforward: this application being bilinear, we only have to prove its continuity, in order to prove that it is $\mathcal{C}^\infty$. To do this, we distinguish two situations: $1\leq k\leq m$ and $0=k<m$ and then use the Sobolev imbeddings and their consequence that $H^k((0,T);\mathbb{R}^n)$ is a Banach algebra for $k\geq 1$.

Lemma 2.3. *Let k be a positive integer. Then the mapping*

$$f\in H^k((0,T);\mathbb{R}^n)\to g\in H^{k+1}((0,T);\mathbb{R}^n),$$

where $g(t)=\displaystyle\int_0^t f(s)ds$, is of class $\mathcal{C}^\infty$.

Proof. The continuity of this mapping is a classical result (see, for example, [6]). Since this mapping is also linear, it is of class $\mathcal{C}^\infty$.

For the sake of completeness, we state the implicit function theorem in the functional setting of Banach spaces.

Theorem 2.1. (Implicit Function Theorem) *Let there be given three Banach spaces X_1, X_2 and Y, an open subset Ω of the space $X_1\times X_2$ containing a point (a_1,a_2), and a mapping $\varphi\colon\Omega\subset X_1\times X_2\to Y$ satisfying*

$$\varphi\in\mathcal{C}^1(\Omega,Y),\qquad \partial_2\varphi(a_1,a_2)\in\operatorname{Isom}(X_2,Y).$$

Let $\varphi(a_1,a_2)=b\in Y$. Then there exist open subsets O_1 and O_2 of the spaces X_1 and X_2 respectively, such that $(a_1,a_2)\in O_1\times O_2\subset\Omega$, and there exists an implicit function $f\colon O_1\subset X_1\to O_2\subset X_2$ such that

$$\{(x_1,x_2)\in O_1\times O_2,\varphi(x_1,x_2)=b\}=\{(x_1,x_2)\in O_1\times O_2,x_2=f(x_1)\},$$

$f\in\mathcal{C}^1(O_1,X_2)$ and $f'(x_1)=-\{\partial_2\varphi(x_1,f(x_1))\}^{-1}\partial_1\varphi(x_1,f(x_1))$ for all $x_1\in O_1$; moreover, f is unique, provided that O_1 is taken sufficiently small.

If in addition the mapping $\varphi\colon\Omega\subset X_1\times X_2\to Y$ is of class $\mathcal{C}^m$, $m\geq 2$, the implicit function $f\colon O_1\to X_2$ is also of class $\mathcal{C}^m$.

Proof. This is a well-known result (see for example [9, Chapter 3, Section 8]).

§ 3 . Existence and Uniqueness of the Curve c

To begin with, we present the geometrical framework of our problem. The integer $n \geq 3$ is fixed throughout this section. If $v_1, \cdots, v_k$, $k \geq 1$, are vectors in $\mathbb{R}^n$, we denote by $\mathrm{sp}\{v_1, \cdots, v_k\}$ the vector space spanned by these vectors.

Let $I = [0, T]$ be a bounded interval of $\mathbb{R}$. Let $c\colon I \to \mathbb{R}^n$ be a regular curve of class $\mathcal{C}^{n-1}$ over I, i.e., the vectors $c^{(1)}(t), c^{(2)}(t), \cdots, c^{(n-1)}(t)$ are linearly independent for all $t \in I$. In this setting, one can show (see, for example, [4]) that there exists a unique Frenet frame associated with this curve, denoted by $\{e_1(t), \cdots, e_n(t)\}$, which is a family of vector fields along the curve c such that

$$\begin{aligned} \langle e_i(t), e_j(t)\rangle &= \delta_{ij}, && \forall\, i, j \in \{1, \cdots, n\},\ \forall\, t \in I, \\ \mathrm{sp}\{e_1(t), \cdots e_k(t)\} &= \mathrm{sp}\{c^{(1)}(t), \cdots, c^{(k)}(t)\}, && \forall\, k \in \{1, \cdots, n-1\},\ \forall\, t \in I, \end{aligned}$$

the two bases having the same orientation (i.e., the matrix of change of basis has positive determinant) and such that $\{e_1(t), \cdots e_n(t)\}$ is positively oriented for all $t \in I$ (i.e., it has the same orientation as the natural ordered basis $\{(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, 0, \cdots, 1)\}$).

Then the Frenet formulas read:

$$e_i'(t) = \sum_{j=1}^{n} a_{ij}(t) e_j(t), \qquad \forall\, i \in \{1, \cdots, n\},$$

where the functions $a_{ij}\colon I \to \mathbb{R}$ satisfy

$$\begin{aligned} a_{ij}(t) + a_{ji}(t) &= 0, && \forall\, i, j \in \{1, \cdots, n\},\ \forall\, t \in I, \\ a_{ij}(t) &= 0, && \forall\, j \geq i + 2,\ \forall\, t \in I. \end{aligned}$$

The curvature functions of c at the point $t \in I$ are then defined by

$$k_i(t) := \frac{a_{i,i+1}(t)}{|c'(t)|}, \qquad \forall\, i \in \{1, \cdots, n-1\} \tag{3.1}$$

or equivalently, by means of the Frenet formulas, as

$$k_i(t) = \frac{\langle e_{i+1}(t), e_i'(t)\rangle}{|c'(t)|}. \tag{3.2}$$

We recall that the curvature functions are invariant under the action of rigid motions, i.e., if $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a proper isometry and $\tilde{c} = \varphi \circ c$, then $\tilde{k}_i(t) = k_i(t)$ for all $i \in \{1, \cdots, n-1\}$ and for all $t \in I$ (with self-explanatory notations).

The objective of this section is to prove the existence and uniqueness up to rigid motions of a curve isometrically immersed in the n-dimensional space and with prescribed curvature functions. While this result is classical if the curvature functions are regular enough (see the assumptions of Corollary 3.2), it will be assumed here that they only belong to some specific Sobolev spaces.

Theorem 3.1. (Existence and Uniqueness) *Let* $(F_1, \cdots, F_{n-1}) \in H^{n-2}(I; \mathbb{R}) \times \cdots \times H^1(I; \mathbb{R}) \times L^2(I; \mathbb{R})$ *be such that* $F_1(t) > 0, \cdots, F_{n-2}(t) > 0$ *for all* $t \in I$*. Then*

(a) *There exists a regular curve* $c \in H^n(I; \mathbb{R}^n)$ *such that* $|c'(t)| = 1$ *for all* $t \in I$ *and its curvature functions are* $F_1, \cdots, F_{n-1}$*, i.e.,* $k_i(t) = F_i(t)$ *for all* $i \in \{1, \cdots, n-1\}$ *and* $t \in I$*.*

(b) *If c and $\tilde{c}$ are two curves satisfying the conditions of part* (a)*, then there exists a rigid motion $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\tilde{c} = \varphi \circ c$.*

(c) *If $x_0 \in \mathbb{R}^n$ is fixed, then there exists a unique curve c satisfying the properties of part* (a) *and such that $c(0) = x_0$ and its Frenet frame at the origin is given by $e_1(0) = (1, 0, \cdots, 0), \cdots, e_n(0) = (0, 0, \cdots, 1)$.*

Proof. The proof is broken into six steps, in order to provide a more clear presentation. Throughout the proof, the elements of the space $H^1(I; \mathbb{R}^n)$, which are classes of functions with respect to the equality almost everywhere, are identified with their continuous representative (as allowed by the Sobolev imbedding theorem).

(i) We show that there exists a unique solution $(e_1, \cdots, e_n) \in H^1(I; \mathbb{R}^n) \times \cdots \times H^1(I; \mathbb{R}^n)$ to the Cauchy problem:

$$e_i'(t) = -F_{i-1}(t) e_{i-1}(t) + F_i(t) e_{i+1}(t) \qquad \text{a.e. in } \ I, \ i \in \{1, \cdots, n\}, \tag{3.3}$$

$$e_1(0) = (1, 0, \cdots, 0), \cdots, e_n(0) = (0, 0, \cdots, 1), \tag{3.4}$$

where $F_0 = F_n := 0, \ e_0 = e_{n+1} := 0$.

We rewrite the system of ordinary differential equations as a matrix equation, viz.,

$$\begin{pmatrix} e_1^1 & \cdots & e_1^n \\ e_2^1 & \cdots & e_2^n \\ \vdots & & \\ e_n^1 & \cdots & e_n^n \end{pmatrix}' = \begin{pmatrix} 0 & F_1 & \cdots & 0 \\ -F_1 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & -F_{n-1} & 0 \end{pmatrix} \begin{pmatrix} e_1^1 & \cdots & e_1^n \\ e_2^1 & \cdots & e_2^n \\ \vdots & & \\ e_n^1 & \cdots & e_n^n \end{pmatrix}.$$

The Cauchy problem given by the relations (3.3) and (3.4) can be written as a linear system of ordinary differential equations of the form

$$Y' = AY \qquad \text{a.e. in } \ I,$$
$$Y(0) = I_n.$$

By assumption, the mapping $A\colon I \to \mathbb{M}^{n \times n}$ belongs to the space $L^2(I; \mathbb{M}^{n\times n})$, since the functions $F_1, \cdots, F_{n-1}$ are at least in $L^2(I; \mathbb{R})$. Then Lemma 2.1 applied to the above system shows that it possesses a unique solution $Y \in H^1(I; \mathbb{M}^{n \times n})$.

Thus, we have obtained the existence of a unique family $\{e_1, \cdots, e_n\}$ that satisfies the Cauchy problem given by (3.3) and (3.4).

(ii) We show that this family is orthonormal at all $t \in I$.

Let $\alpha_{ij}(t) := \langle e_i(t), e_j(t) \rangle$ for all $t \in I$ and for all $i, j \in \{1, \cdots, n\}$. Then the system (3.3) shows that

$$\begin{aligned} \alpha_{ij}'(t) &= \langle e_i(t)', e_j(t) \rangle + \langle e_i(t), e_j'(t) \rangle \\ &= \langle -F_{i-1}(t) e_{i-1}(t) + F_i(t) e_{i+1}(t), e_j(t) \rangle + \langle e_i(t), -F_{j-1}(t) e_{j-1}(t) + F_j(t) e_{j+1}(t) \rangle \end{aligned}$$

with the convention made in step (i) that $F_0 = F_n = 0$ and $e_0 = e_{n+1} = 0$. This implies on the one hand that the functions $\alpha_{ij}\colon I \to \mathbb{R}$ satisfy the following system

$$\alpha_{ij}'(t) = -F_{i-1}(t)\alpha_{i-1,j}(t) + F_i(t)\alpha_{i+1,j}(t) - F_{j-1}(t)\alpha_{i,j-1}(t) + F_j(t)\alpha_{i,j+1}(t),$$

$$\alpha_{ij}(0) = \delta_{ij}.$$

On the other hand, one can see that the functions $\beta\colon I \to \mathbb{R}$, $\forall\, i,j \in \{1,\cdots,n\}$, defined by $\beta_{ij}(t) = \delta_{ij}$ for all $t \in I$ satisfy the same system (to this end, one can distinguish the following three possible situations: $i-1=j$ or $i+1=j$ or $i-1 \neq j$ and $i+1 \neq j$). Consequently, the uniqueness of the solution to the above system implies that $\alpha_{ij}(t) = \delta_{ij}$ for all $t \in I$ and for all $i,j \in \{1,\cdots,n\}$. In other words, $\langle e_i(t), e_j(t)\rangle = \delta_{ij}$ for all $t \in I$ and for all $i,j \in \{1,\cdots,n\}$. Hence the family $\{e_1(t),\cdots,e_n(t)\}$ is orthonormal for all $t \in I$.

(iii) We show a regularity result for the solution $\{e_1,\cdots,e_n\}$ to the system (3.3).

Since for all $m > \frac{1}{2}$, $H^m(I;\mathbb{R})$ is an algebra, it follows that the product (fe) between $f \in H^m(I;\mathbb{R})$ and $e \in H^m(I;\mathbb{R}^n)$ belongs to $H^m(I;\mathbb{R}^n)$. We shall use this fact in the proof below.

We first infer from (3.3) that

$$e'_m = -F_{m-1}e_{m-1} + F_m e_{m+1} \qquad \text{for } \ m = 1,2,\cdots,n-2.$$

Since $e_{m-1}, e_{m+1} \in H^1(I;\mathbb{R}^n)$ and $F_{m-1}, F_m \in H^1(I;\mathbb{R})$, we obtain that $e_m \in H^2(I;\mathbb{R}^n)$ for all $m \in \{1,2,\cdots,n-2\}$.

Next, we again infer from (3.3) that

$$e'_m = -F_{m-1}e_{m-1} + F_m e_{m+1} \qquad \text{for } \ m = 1,2,\cdots,n-3.$$

Since $e_{m-1}, e_{m+1} \in H^2(I;\mathbb{R}^n)$ and F_{m-1} and F_m are at least in $H^2(I;\mathbb{R})$, we deduce that $e_{n-3} \in H^3(I;\mathbb{R})$ for all $m \in \{1,2,\cdots,n-3\}$.

We continue the same argument, using each time relation (3.3) for $m = 1,2,\cdots,n-k$ and $k = 2,3,\cdots,n-1$. In this fashion, we eventually obtain that

$$e'_1 = F_1 e_2.$$

Noting that $F_1 \in H^{n-2}(I;\mathbb{R})$ and $e_2 \in H^{n-2}(I;\mathbb{R}^n)$, we finally deduce that $e_1 \in H^{n-1}(I;\mathbb{R}^n)$.

In conclusion, we have shown that $e_i \in H^{n-i}(I;\mathbb{R}^n)$ and $e_n \in H^1(I;\mathbb{R}^n)$, where $i = 1,2,\cdots,n-1$.

(iv) We establish the existence of a curve satisfying the part (a) of the theorem.

Define the function $c\colon [0,T] \to \mathbb{R}^n$ by $c(t) := \int_0^t e_1(s)ds + x_0$, $0 \le t \le T$. This integral is well defined since we know from step (iii) that $e_1 \in H^{n-1}(I;\mathbb{R}^n)$, which also implies that $c \in H^n(I;\mathbb{R}^n)$.

This allows to compute the successive derivatives, by proceeding recursively. For $k = 1$ we have

$$c^{(1)} = e_1.$$

Assume now that for an arbitrary $k \le n-1$, the k-th derivative is given by

$$c^{(k)} = \sum_{i=1}^{k} a_i^k e_i, \tag{3.5}$$

where $a_i^k \in H^{n-k}(I;\mathbb{R}^n)$. We deduce that

$$c^{(k+1)} = \sum_{i=1}^{k} (a_i^k)' e_i + \sum_{i=1}^{k} a_i^k(-F_{i-1}e_{i-1} + F_i e_{i+1}) = \sum_{i=1}^{k+1} ((a_i^k)' - a_{i+1}^k F_i + a_{i-1}^k F_{i-1}) e_i$$

with the convention that $a_0^k = 0$ and $a_k^i = 0$ for all $i > k$. Hence

$$c^{(k+1)} = \sum_{i=1}^{k+1} a_i^{k+1} e_i,$$

where the functions $a_i^{k+1} := (a_i^k)' - a_{i+1}^k F_i + a_{i-1}^k F_{i-1}$ belong to the space $H^{n-k-1}(I;\mathbb{R}^n)$ for all $i \in \{1, \cdots, k+1\}$. Consequently, relation (3.5) holds for all $k \in \{1, \cdots, n-1\}$.

From these relations, we first infer that $|c'(t)| = |e_1(t)| = 1$ for all $t \in I$. Then, if $A^k(t)$ denotes the lower triangular matrix in $\mathbb{M}^{n\times n}$ whose entries are a_i^k if $i \le k$, these relations can be written as the following matrix equation

$$\begin{pmatrix} c^{(1)}(t) \\ c^{(2)}(t) \\ \vdots \\ c^{(k)}(t) \end{pmatrix} = A^k(t) \begin{pmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_k(t) \end{pmatrix}.$$

Note that the matrices $A^k(t)$, $k \le n-1$, have only positive terms on the principal diagonal, viz., 1, $F_1(t) > 0$, $F_1(t)F_2(t) > 0, \cdots$, $F_1(t)F_2(t)\cdots F_{k-1}(t) > 0$ for all $t \in I$. This shows that these matrices are invertible and their determinant is strictly positive. Then, since the matrix $A^{n-1}(t)$ is invertible, the vectors $\{c^{(1)}(t), \cdots, c^{(n-1)}(t)\}$ are linearly independent for all $t \in I$, so that the curve c is regular.

We now show that the orthonormal family of vectors $\{e_1(t), \cdots, e_n(t)\}$ constitutes the Frenet frame of the curve c at $t \in I$. First, since the matrices $A^k(t)$, $k \le n-1$ are invertible and with strictly positive determinant, it follows that

$$\mathrm{sp}\{e_1(t), \cdots, e_k(t)\} = \mathrm{sp}\{c^{(1)}(t), \cdots, c^{(k)}(t)\}, \qquad \forall\, k \in \{1, \cdots, n-1\},\ \forall\, t \in I \tag{3.6}$$

and the two bases have the same orientation.

Let $\Delta(t)$ be the determinant of the $n \times n$ matrix whose k-th row is the vector $e_k(t)$, $1 \le k \le n$. We notice that $\Delta\colon I \to \mathbb{R}$ is a continuous function, since $Y \in H^1(I;\mathbb{M}^{n\times n}) \subset \mathcal{C}^0(\bar{I};\mathbb{M}^{n\times n})$. Then, thanks to step (ii), the family $\{e_1(t), \cdots, e_n(t)\}$ is orthonormal for all $t \in I$, hence in particular these vectors form a linearly independent system for each t. This implies on the one hand that $\Delta(t) \ne 0$ for all $t \in I$. On the other hand, by using relation (3.4), we obtain that $\Delta(0) = 1$. Consequently, $\Delta(t) > 0$ for all $t \in I$, which means that the basis $\{e_1(t), \cdots, e_n(t)\}$ is positively oriented for all $t \in I$.

From all these relations, we conclude that $\{e_1(t), \cdots, e_n(t)\}$ is the Frenet frame of the curve c. Consequently, its curvatures are given by (see relation (3.2))

$$k_i(t) = \frac{\langle e_{i+1}(t), e_i'(t)\rangle}{|c'(t)|} = \langle e_{i+1}(t), -F_{i-1}(t)e_{i-1}(t) + F_i(t)e_{i+1}(t)\rangle = F_i(t),$$

where we used relation (3.3) and step (ii). This establishes part (a) of Theorem 3.1.

(v) We prove part (c) of Theorem 3.1.

Let c and $\tilde{c}$ be two regular curves of class H^n over I, parametrized by their arc length (i.e., $|c'(t)| = |\tilde{c}'(t)| = 1$ for all $t \in I$), such that $k_i = \tilde{k}_i$ for all $i \in \{1, \cdots, n-1\}$. The Frenet equations for the curves c and $\tilde{c}$ are respectively given by

$$e_i' = \sum_{j=1}^{n} a_{ij} e_j, \qquad \text{a.e. in } I,\ \forall\, i \in \{1, \cdots, n\},$$

$$\tilde{e}_i' = \sum_{j=1}^{n} \tilde{a}_{ij} \tilde{e}_j, \qquad \text{a.e. in } I,\ \forall\, i \in \{1, \cdots, n\}.$$

Since $k_i = \tilde{k}_i$, formula (3.1) shows that $a_{ij} = \tilde{a}_{ij}$ a.e. in I and for all $i, j \in \{1, \cdots, n\}$. Noting that $e_i(0) = \tilde{e}_i(0)$ for all $i \in \{1, \cdots, n\}$ (thanks to relation (3.4)), Lemma 2.1 implies that $e_i = \tilde{e}_i$ for all $i \in \{1, \cdots, n\}$. In particular, $e_1 = \tilde{e}_1$, hence $c' = \tilde{c}'$. Since $c(0) = \tilde{c}(0)$, we finally obtain that $c = \tilde{c}$ in I.

(vi) We establish part (b) of Theorem 3.1.

Let c and $\tilde{c}$ be two regular curves parametrized by their arc length, such that $k_i = \tilde{k}_i$ for all $i \in \{1, \cdots, n-1\}$. Let $\{e_1(0), \cdots, e_n(0)\}$ be the Frenet frame of c at $c(0)$ and let $\{\tilde{e}_1(0), \cdots, \tilde{e}_n(0)\}$ be the Frenet frame of the curve $\tilde{c}$ at $\tilde{c}(0)$. Clearly, there exist a vector $a \in \mathbb{R}^n$ and a matrix $Q \in \mathbb{O}^n_+$, such that

$$\tilde{c}(0) = a + Qc(0), \tag{3.7}$$

$$\tilde{e}_i(0) = Qe_i(0) \qquad \text{for all } \ i \in \{1, \cdots, n\}. \tag{3.8}$$

The Frenet equations for the curves c and $\tilde{c}$ respectively read

$$e_i' = \sum_{j=1}^{n} a_{ij} e_j, \qquad \forall\, i \in \{1, \cdots, n\} \tag{3.9}$$

and

$$\tilde{e}_i' = \sum_{j=1}^{n} \tilde{a}_{ij} \tilde{e}_j, \qquad \forall\, i \in \{1, \cdots, n\}. \tag{3.10}$$

Since

$$a_{ij} = \begin{cases} -k_i, & \text{if } \ i = j-1, \\ k_i, & \text{if } \ i = j+1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\tilde{a}_{ij} = \begin{cases} -\tilde{k}_i, & \text{if } \ i = j-1, \\ \tilde{k}_i, & \text{if } \ i = j+1, \\ 0, & \text{otherwise,} \end{cases}$$

we deduce that $a_{ij} = \tilde{a}_{ij}$. Consequently

$$Qe_i' = \sum_{j=1}^{n} a_{ij} Q e_j \Rightarrow (Qe_i)' = \sum_{j=1}^{n} a_{ij} (Qe_j).$$

This last relation, combined with the relations (3.8) and (3.10) show that $(\tilde{e}_i)$ and (Qe_i) satisfy the same Cauchy problem. Then the uniqueness result of Lemma 2.1 implies that

$$Qe_i = \tilde{e}_i, \qquad \forall\, i \in \{1, \cdots, n\}.$$

In particular, the first relation (i.e., corresponding to $i = 1$ in the above relation) shows that

$$(Qc)' = Qc' = Qe_1 = \tilde{e}_1 = (\tilde{c})'.$$

Therefore, there exists a vector $V \in \mathbb{R}^n$ such that $Qc(t) = \tilde{c}(t) + V$ for all $t \in I$. Then relation (3.7) shows that $V = -a$, so that $\tilde{c}(t) = Qc(t) + a$ for all $t \in I$. This means that $\tilde{c} = \varphi \circ c$, where $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $\varphi(x) = a + Qx$ is a rigid motion.

We now restate the above result in the special, but most commonly encountered in practice, case of dimension 3.

Corollary 3.1. *Let $k \in H^1(I;\mathbb{R}_+^*)$ and $\tau \in L^2(I;\mathbb{R})$.*

(a) *There exists a curve $c \in H^3(I;\mathbb{R}^3)$, unique up to rigid motions in $\mathbb{R}^3$, parametrized by its arc length, such that k and τ are its curvature and torsion functions.*

(b) *If $x_0 \in \mathbb{R}^3$ is fixed, then there exists a unique curve c satisfying the properties of part* (a) *and such that $c(0) = x_0$ and its Frenet frame at the origin is given by $e_1(0) = (1,0,0)$, $e_2(0) = (0,1,0)$, $e_3(0) = (0,0,1)$.*

We also can use Theorem 3.1 to obtain the analogous statement for curves of class $\mathcal{C}^n$. More specifically, the following result holds:

Corollary 3.2. *Let $(F_1,\cdots,F_{n-1}) \in \mathcal{C}^{n-2}(I;\mathbb{R}) \times \cdots \times \mathcal{C}^1(I;\mathbb{R}) \times \mathcal{C}^0(I;\mathbb{R})$ be such that $F_1(t) > 0,\cdots,F_{n-2}(t) > 0$ for all $t \in I$. Then*

(a) *There exists a regular curve $c \in \mathcal{C}^n(I;\mathbb{R}^n)$ such that $|c'(t)| = 1$ for all $t \in I$ and its curvature functions are $F_1,\cdots,F_{n-1}$, i.e., $k_i(t) = F_i(t)$ for all $i \in \{1,\cdots,n-1\}$ and $t \in I$.*

(b) *If c and $\tilde{c}$ are two curves satisfying the conditions of part* (a), *then there exists a rigid motion $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\tilde{c} = \varphi \circ c$.*

(c) *If $x_0 \in \mathbb{R}^n$ is fixed, then there exists a unique curve c satisfying the conditions of part* (a) *and such that $c(0) = x_0$ and its Frenet frame at the origin is given by $e_1(0) = (1,0,\cdots,0),\cdots,e_n(0) = (0,0,\cdots,1)$.*

Sketch of Proof. In order to prove this corollary, we can use two different approaches: we can either carry out the same computations as in the proof of Theorem 3.1 and use a classical result of existence and uniqueness for ordinary differential equations (instead of Lemma 2.1), or we can derive these results from Theorem 3.1, by using in particular the Sobolev imbedding $H^m(I;\mathbb{R}^n) \subset \mathcal{C}^{m-1}(I;\mathbb{R}^n)$. For this second approach, we also need some further analysis which makes the proof rather lengthy. By contrast, the first approach leads to the result in a simpler way.

§ 4. Regularity of a Curve as a Mapping of Its Curvatures

In order to simplify the presentation, we introduce the following notations:

$$\begin{aligned}
\mathcal{H}(I;\mathbb{R}) &:= \prod_{k=0}^{n-2} H^{n-k-2}(I;\mathbb{R}),\\
\mathcal{H}(I;\mathbb{R})_> &:= \{(F_1,\cdots,F_{n-1}) \in \mathcal{H}(I;\mathbb{R});\ F_i(t) > 0,\ \forall\, t \in I,\ \forall\, i \in \{1,\cdots,n-2\}\},\\
\mathbf{H}(I;\mathbb{R}^n) &:= \Big(\prod_{k=1}^{n-1} H^{n-k}(I;\mathbb{R}^n)\Big) \times H^1(I;\mathbb{R}^n).
\end{aligned}$$

The set $\mathcal{H}(I;\mathbb{R})_>$ is open in the Hilbert space $\mathcal{H}(I;\mathbb{R})$, endowed with the inner product

$$\langle (F_1,\cdots,F_{n-1}),(G_1,\cdots,G_{n-1})\rangle_{\mathcal{H}(I;\mathbb{R})} := \sum_{k=0}^{n-2} \langle F_k, G_k\rangle_{H^{n-k-2}(I;\mathbb{R})}.$$

The space $\mathbf{H}(I;\mathbb{R}^n)$, endowed with the inner product

$$\langle (e_1,\cdots,e_n),(f_1,\cdots,f_n)\rangle_{\mathbf{H}(I;\mathbb{R}^n)} := \sum_{k=1}^{n-1} \langle e_k, f_k\rangle_{H^{n-k}(I;\mathbb{R}^n)} + \langle e_n, f_n\rangle_{H^1(I;\mathbb{R}^n)},$$

is a Hilbert space.

In the previous section, under some appropriate assumptions, we have proved the existence and uniqueness of a curve c with prescribed curvature functions. More specifically, Theorem 3.1 asserts that with each $(n-1)$-tuple of functions $(F_1, \cdots, F_{n-1}) \in \mathcal{H}(I;\mathbb{R})_>$, one can associate a unique curve $c \in H^n(I;\mathbb{R}^n)$ parametrized by its arc length, satisfying some ad hoc "initial" conditions, and whose curvatures are the given functions $(F_1, \cdots, F_{n-1})$. In this way, we have constructed a mapping

$$\mathcal{F}\colon (F_1, \cdots, F_{n-1}) \in \mathcal{H}(I;\mathbb{R})_> \to c \in H^n(I;\mathbb{R}^n).$$

The aim of this section is to study the regularity properties of this mapping. Our main result is the following

Theorem 4.1. *Define the mapping*

$$\mathcal{F}\colon (F_1, \cdots, F_{n-1}) \in \mathcal{H}(I;\mathbb{R})_> \to c \in H^n(I;\mathbb{R}^n),$$

where the curve c is defined in part (c) *of Theorem* 3.1*. Then the mapping $\mathcal{F}$ is of class $\mathcal{C}^\infty$.*

Proof. For clarity, we break the proof in three steps: in step (i) we construct a function f (related to $\mathcal{F}$) and prove that it is of class $\mathcal{C}^\infty$, in step (ii) we apply the implicit function theorem 2.1 to this function, and in step (iii) we conclude the proof.

(i) *Let* $e_1^0 = (1, 0, \cdots, 0)$, $e_2^0 = (0, 1, \cdots, 0), \cdots, e_n^0 = (0, 0, \cdots, 1)$. *Define the function*

$$f\colon \mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n) \to \mathbf{H}(I;\mathbb{R}^n)$$

by

$$f((F_1, \cdots, F_{n-1}), (e_1, \cdots, e_n)) = (w^1, \cdots, w^n),$$

where, for all $t \in I$,

$$\begin{aligned} w^1(t) &:= -e_1(t) + e_1^0 + \int_0^t F_1(s)e_2(s)ds, \\ w^2(t) &:= -e_2(t) + e_2^0 + \int_0^t (F_2(s)e_3(s) - F_1(s)e_1(s))ds, \\ &\vdots \\ w^n(t) &:= -e_n(t) + e_n^0 + \int_0^t (-F_{n-1}(s)e_{n-1}(s))ds. \end{aligned} \tag{4.1}$$

Then the function f is well defined and of class $\mathcal{C}^\infty$.

To see this, it suffices to prove that each component of f is well defined and of class $\mathcal{C}^\infty$. More specifically, we have to show that, for each $k \in \{1, 2, \cdots, n\}$, the function defined by

$$\begin{aligned} &f^k\colon \mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n) \to H^{n-k}(I;\mathbb{R}^n), \\ &f^k((F_1, \cdots, F_{n-1}), (e_1, \cdots, e_n)) = w^k, \end{aligned}$$

where

$$w^k(t) = -e_k(t) + e_k^0 + \int_0^t (F_k(s)e_{k+1}(s) - F_{k-1}(s)e_{k-1}(s))ds \qquad \text{for all} \quad t \in I$$

is well defined and of class $\mathcal{C}^\infty$. Note that we use the convention that $F_0 = 0$ and $e_0 = e_{n+1} = 0$.

Since $F_k \in H^{n-k-1}(I;\mathbb{R})$ and $e_{k+1} \in H^{n-k-1}(I;\mathbb{R}^n)$, we deduce that

$$F_k e_{k+1} \in H^{n-k-1}(I;\mathbb{R}^n).$$

In the same manner, we infer from the relations $F_{k-1} \in H^{n-k}(I;\mathbb{R})$ and $e_{k-1} \in H^{n-k+1}(I;\mathbb{R}^n)$ that

$$F_{k-1}e_{k-1} \in H^{n-k}(I;\mathbb{R}^n).$$

Consequently, the last two relations imply that

$$F_k e_{k+1} - F_{k-1}e_{k-1} \in H^{n-k-1}(I;\mathbb{R}^n).$$

We infer from Lemma 2.3 and from the relations $e_k \in H^{n-k}(I;\mathbb{R}^n)$ that $w^k \in H^{n-k}(I;\mathbb{R}^n)$. This shows that the functions f^k are well defined for all $k \in \{1, \cdots, n\}$.

We next show that the mapping f^k is of class $\mathcal{C}^\infty$. We start by writing f^k as a sum of four terms, viz., $f^k = f_1^k + f_2^k + f_3^k + f_4^k$, where

$$\begin{aligned}
&f_1^k((F_1, \cdots, F_{n-1}), (e_1, \cdots, e_n)) = -e_k,\\
&f_2^k((F_1, \cdots, F_{n-1}), (e_1, \cdots, e_n)) = e_k^0,\\
&f_3^k((F_1, \cdots, F_{n-1}), (e_1, \cdots, e_n)) = \Big\{t \mapsto \int_0^t F_k(s)e_{k+1}(s)ds\Big\},\\
&f_4^k((F_1, \cdots, F_{n-1}), (e_1, \cdots, e_n)) = \Big\{t \mapsto -\int_0^t F_{k-1}(s)e_{k-1}(s)ds\Big\}.
\end{aligned}$$

The mappings f_k^1 and f_k^2 are of class $\mathcal{C}^\infty$ since the first one is a projection and the second one is constant.

In order to prove that f_3^k is of class $\mathcal{C}^\infty$, note that the projection

$$\begin{aligned}
&\mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n) \to H^{n-k-1}(I;\mathbb{R}) \times H^{n-k-1}(I;\mathbb{R}^n),\\
&((F_1, \cdots, F_{n-1}), (e_1, \cdots, e_n)) \mapsto (F_k, e_{k+1})
\end{aligned}$$

is of class $\mathcal{C}^\infty$. Then, applying succesively Lemma 2.2 and Lemma 2.3, one can show that the mapping

$$H^{n-k-1}(I;\mathbb{R}) \times H^{n-k-1}(I;\mathbb{R}^n) \to H^{n-k-1}(I;\mathbb{R}^n) \to H^{n-k}(I;\mathbb{R}^n),$$

defined by

$$(F_k, e_{k+1}) \mapsto F_k e_{k+1} \mapsto \Big\{t \mapsto \int_0^t F_k(s)e_{k+1}(s)ds\Big\},$$

is of class $\mathcal{C}^\infty$. The mapping f_3^k, being the composition of two mappings of class $\mathcal{C}^\infty$, is thus also of class $\mathcal{C}^\infty$.

The same argument can be used to show that the function f_4^k is of class $\mathcal{C}^\infty$. We write f_4^k as a composite mapping made of a projection and a mapping that is of class $\mathcal{C}^\infty$, as shown by applying successively Lemmas 2.2 and 2.3. Hence the mapping f_4^k is also of class $\mathcal{C}^\infty$.

The mapping f^k being the sum of four applications of class $\mathcal{C}^\infty$, is of class $\mathcal{C}^\infty$ too. Letting k vary in the set $\{1, \cdots, n\}$ shows that the mapping f is of class $\mathcal{C}^\infty$, as claimed.

(ii) The implicit function theorem can be applied to the function f defined in step (i).

The functional framework is that presented in Theorem 2.1. Let $X_1 = \mathcal{H}(I;\mathbb{R})_>$, let $X_2 = \mathbf{H}(I;\mathbb{R}^n)$, and let

$$\tilde{p} := ((\widetilde{F}_1, \cdots, \widetilde{F}_{n-1}), (\tilde{e}_1, \cdots, \tilde{e}_n)) \in \mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n)$$

such that $f(\tilde{p}) = 0$. Note that such an element $\tilde{p}$ always exists, as showed in steps (i) and (iii) of the proof of Theorem 3.1. More specifically, it was shown there that for any $(\widetilde{F}_1, \cdots, \widetilde{F}_{n-1}) \in \mathcal{H}(I;\mathbb{R})_>$, there exists a unique n-tuple $(\tilde{e}_1, \cdots, \tilde{e}_n) \in \mathbf{H}(I;\mathbb{R}^n)$ such that $f((\widetilde{F}_1, \cdots, \widetilde{F}_{n-1}), (\tilde{e}_1, \cdots, \tilde{e}_n)) = 0$.

The partial derivatives of the function f are denoted by

$$f_{F_i} := \frac{\partial f}{\partial F_i} \colon \mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n) \to \mathcal{L}(H^{n-i-1}(I;\mathbb{R}), \mathbf{H}(I;\mathbb{R}^n))$$

for $i = 1, \cdots, n-1$, and

$$f_{e_j} := \frac{\partial f}{\partial e_j} \colon \mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n) \to \mathcal{L}(H^{n-j}(I;\mathbb{R}^n), \mathbf{H}(I;\mathbb{R}^n))$$

for $j = 1, \cdots, n$.

The gradient matrix of the function f is denoted by

$$Df = \begin{pmatrix} f^1_{F_1} & \cdots & f^1_{F_{n-1}} & f^1_{e_1} & \cdots & f^1_{e_n} \\ \vdots & & & & & \\ f^n_{F_1} & \cdots & f^n_{F_{n-1}} & f^n_{e_1} & \cdots & f^n_{e_n} \end{pmatrix}_{n \times (2n-1)},$$

and the derivative

$$D_{(e_1,\cdots,e_n)} f \colon \mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n) \to \mathcal{L}(\mathcal{H}(I;\mathbb{R})_>, \mathbf{H}(I;\mathbb{R}^n))$$

can be written by using matrix notation under the form

$$D_{(e_1,\cdots,e_n)} f = \begin{pmatrix} f^1_{e_1} & \cdots & f^1_{e_n} \\ \vdots & & \\ f^n_{e_1} & \cdots & f^n_{e_n} \end{pmatrix}_{n \times n}.$$

We have already seen that the mapping f is of class $\mathcal{C}^\infty$. In order to apply the implicit function theorem (see Theorem 2.1), we have to prove that $D_{(e_1,\cdots,e_n)} f(\tilde{p})$ is an isomorphism between the spaces $\mathbf{H}(I;\mathbb{R}^n)$ and $\mathbf{H}(I;\mathbb{R}^n)$.

First, we claim that this mapping is one-to-one, which means that for any $(w_1, \cdots, w_n) \in \mathbf{H}(I;\mathbb{R}^n)$, there exists a unique $(v_1, \cdots, v_n) \in \mathbf{H}(I;\mathbb{R}^n)$ such that

$$\begin{pmatrix} f^1_{e_1}(\tilde{p}) & \cdots & f^1_{e_n}(\tilde{p}) \\ \vdots & & \\ f^n_{e_1}(\tilde{p}) & \cdots & f^n_{e_n}(\tilde{p}) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}. \tag{4.2}$$

Equivalently, this can be written as

$$\begin{aligned} & f^1_{e_1}(\tilde{p}) v_1 + f^1_{e_2}(\tilde{p}) v_2 + \cdots + f^1_{e_n}(\tilde{p}) v_n = w_1, \\ & \vdots \\ & f^n_{e_1}(\tilde{p}) v_1 + f^n_{e_2}(\tilde{p}) v_2 + \cdots + f^n_{e_n}(\tilde{p}) v_n = w_n, \end{aligned}$$

or, according to the definition of f given in (4.1), as

$$\begin{aligned}
-v_1(t) + \int_0^t \widetilde{F}_1(s)v_2(s)ds &= w_1(t),\\
-\int_0^t \widetilde{F}_1(s)v_1(s)ds - v_2(t) + \int_0^t \widetilde{F}_2(s)v_3(s)ds &= w_2(t),\\
&\vdots\\
-\int_0^t \widetilde{F}_{n-2}(s)v_{n-2}(s)ds - v_{n-1}(t) + \int_0^t \widetilde{F}_{n-1}(s)v_n(s)ds &= w_{n-1}(t),\\
-\int_0^t \widetilde{F}_{n-1}(s)v_{n-1}(s)ds - v_n(t) &= w_n(t)
\end{aligned}$$

for all $t \in I$.

By derivation, this implies that the n-tuple $(v_1, \cdots, v_n)$ satisfies the following system

$$\begin{aligned}
v_1' &= \widetilde{F}_1 v_2 - w_1',\\
v_2' &= \widetilde{F}_2 v_3 - \widetilde{F}_1 v_1 - w_2',\\
&\vdots\\
v_{n-1}' &= \widetilde{F}_{n-1} v_n - \widetilde{F}_{n-2} v_{n-2} - w_{n-1}',\\
v_n' &= -\widetilde{F}_{n-1} v_{n-1} - w_n',
\end{aligned}$$

and the initial conditions

$$\begin{aligned}
v_1(0) &= -w_1(0),\\
v_2(0) &= -w_2(0),\\
&\vdots\\
v_{n-1}(0) &= -w_{n-1}(0),\\
v_n(0) &= -w_n(0).
\end{aligned}$$

Note that this is a system of ordinary differential equations, which in matrix form becomes

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}' = \begin{pmatrix} 0 & \widetilde{F}_1 & \cdots & 0 & 0 \\ -\widetilde{F}_1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & \widetilde{F}_{n-1} \\ 0 & 0 & \cdots & -\widetilde{F}_{n-1} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} + \begin{pmatrix} -w_1' \\ -w_2' \\ \vdots \\ -w_{n-1}' \\ -w_n' \end{pmatrix}$$

with the initial condition

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}(0) = \begin{pmatrix} -w_1 \\ -w_2 \\ \vdots \\ -w_n \end{pmatrix}(0).$$

Thanks to Lemma 2.1, this system has a unique solution in $H^1(I; \mathbb{M}^{n\times n})$. Applying the same method as that used in step (iii) of the proof of Theorem 3.1, and taking into account the fact that $(w_1, \cdots, w_n) \in \mathbf{H}(I;\mathbb{R}^n)$, one can see that $(v_1, \cdots, v_n) \in \mathbf{H}(I;\mathbb{R}^n)$. Thus, we have proved that $D_{(e_1,\cdots,e_n)} f(\tilde{p})$ is one-to-one, hence an isomorphism, between the spaces $\mathbf{H}(I;\mathbb{R}^n)$ and $\mathbf{H}(I;\mathbb{R}^n)$.

We are now in a position to apply the implicit function theorem 2.1 to the function f, which is of class $\mathcal{C}^\infty$. Accordingly, there exist an open subset U of $\mathcal{H}(I;\mathbb{R})_>$ containing $(\widetilde{F}_1, \cdots, \widetilde{F}_{n-1})$, an open subset V of $\mathbf{H}(I;\mathbb{R}^n)$ containing $(\tilde{e}_1, \cdots, \tilde{e}_n)$, and an implicit function $g\colon U \to V$ such that

$$f((F_1, \cdots, F_{n-1}), (e_1, \cdots, e_n)) = 0 \quad \text{and} \quad ((F_1, \cdots, F_{n-1}), (e_1, \cdots, e_n)) \in U \times V$$

is equivalent to

$$(e_1, \cdots, e_n) = g(F_1, \cdots, F_{n-1}) \qquad \text{for all } \ (F_1, \cdots, F_{n-1}) \in U.$$

Moreover, the same theorem shows that the mapping $g\colon U \to V$ is of class $\mathcal{C}^\infty$.

But, in the proof of Theorem 3.1, we have seen that for any $(F_1, \cdots, F_{n-1}) \in \mathcal{H}(I;\mathbb{R})_>$, the equation

$$f((F_1, \cdots, F_{n-1}), (e_1, \cdots, e_n)) = 0$$

has a unique solution $(e_1, \cdots, e_n) \in \mathbf{H}(I;\mathbb{R}^n)$. This shows that the mapping $\bar{g}\colon \mathcal{H}(I;\mathbb{R})_> \to \mathbf{H}(I;\mathbb{R}^n)$ defined by $\bar{g}(F_1, \cdots, F_{n-1}) = (e_1, \cdots, e_n)$, is well defined. Therefore, the uniqueness part of the implicit function theorem 2.1, shows that $\bar{g} = g$ on U, hence that $\bar{g}$ is of class $\mathcal{C}^\infty$ over U. Since the $(n-1)$-tuple $(\widetilde{F}_1, \cdots, \widetilde{F}_{n-1})$ was arbitrarily chosen in $\mathcal{H}(I;\mathbb{R})_>$, we deduce that the mapping $\bar{g}$ is of class $\mathcal{C}^\infty$ over $\mathcal{H}(I;\mathbb{R})_>$.

(iii) We now conclude our proof.

First, the previous step shows that the mapping

$$\bar{g}\colon (F_1, \cdots, F_{n-1}) \in \mathcal{H}(I;\mathbb{R})_> \to (e_1, \cdots, e_n) \in \mathbf{H}(I;\mathbb{R}^n)$$

is of class $\mathcal{C}^\infty$.

Second, the mapping

$$(e_1, \cdots, e_n) \in \mathbf{H}(I;\mathbb{R}^n) \to e_1 \in \mathbf{H}(I;\mathbb{R}^n)$$

is a projection, hence of class $\mathcal{C}^\infty$.

Third, the mapping

$$\phi\colon H^{n-1}(I;\mathbb{R}^n) \to H^n(I;\mathbb{R}^n)$$

defined by

$$e_1 \mapsto \left\{ t \mapsto c(t) = \int_0^t e_1(s)ds + x_0 \right\}$$

is also of class $\mathcal{C}^\infty$ thanks to Lemma 2.3 (a translation by a constant vector x_0 is clearly of class $\mathcal{C}^\infty$).

Since the mapping

$$\mathcal{F}\colon (F_1, \cdots, F_{n-1}) \in \mathcal{H}(I;\mathbb{R})_> \to c \in H^n(I;\mathbb{R}^n)$$

is the composition of the three above mappings, it is also of class $\mathcal{C}^\infty$. The proof is now complete.

In the special case of dimension 3, which is the most encountered in practice, the theorem above leads to the following corollary:

Corollary 4.1. *The mapping*

$$\mathcal{F}\colon (k,\tau) \in H^1(I;\mathbb{R}^*_+) \times L^2(I;\mathbb{R}) \to c \in H^3(I;\mathbb{R}^3),$$

where the curve c is defined in part (b) *of Corollary* 3.1*, is of class $\mathcal{C}^\infty$.*

For curves of class $\mathcal{C}^n$, the following result, similar to that of Theorem 4.1, holds:

Corollary 4.2. *Let the mapping*

$$\mathcal{F}\colon \mathcal{C}^{n-2}(I;\mathbb{R}^*_+) \times \mathcal{C}^{n-3}(I;\mathbb{R}^*_+) \times \cdots \times \mathcal{C}^1(I;\mathbb{R}^*_+) \times \mathcal{C}^0(I;\mathbb{R}) \to \mathcal{C}^n(I;\mathbb{R}^n)$$

be defined by

$$(F_1, \cdots, F_{n-1}) \to c,$$

where the curve c is defined in part (c) *of Corollary* 3.2*. Then $\mathcal{F}$ is of class $\mathcal{C}^\infty$.*

Sketch of Proof. The two methods that we have already mentioned for the proof of Corollary 3.2 can be also used to prove this result. Accordingly, we can either use the classical theory for ordinary differential equations, or we can derive the result from Theorem 4.1 and from the fact that the imbedding $H^m(I;\mathbb{R}^n) \subset \mathcal{C}^{m-1}(I;\mathbb{R}^n)$ is continuous for all $m > \frac{1}{2}$ and linear, hence of class C^∞.

§ 5 . Commentaries

(1) In the statements of Theorems 3.1 and 4.1, we have considered the case of a curve parametrized by its arc length. This restriction is not essential however. To see this, let $\alpha\colon I \to \mathbb{R}^n$ be a given curve, not necessarily parametrized by its arc length, such that $\alpha'(t) \neq 0$ for all $t \in I$. Then, it is always possible to obtain another curve $\beta\colon J \to \mathbb{R}^n$ this time parametrized by its arc length, which has the same image and the same curvature functions as the curve α. Indeed, let

$$s(t) := \int_0^t |\alpha'(\tau)| d\tau \qquad \text{for all } \ t \in I.$$

Since $s'(t) = |\alpha'(t)| \neq 0$, the inverse function theorem shows that there exists an inverse function $s \mapsto t(s)$, defined on $J := s(I)$. It is then easily seen that the curve $\beta := \alpha \circ t\colon J \to \mathbb{R}^n$ satisfies the required properties.

(2) In this paper, we have restricted our attention to curves such that $\{c^{(1)}(t), \cdots, c^{(n-1)}(t)\}$ are linearly independent at each point $t \in I$. In fact, if $c^{(k)}(t)$ is linearly dependent on $\{c'(t), \cdots, c^{(k-1)}(t)\}$ along a whole interval $[a,b] \subset I$, then one can prove that the image of c lies in a $(k-1)$-dimensional subspace of $\mathbb{R}^n$, so that we can establish a result similar to that of Theorems 3.1 and 4.1 on this interval, but in a lower dimension. More difficulties arise in some other cases (for example, if the property above holds only at isolated points or at some sequence of points); for details, see [10, Chapter 1].

(3) Another natural question arises: What happens (for curves immersed in the three-dimensional space, for simplicity) at the points where the curvature vanishes and consequently the torsion is not defined? More specifically, assume that $\alpha : [a,b] \to \mathbb{R}^3$ is a curve whose torsion vanishes everywhere, save at one point $t_0 \in]a,b[$, where the torsion is undefined; then it seems reasonable to say that α has zero torsion everywhere, by extension. However, if we accept this convention, we can see that the hypothesis $k > 0$ of Corollary 3.1 is an essential one, as shown by the following example:

Let

$$\alpha_1(t) = \begin{cases} 0, & \text{if } t = 0, \\ (t, 0, 5e^{-\frac{1}{t^2}}), & \text{if } t \neq 0, \end{cases}$$

$$\alpha_2(t) = \begin{cases} (t, 5e^{-\frac{1}{t^2}}, 0), & \text{if } t < 0, \\ 0, & \text{if } t = 0, \\ (t, 0, 5e^{-\frac{1}{t^2}}), & \text{if } t > 0. \end{cases}$$

Then α_1 and α_2 have the same curvature and the same torsion, but there is no rigid transformation mapping α_1 onto α_2. To see this, note that the curvatures of α_1 and α_2 are also vanishing in $t = 0$, since the function $\varphi\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) := \begin{cases} e^{-\frac{1}{t^2}}, & \text{if } t \neq 0, \\ 0, & \text{if } t = 0 \end{cases}$$

has the property that all its derivatives vanish at 0.

The curve α_1 is a planar curve, hence its torsion vanishes, i.e., $\tau_{\alpha_1}(t) = 0$ for all $t \neq 0$ and, by the convention above, $\tau_{\alpha_1}(t) = 0$ for all $t \in \mathbb{R}$. In the same way, one can see that $\tau_{\alpha_2}(t) = 0$ for all $t \in \mathbb{R}$, by the same convention. Therefore, the curves α_1 and α_2 have the same curvature and the same torsion. On the other hand, any rigid motion would have to be the identity on one portion of $\mathbb{R}^3$ and a rotation on the other one, which is impossible.

Acknowledgements. The author is grateful to P. G. Ciarlet for bringing this problem to her attention and for his interest in her work. The author would also like to thank C. Mardare for interesting and helpful discussions.

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