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HARMONIC FUNCTIONS ON A COMPLETE NONCOMPACT MANIFOLD WITH ASYMPTOTICALLY NONNEGATIVE CURVATURE***

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Abstract

The authors prove the space of harmonic functions with polynomial growth of a fixed rate on a complete noncompact Riemannian manifold with asymptotically nonnegative curvature is finite dimensional.

Keywords Harmonic function, Asymptotically nonnegative curvature, Polynomial growth
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§ 1 . Introduction

Recently Tobias H. Colding and Willian P. Minicozzi II proved a conjecture of Yau [1].

Yau's Conjecture. *For an open manifold with nonnegative Ricci curvature, the space of harmonic functions with polynomial growth of a fixed rate is finite dimensional.*

Now we will prove the conjecture of Yau is true under the condition of asymptotically nonnegative curvature, that is,

Theorem 1.1. *Let M^n be a complete noncompact Riemannian manifold with asymptotically nonnegative curvature, then the space of harmonic functions with polynomial growth of a fixed rate is finite dimensional.*

§ 2 . Preliminary

First, we give some definitions and notations.

Definition 2.1. *We say that the curvature is asymptotically nonnegative if $K_M(x) \geq -\lambda(r(x))$, where $\lambda(\cdot)$ is a nonnegative and nonincreasing function on $[0, +\infty)$ and*

$$\int_0^\infty r\lambda(r)dr < +\infty,$$

$r(x) = \mathrm{dist}(p, x)$ and p is a fixed point in M.

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Definition 2.2. *We set* $I_u(a,r)=\int_{B_a(r)}u^2$, $J_{a,r}(u,v)=\int_{B_a(r)}uv$, *where* a *is a point in* M, $B_a(r)$ *is a geodesic ball of radius* r *centered at* p.

Definition 2.3. *Let*

$$H_d(M)=\{u \text{ is harmonic, and } |u|\le c(r^d+1)\},$$

$$P_d(M)=\{u \mid I_u(a,r)\le K(r^{2d+n_0}+1),\ n_0 \text{ is some constant}\},$$

$$HP_d(M)=\{u \text{ is harmonic}\}\cap P_d(M),\quad \text{where } r(x)=\mathrm{dist}(p,x).$$

The key to this paper is to prove that the following Property 2.1 and Property 2.2 are true on a complete noncompact manifold with asymptotically nonnegative curvature.

Property 2.1. *Let* M *be an* n*-dimensional complete noncompact manifold with asymptotically nonnegative curvature, then there exists* $C_D<\infty$ *such that* $\forall\, x\in M$, $\forall\, r\ge 0$,

$$V_x(2r)\le C_D V_x(r),$$

where $V_x(r)=\mathrm{Vol}(B_x(r))$, $B_x(r)=\{y\in M \mid \mathrm{dist}(x,y)<r\}$.

Property 2.2. *We say* M^n *satisfies a local Neumann-Poicaré inequality if there exists* $C_N<\infty$ *such that* $\forall x\in M$ *and* $r(p,x)\ge r^{m_0}+4r$ $(r>0)$, $f\in W^{1,2}_{\mathrm{loc}}(M)$,

$$\int_{B_x(2r)}(f-A)^2\le C_N\times(2r)^{2m_0}\int_{B_x(2r)}|\nabla f|^2,$$

where $A=\int_{B_x(2r)}f\Big/V_x(2r)$, *some constant* $m_0>1$ *depending on* n.

We will prove them in §3, §4 respectively.

In addition, we will use the following property which has been proved by S. T. Yau [2].

Property 2.3. *If* $\lambda>1$, u *is harmonic on* M^n, $\forall\, x\in M$, *and* $r>0$, *then there exists* $C_F=C_F(\lambda)<\infty$ *such that*

$$r^2\int_{B_x(r)}|\nabla u|^2\le C_F\int_{B_x(\lambda r)}u^2.$$

§ 3 . Proof of Property 2.1

Before the proof of Property 2.1, we shall prove

Lemma 3.1. *Suppose* $f(r)$ *is a* C^2 *function on* $(0,T]$,

$$\frac{d}{dr}(-f(r))-\frac{f^2(r)}{n-1}\ge-\frac{(n-1)k}{r^2},\qquad \lim_{r\to 0^+}f(r)=+\infty.$$

Then

$$0\le f(r)\le\frac{(n-1)(1+\sqrt{1+4k})}{2r},\tag{3.1}$$

where $k>0$, T *can tend to* $+\infty$.

Proof. We use comparison method to discuss the inequality.

Assume

$$b_1(r) = \frac{-1+\sqrt{1+4k}}{2(n-1)k}r, \qquad b_2(r) = \frac{-1-\sqrt{1+4k}}{2(n-1)k}r. \tag{3.2}$$

Then

$$\frac{db_i}{dr} + \frac{(n-1)k}{r^2}b_i^2 - \frac{1}{n-1} = 0, \qquad i = 1, 2 \tag{3.3}$$

and $b_1(0) = b_2(0) = 0$.

$$\dot{b}_1(r) = \frac{1}{n-1} - \frac{(n-1)k}{r^2}b_1^2 = \frac{-1+\sqrt{1+4k}}{2(n-1)k}, \tag{3.4}$$

$$\dot{b}_2(r) = \frac{1}{n-1} - \frac{(n-1)k}{r^2}b_2^2 = \frac{-1-\sqrt{1+4k}}{2(n-1)k}. \tag{3.5}$$

Let $A = \frac{1}{f(r)}$. Then

$$\frac{dA}{dr} + \frac{(n-1)k}{r^2}A^2 - \frac{1}{n-1} \geq 0 \tag{3.6}$$

and $A(0) = 0$.

From the Taylor expansion of A at $r = 0$, we know that

$$A^2 = [\dot{A}(0)r + O(r^2)]^2 = \dot{A}^2(0)r^2 + O(r^3), \tag{3.7}$$

then

$$\dot{A}(0) \geq \frac{1}{n-1} - (n-1)k\frac{A^2}{r^2}\Big|_{r=0} = \frac{1}{n-1} - (n-1)k\dot{A}^2(0), \tag{3.8}$$

i.e., $\dot{A}_1(0) \geq \dot{b}_1(0)$ or $\dot{A}_2(0) \leq \dot{b}_2(0)$.

We will prove $\dot{A}_2(0) \leq \dot{b}_2(0)$ is not valid.

If not, then there exists $\eta_1 \leq T$, such that for $0 \leq r \leq \eta_1$, $A_2(r) \leq b_2(r) < 0$,

$$-\frac{(n-1)(1+\sqrt{1+4k})}{2r} \leq f(r) \leq 0. \tag{3.9}$$

This contradicts the fact that $\lim\limits_{r\to 0^+} f(r) = +\infty$.

We know from $\dot{A}_1(0) \geq \dot{b}_1(0)$ that there exists $\eta_2 \leq T$, such that for $0 \leq r \leq \eta_2$, $A_1(r) \geq b_1(r)$.

If $\exists\, \eta_2 = T$, then we complete the proof; if $\exists\, \eta_2 < T$, we can prove that $\forall\, r \in (0, T]$, $A_1(r) \geq b_1(r)$, if not, $(0, r_1]$ is the maxium connected closed interval, where $r_1 < T$, $A_1(r_1) = b_1(r_1)$ and $\dot{A}_1(r_1) < \dot{b}_1(r_1)$,

$$\dot{b}_1(r_1) > \dot{A}_1(r_1) \geq \frac{1}{n-1} - (n-1)k\frac{A_1^2}{r^2}\Big|_{r=r_1} = \frac{1}{n-1} - (n-1)k\frac{b_1^2}{r^2}\Big|_{r=r_1},$$

this contradicts the definition of b_1. So

$$0 \leq f(r) \leq \frac{(n-1)(1+\sqrt{1+4k})}{2r}.$$

Lemma 3.2. *Suppose M is an n-dimensional complete noncompact Riemannian manifold, whose radial Ricci curvature $\geq -\dfrac{(n-1)k}{r(p,x)^2}$, where $k>0$, p is a fixed point, $r(p,x)$ is the distance between p and x. Then its metric can be written as $ds^2 = dr^2 + \sum\limits_{i,j=1}^{n-1} g_{ij}(r,\theta)$, we have*

$$0 \leq \frac{\partial \log \sqrt{g}}{\partial r} \leq \frac{(n-1)(1+\sqrt{1+4k})}{2r}, \tag{3.10}$$

where $g = \det(g_{ij})$.

Proof. We divide the proof into three steps.

The first step: when x is not in the cut-locus of p, that is, x is differentiable, assume $\gamma(x)$ is a normal geodesic which issues from p, choose an orthonormal frame $\{e_i\}_{i=1}^n$ by parallel transport along the geodesic, $e_n = \nabla r$ and $\nabla_{e_n} e_n = 0$.

Let $\nabla_{e_i} e_n = u_{ij} e_j,\ \nabla_{e_n} e_i = \lambda_{ij} e_j$,

$$\begin{aligned} u_{ij} &= \langle \nabla_{e_i} e_n, e_j\rangle = e_i\langle e_n, e_j\rangle - \langle e_n, \nabla_{e_i} e_j\rangle = -\nabla_{e_i} e_j(r) \\ &= e_i e_j r - \nabla_{e_i} e_j(r) = Hr(e_i, e_j) = u_{ji}, \end{aligned} \tag{3.11}$$

$$\lambda_{ij} = \langle \nabla_{e_n} e_i, e_j\rangle = -\langle e_i, \nabla_{e_n} e_j\rangle = -\lambda_{ij}, \tag{3.12}$$

$$\begin{aligned} -\frac{(n-1)k}{r^2} &\leq \mathrm{Ric}(e_n, e_n) = \sum_{i=1}^{n-1} \langle R(e_i, e_n)e_n, e_i\rangle \\ &= -\sum_{i=1}^{n-1}\langle \nabla_{e_n}\nabla_{e_i} e_n, e_i\rangle - \sum_{i=1}^{n-1}\langle \nabla_{\nabla_{e_i} e_n} e_n, e_i\rangle + \sum_{i=1}^{n-1}\langle \nabla_{\nabla_{e_n} e_i} e_n, e_i\rangle \\ &= \frac{d}{dr}\Big(-\sum_{i=1}^{n-1} u_{ii}\Big) - \sum_{i,j=1}^{n-1} u_{ij}\lambda_{ji} - \sum_{i,j=1}^{n-1} u_{ij}u_{ji} + \sum_{i,j=1}^{n-1}\lambda_{ij}u_{ji} \\ &\leq \frac{d}{dr}\Big(-\sum_{i=1}^{n-1} u_{ii}\Big) - \frac{1}{n-1}\Big(\sum_{i=1}^{n-1} u_{ii}\Big)^2, \end{aligned} \tag{3.13}$$

where $\triangle r = \sum\limits_{i=1}^{n-1} u_{ii}$, and $\lim\limits_{t\to 0^+} \triangle r(t) = +\infty$.

So by Lemma 3.1, at differentiable points, we have

$$0 \leq \triangle r \leq \frac{(n-1)(1+\sqrt{1+4k})}{2r}. \tag{3.14}$$

The second step: when x is a cut-point to p, we will prove (3.14) is still valid.

We let E include all cut-locuses of p, $M = \Omega \cup E$, where Ω is a starlike domain, the Lipschitz function r is differentiable in Ω, so by (3.14), in Ω we have

$$0 \leq \triangle r \leq \frac{(n-1)(1+\sqrt{1+4k})}{2r}. \tag{3.15}$$

We choose $\forall \varphi \in C_0^\infty(M),\ \varphi \geq 0$, since $\mathrm{mess}(E) = 0$, then

$$\int_M r\triangle\varphi = \int_\Omega r\triangle\varphi.$$

We can find a sequence of starlike subdomains Ω_ϵ such that $\Omega_\epsilon \subset\subset \Omega$, $\lim\limits_{\epsilon\to 0}\Omega_\epsilon = \Omega$, and Ω_ϵ is interior contracting from Ω along r direction.

Since Stokes formula is true to Lipschitz function, and $\varphi \in C_0^\infty(M)$, we have

$$\int_M r\triangle\varphi = -\int_M \nabla\varphi\cdot\nabla r = \lim_{\epsilon\to 0}(-1)\int_{\Omega_\epsilon}\nabla\varphi\cdot\nabla r, \tag{3.16}$$

the last equality is because $|\nabla r| = 1$ is almost valid and $\nabla\varphi$ is bounded.

By Green formula, one has

$$-\int_{\Omega_\epsilon}\nabla\varphi\cdot\nabla r = \int_{\Omega_\epsilon}\triangle r\cdot\varphi - \int_{\partial\Omega_\epsilon}\varphi\cdot\frac{\partial r}{\partial v},$$

where v is an exterior normal direction. Because $\varphi \geq 0$ and Ω_ϵ is interior contracting from Ω along r direction, then $\frac{\partial r}{\partial v} > 0$, and

$$-\int_{\Omega_\epsilon}\nabla\varphi\cdot\nabla r \leq \int_{\Omega_\epsilon}\triangle r\cdot\varphi \leq \int_{\Omega_\epsilon}\frac{(n-1)(1+\sqrt{1+4k})}{2r}\varphi, \tag{3.17}$$

then we have

$$\begin{aligned}\int_M r\cdot\triangle\varphi &\leq \lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\frac{(n-1)(1+\sqrt{1+4k})}{2r}\varphi\\ &= \int_\Omega\frac{(n-1)(1+\sqrt{1+4k})}{2r}\varphi = \int_M\frac{(n-1)(1+\sqrt{1+4k})}{2r}\varphi.\end{aligned} \tag{3.18}$$

So in the sense of distribution, $\forall\, x$, that is, $\forall\, r \in [0, +\infty)$, we have

$$0 \leq \triangle r \leq \frac{(n-1)(1+\sqrt{1+4k})}{2r}. \tag{3.19}$$

The last step: the proof of (3.10). Because $\triangle = \frac{\partial^2}{\partial r^2} + \frac{\partial\log\sqrt{g}}{\partial r}\frac{\partial}{\partial r} + \triangle'$, where $\triangle'$ is a differential operator only consisting of $\frac{\partial}{\partial\theta^i}$, θ^i are local coordinates on $S^{n-1}(1)$, for $1 \leq i \leq n-1$, then $\triangle r = \frac{\partial\log\sqrt{g}}{\partial r}$. By (3.19), we have

$$0 \leq \frac{\partial\log\sqrt{g}}{\partial r} \leq \frac{(n-1)(1+\sqrt{1+4k})}{2r}. \tag{3.20}$$

Lemma 3.3. *Let M^n be a complete noncompact manifold. If its radial Ricci curvature $\geq -\dfrac{(n-1)k}{r^2}$, then $\forall\,\alpha,\ 0<\alpha<1$, we have*

$$V_p(r) \leq \Big(\frac{1}{\alpha}\Big)^{m+1}V_p(\alpha r), \tag{3.21}$$

$$V_y\Big(\frac{r}{2}\Big) \leq \Big(\frac{1}{\alpha}\Big)^{m+1}V_y\Big(\frac{\alpha r}{2}\Big), \qquad \forall\, y\in\partial B_p(r), \tag{3.22}$$

$$V_p(r) \leq \mathrm{Vol}(B_p(k_1 r) - B_p((k_1-1)r)), \qquad \forall\, k_1 \geq 2, \tag{3.23}$$

where $k > 0$, $m = \dfrac{(n-1)(1+\sqrt{1+4k})}{2}$.

Proof. Because
$$0 \le \frac{\partial \log \sqrt{g}}{\partial r} \le \frac{(n-1)(1+\sqrt{1+4k})}{2r},$$
let $m = \frac{(n-1)(1+\sqrt{1+4k})}{2}$. Then $\forall\, 0 < \alpha < 1$, we have
$$|\log \sqrt{g(t,\theta)} - \log \sqrt{g(\alpha t,\theta)}| \le \int_{\alpha t}^{t} \Big|\frac{\partial \log \sqrt{g}}{\partial r}\Big| dr \le m \ln \frac{1}{\alpha}.$$

Then
$$\sqrt{g(t,\theta)} \le \Big(\frac{1}{\alpha}\Big)^m \sqrt{g(\alpha t,\theta)}, \tag{3.24}$$
$$\begin{aligned} V_p(r) &= \int_{S^{n-1}} \int_0^r \sqrt{g(t,\theta)} dt d\theta \le \int_{S^{n-1}} \int_0^r \Big(\frac{1}{\alpha}\Big)^m \sqrt{g(\alpha t,\theta)} dt d\theta \\ &= \int_{S^{n-1}} \int_0^{\alpha r} \Big(\frac{1}{\alpha}\Big)^{m+1} \sqrt{g(t,\theta)} dt d\theta = \Big(\frac{1}{\alpha}\Big)^{m+1} V_p(\alpha r), \end{aligned} \tag{3.25}$$
so (3.21) is valid.

$\forall\, y \in \partial B_p(r)$ and $\forall\, x \in B_y(\frac{r}{2})$, $r(p,x) \ge r(y,x)$, so
$$\mathrm{Ric}(M) \ge -\frac{(n-1)k}{r(p,x)^2} \ge -\frac{(n-1)k}{r(y,x)^2},$$
then we can have (3.22) by proving similarly as (3.21).

Because $\frac{\partial \log \sqrt{g}}{\partial r} \ge 0$, then $\sqrt{g(r,\theta)}$ is increasing with r.
$$\begin{aligned} V_p(r) &= \int_{S^{n-1}(1)} \int_0^r \sqrt{g(t,\theta)} dt d\theta \le \int_{S^{n-1}(1)} \sqrt{g(r,\theta)} d\theta \\ &\le \int_{S^{n-1}(1)} \int_{(k_1-1)r}^{k_1 r} \sqrt{g(t,\theta)} dt d\theta = \mathrm{Vol}(B_p(k_1 r) - B_p((k_1-1)r)). \end{aligned} \tag{3.26}$$

Lemma 3.4. *Suppose that M is a complete noncompact manifold with asymptotically nonnegative curvature. Then $\forall\, r > 0,\ 0 < \alpha < 1$, we have*
$$B_p(2r) \backslash B_p(r) \subset \bigcup_{i=1}^{N} B_{x_i}(\alpha r),$$
where $x_i \in B_p(2r)\backslash B_p(r)$, and N is a constant independent of r.

Proof. Before proving this lemma, we quote [3, Theorem 2.2.B]: Let V^n be a compact manifold of diameter D and $\inf K \ge -Q^2$, where K is the curvature of V, $\{B_{x_i}(\epsilon)\},\ i = 1, \cdots, N$, is a minimal covering, where $x_i \in V$. Then $N \le 80^n D^n \epsilon^{-n} \exp(nQD)$.

If $V = B_p(2r)\backslash B_p(r)$, then $D = 4r$. Since the curvature of M is asymptotically nonnegative, there exists $k > 0$, such that the curvature at $x \ge -\frac{k}{r(p,x)^2}$, so $Q = \frac{\sqrt{k}}{r}$, and we let $\epsilon = \alpha r,\ 0 < \alpha < 1$, then
$$N \le 80^n \Big(\frac{4r}{\alpha r}\Big)^n \exp\Big(n \times \frac{\sqrt{k}}{r} \times 4r\Big) = 320^n \alpha^{-n} \exp(4n\sqrt{k}).$$
We complete the proof of this lemma.

Proof of Property 2.1. By Lemma 3.4,

$$B_p\Big(\frac{9}{8}r\Big)\Big\backslash B_p\Big(\frac{7}{8}r\Big) \subset \bigcup_{i=1}^{N} B_{x_i}\Big(\frac{r}{8}\Big),$$

where $x_i \in B_p(\frac{9}{8}r)\backslash B_p(\frac{7}{8}r)$ and N is a constant independent of r, and for each $B_{x_i}(\frac{r}{8})$, there exists $y_i \in \partial B_p(r)$ such that $B_{y_i}(\frac{r}{2}) \supset B_{y_i}(\frac{r}{4}) \supset B_{x_i}(\frac{r}{8})$, so

$$B_p\Big(\frac{9}{8}r\Big)\Big\backslash B_p\Big(\frac{7}{8}r\Big) \subset \bigcup_{i=1}^{N} B_{y_i}\Big(\frac{r}{2}\Big),$$

where N is a constant independent of r.

Because the curvature is asymptotically nonnegative, then there exists $k > 0$, such that the curvature at $x \geq -\frac{k}{r(p,x)^2}$, so Ricci curvature at $x \geq -\frac{(n-1)k}{r(p,x)^2}$, and by Lemma 3.3,

$$V_p(r) \leq C_1 V_p\Big(\frac{r}{4}\Big) \leq C_1 \mathrm{Vol}\Big(B_p\Big(\frac{9}{8}r\Big) - B_p\Big(\frac{7}{8}r\Big)\Big) \leq C_1 \sum_{i=1}^{N} V_{y_i}\Big(\frac{r}{2}\Big) \leq C_2 V_z\Big(\frac{r}{2}\Big)$$

for some $z \in \partial B_p(r)$. And

$$V_z\Big(\frac{r}{2}\Big) \leq C_3 V_z\Big(\frac{r}{4}\Big) \leq C_4 V_{z_1}\Big(\frac{r}{2}\Big), \tag{3.27}$$

where $z_1 \in \partial B_z(\frac{r}{4}) \cap \partial B_p(r)$.

By Lemma 3.4, $\forall\, y \in \partial B_p(r)$, there is a piecewise smooth curve from y to z in $B_p(\frac{9}{8}r)\backslash B_p(\frac{7}{8}r)$ with length not greater than $C_5 r$, where some constant C_5 is independent of $y,\ z,\ r$. So by continuing the way of (3.27), we can find C_6 so that

$$V_z\Big(\frac{r}{2}\Big) \leq C_6 V_y\Big(\frac{r}{2}\Big), \qquad \forall\, y \in \partial B_p(r),$$

that is,

$$V_p(r) \leq C_7 V_y\Big(\frac{r}{2}\Big), \qquad \forall\, y \in \partial B_p(r).$$

Now we divide the proof of $V_x(2r) \leq C_D V_x(r),\ \forall x \in M$, into four cases.

Let $r(p,x) = s$.

(1) When $r \geq 2s$.

By the triangle inequality, we know the geodesic ball $B_x(2r) \subseteq B_p(2r+s)$, then

$$V_x(2r) \leq V_p(2r+s). \tag{3.28}$$

By Lemma 3.3,

$$V_p(2r+s) \leq \Big(\frac{2r+s}{r-s}\Big)^{m+1} V_p(r-s), \tag{3.29}$$

but $\frac{2r+s}{r-s} \leq 8$, and $B_p(r-s) \subseteq B_x(r)$, so

$$V_x(2r) \leq 2^{3m+3} V_x(r). \tag{3.30}$$

(2) When $\frac{s}{2} \leq r \leq 2s$.

$$V_x(2r) \leq V_p(2r+s) \leq \Big(\frac{2r+s}{s}\Big)^{m+1} V_p(s). \tag{3.31}$$

Because $\frac{s}{2} \le r \le 2s$, then $\frac{2r+s}{s} \le 5 < 8$, and by (3.27),

$$V_x(2r) \le 2^{3m+3} V_p(s) \le 2^{3m+3} C_5 V_x\left(\frac{s}{2}\right) \le 2^{3m+3} C_5 V_x(r) \le C_D V_x(r). \tag{3.32}$$

(3) When $\frac{s}{4} \le r \le \frac{s}{2}$.

$$V_x(2r) \le V_p(2r+s) \le \left(\frac{2r+s}{s}\right)^{m+1} V_p(s). \tag{3.33}$$

Because $\frac{s}{4} \le r \le \frac{s}{2}$, then $\frac{2r+s}{s} \le 2$. By (3.27) and (3.22) in Lemma 3.3, we have

$$V_p(s) \le C_5 V_x\left(\frac{s}{2}\right) \le C_5 \left(\frac{s}{2r}\right)^{m+1} V_x(r), \tag{3.34}$$

so

$$V_x(2r) \le 2^{m+1} C_5 2^{m+1} V_x(r) = 2^{2m+2} C_5 V_x(r) \le C_D V_x(r). \tag{3.35}$$

(4) When $r \le \frac{s}{4}$, i.e., $4r \le s$, $\forall\, y \in B_x(2r)$, $r(p,y) \ge r(x,y) = 2r$, so

$$\mathrm{Ric}(M) \ge -\frac{(n-1)k}{r(p,y)^2} \ge -\frac{(n-1)k}{r(x,y)^2}. \tag{3.36}$$

By (3.22) in Lemma 3.3,

$$V_x(2r) \le 2^{m+1} V_x(r) \le C_D V_x(r). \tag{3.37}$$

So Property 2.1 indicates the volumn of M is polynomial growth whose growth degree at most $\frac{\log C_D}{\log 2}$.

§ 4. Proof of Property 2.2

Before the proof of this property, we will quote a theorem (see [4]): Let X^n be a compact Riemannian manifold, whose Ricci curvature is bounded below by $-(n-1)R^2$, where $R \ge 0$, and ρ denotes injectivity radius of X, then $\forall f \in W^{1,2}(X)$, there only exists a constant $c_1(n) > 1$ dependent on n, such that

$$\int_X (f-A)^2 \le (\rho^{1-n}\mathrm{Vol}(X))^2 c_1^{1+R\rho^{1-n}\mathrm{Vol}(X)} \int_X |\nabla f|^2,$$

where $A = \frac{\int_X f}{\mathrm{Vol}(X)}$.

By the assumption of curvature, from Property 2.1, we have

$$V_x(2r) \le C_D V_x(r),$$

i.e.,

$$V_x(r) \le V r^{n_0}, \tag{4.1}$$

where $V = C_D V_x(1)$, $n_0 = \frac{\log C_D}{\log 2}$. By the proof in §3, we know that $n_0 \ge m+1 > n$. Let

$$m_0 = n_0 - n + 1, \tag{4.2}$$

so this constant $m_0 > 1$.

Due to the assumption of curvature, there exists $k > 0$, such that the curvature at $y \geq -\frac{k}{r(p,y)^2}$ and Ricci curvature $\geq -\frac{(n-1)k}{r(p,y)^2}$. When $r(p,x) \geq r^{m_0} + 4r$, and $y \in B_x(2r)$, then $r(p,y) \geq r^{m_0}$, so $\forall\, y \in B_x(2r)$,

$$\mathrm{Ric} \geq -\frac{(n-1)k}{r(p,y)^2} \geq -\frac{(n-1)k}{r^{2m_0}}.$$

If $X = B_x(2r)$, by the theorem in [4], then there exists a constant $c_1(n) > 1$, such that

$$\begin{aligned}\int_{B_x(2r)} (f-A)^2 &\leq [V \times (2r)^{n_0-n+1}]^2 c_1^{1+\frac{\sqrt{k}}{r^{m_0}}\times(2r)^{1-n}\times V\times(2r)^{n_0}} \int_{B_x(2r)} |\nabla f|^2 \\ &\leq C_N \times (2r)^{2m_0} \int_{B_x(2r)} |\nabla f|^2,\end{aligned}$$

where the definitions of $A,\ f$ are the same as the above.

§ 5. Proof of Theorem 1.1

Because the curvature of M is asymptotically nonnegative, for a given point p, we now choose a point $a \in M$. If we have no special expression in the following, we usually use r to denote the distance to p, R the distance to a.

Since $u \in H_d(M)$, there exists c such that $|u| \leq c(r^d + 1)$, of course, there exists c' such that

$$|u| \leq c'(R^d + 1). \tag{5.1}$$

But we still denote it by $u \in H_d(M)$.

By (4.2),

$$V_x(r) \leq V r^{n_0} \leq (r^{n_0} + 1)V, \tag{5.2}$$

so by (5.1) and (5.2),

$$I_u(a,R) \leq {c'}^2 (R^d+1)^2 (R^{n_0}+1)V \leq 4V{c'}^2(R^{2d+n_0}+1),$$

then

$$H_d(M) \subset HP_d(M) \subset P_d(M).$$

We will use the following two propositions to prove the theorem directly.

Proposition 5.1. *Let (Y, d, μ) be a complete metric space with a locally finite positive Borel measure μ. For $a \in Y$, let $X = B_a(R)$, where $r(p,a) \geq R^{m_0} + 4R$, be a metric ball with $\mu(X) = 1$. Suppose that Y satisfies Property* 2.1 *and Property* 2.2. *Given $\beta > 0$, there exist at most $N-1$ orthonormal (on $B_a(R)$) functions in $W_{\beta^2}(B_a(2R))$, where $N = N(\beta^2, C_D, C_N)$, $\beta > R > 2$, the definition of $m_0 = n_0 - n + 1$ is the same as the above.*

$$W_{\beta^2}(B_a(R)) := \Big\{ f \in W^{2,1}(B_a(R)) \Big| \int_{B_a(R)} f^2 + R^2 \int_{B_a(R)} |\nabla f|^2 \leq \beta^2 \Big\},$$

functions $f_1, f_2, \cdots, f_m$ are orthonormal on $B_a(R)$, if

$$\int_{B_a(R)} f_i f_j = \delta_{ij}, \qquad \forall\, 1 \leq i,j \leq m.$$

Proposition 5.2. *Let $u_1, u_2, \cdots, u_{2k} \in HP_d(M)$ be linearly independent. Given $\Omega > 2$, there exist constants $C_H > \Omega^{2(2d+n_0)}$, $l \geq \frac{k}{2}C_H^{-1}$, and functions $v_1, v_2, \cdots, v_l$ in the linear span of u_i such that for $j = 1, 2, \cdots, l$,*

$$\begin{aligned} &I_{v_j}(a, \Omega^2) \leq 2C_H I_{v_j}(a, \Omega) = 2C_H, \\ &J_{a,\Omega}(v_i, v_j) = \delta_{ij}, \end{aligned} \tag{5.3}$$

where a is a point satisfying $r(p, a) \geq \Omega^{m_0} + 4\Omega$, the definitions of m_0, d, n_0 are the same as the above.

The proofs of these propositions are as similar as those of Proposition 2.5 and Proposition 4.16 in [1] respectively.

Likewise, Proposition 5.1, Proposition 5.2 and Property 2.3 assure that the proof of Theorem 1.1 is as similar as that of Theorem 0.7 in [1], except that in this paper we let $X = B_a(\Omega)$, such that $r(p, a)$ is sufficiently large.

References

[1] Colding, T. H. & Minicozzi II, W. P., Harmonic function on manifolds, *Ann. Math.,* **146**(1997), 725–747.

[2] Yau, S. T., Some function theoretic properties of complete Riemannian manifolds and their applications to geometry, *Indiana Univ. J.,* **25**(1976), 659–670.

[3] Gromov, M., Curvature, diameter and Betti numbers, *Comment. Math. Hel.,* **56**(1981), 179–195.

[4] Buser, P., A note on the isoperimetric constant, *Ann. Scient. Ec. Norm. Sup.,* **15**(1982), 213–230.