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α-FUZZY PAIRWISE RETRACT OF L-VALUED PAIRWISE STRATIFICATION SPACES

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Abstract

The notion of a fuzzy retract was introduced by Rodabaugh (1981). The notion of a fuzzy pairwise retract was introduced in 2001. Some weak forms and some strong forms of α-continuous mappings were introduced in 1988 and 1997. The authors extend some of these forms to the L-fuzzy bitopological setting and construct various α-fuzzy pairwise retracts. The concept of weakly induced spaces in the case $L = [0, 1]$ was introduced by Martin (1980). Liu and Luo (1987) generalized this notion to the case that L is an arbitrary F-lattice and introduced the notion of induced L-fts. Several results are obtained, especially, for L-valued pairwise stratification spaces.

Keywords L-fuzzy pairwise continuous mappings, α-Pairwise continuous mappings, α-Fuzzy pairwise retracts, L-valued pairwise stratification spaces
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§ 1. Introduction

Throughout this paper, $(L, \leq, ')$ (for short L) is a fuzzy lattice, i.e., a completely distributive complete lattice with an order-reversing involution $'$ on it, and with a smallest element 0 and a largest element 1 ($0 \neq 1$). An element a of L is called a prime element iff $a \neq 1$ and whenever $b, c \in L$ with $b \wedge c \leqslant a$ then $b \leqslant a$ or $c \leqslant a$, the set of all prime elements of L will be denoted by $\mathrm{pr}(L)$. $a \in L - \{0\}$ is said to be a molecule (see [15]) iff $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The set of all molecules of L is denoted by $M(L)$.

Let X be a non-empty set. L^X denotes the collection of all mappings from X into L. The elements of L^X are called L-fuzzy sets on X. L^X can be made into a fuzzy lattice by inducing the order and involution from $(L, \leq, ')$. For $A \in L^X$ and $a \in L$, we use the notation $A_{(a)} = \{x \in X \mid A(x) \nleqslant a\}$ and $\mathrm{supp}A = \{x \in X \mid A(x) > 0\}$. $\mathrm{supp}A$ is called the support of A. When $\mathrm{supp}A$ is a singleton, A is called an L-fuzzy point on X and denoted by x_a where $x = \mathrm{supp}A$ and $a = A(x)$. We define $M(L^X) = \{x_a \mid x \in X, a \in M(L)\}$. It is easy to check that $M(L^X)$ is just a set of all molecules of L^X. We denote by $\underline{a}_X$ (for short $\underline{a}$) an L-fuzzy set which takes the constant value $a \in L$ on X.

An L-fuzzy topology on X is a subfamily δ of L^X which contains $\underline{0}$ and $\underline{1}$ and is closed under arbitrary suprema and finite infima (see [6]). The pair (L^X, δ) is called an L-fuzzy topological space (or L-fts, for short). The members of δ are called L-fuzzy open sets and the members of δ' are called L-fuzzy closed sets where $\delta' = \{A' \mid A \in \delta\}$.

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Obviously, in the case $L = [0,1]$, L-fuzzy topological space $([0,1]^X, \delta)$ is just the fuzzy topological space in the sense of Chang and is denoted by (X, δ) (see [2]).

$A \in L^X$ is called a crisp subset on X, if there exists an ordinary subset $U \subset X$ such that $A = 1_U : X \to \{0,1\} \subset L$, i.e. if A is a characteristic function of some ordinary subset of X. For a family $\mathcal{A} \subset L^X$ of L-fuzzy sets, denote the family of all the crisp subsets contained in $\mathcal{A}$ by $\mathrm{crs}(\mathcal{A})$, and denote $[\mathcal{A}] = \{A \subset X : 1_A \in \mathrm{crs}(\mathcal{A})\}$. It is clear that for every L-fts (L^X, δ), $(X, [\delta])$ is a topological space and is called the background space of (L^X, δ) (see [9]).

We say that the fuzzy point x_a belongs to a fuzzy set U, i.e., $x_a \in U$ iff $a \leqslant U(x)$, and the set of all fuzzy points in L^X is denoted by $\mathrm{Pt}(L^X)$. A fuzzy point x_a is said to be quasi-coincident with a fuzzy set $U \in L^X$ denoted by $x_a \hat{q} U$, if $a \nleq U'(x)$. For $U, V \in L^X$, U is quasi-coincident with V, denoted by $U\, \hat{q} V$, if there exists $x \in X$ such that $U(x) \nleqslant V'(x)$. If U is not quasi-coincident with V, we denote $U \neg \hat{q} V$ (see [9]).

Let (L^X, δ) be an L-fts, $A \in L^X$, $x_\lambda \in M(L^X)$. x_λ is called an adherent point of A, if for every $U \in Q\,(x_\lambda)$, U quasi-coincides with A, i.e., $U\, \hat{q} A$ (see [9]).

Let (L^X, δ) be an L-fts, $A \subset X$, $\alpha \in \mathrm{pr}(L)$. Then A is called α-closed, iff for each $x \in X - A$, there exists $U \in \delta$ such that $U(x) \nleq \alpha$ and $U \wedge 1_A = \underline{0}$ (see [5]).

An L-fuzzy mapping $f^{\to} : (L^X, \delta) \to (L^Y, \sigma)$, $\alpha \in \mathrm{pr}(L)$ is called α-continuous, (α-c for short), if for each $x \in X$ and each open set V of L^Y with $V(f(x)) \nleqslant \alpha$, there exists an open set U of L^X with $U(x) \nleqslant \alpha$ such that $f^{\to}(U) \leqslant V$ (see [5]).

Let (L^X, δ), (L^Y, σ) be L-fts's, $f^{\to} : (L^X, \delta) \to (L^Y, \sigma)$ an L-fuzzy mapping, $\alpha \in \mathrm{pr}(L)$, $f^{\to}$ is called Δ-continuous, (Δ-c for short), if its L-fuzzy reverse mapping $f^{\leftarrow}$: $(L^Y, \sigma) \to (L^X, \delta)$ maps every α-closed (resp. α-open) in (L^Y, σ) as an α-closed (resp. α-open) one in (L^X, δ) (see [5]).

Let L be a complete lattice. The co-topology on L generated by the subbase $\{\downarrow a : a \in L\}$ is called the lower co-topology of L and we denote it by $\underline{\Omega_*}(L)$. The correspondent topology of $\underline{\Omega_*}(L)$ is called the lower topology of L and we denote it by $\Omega_*(L)$ (Ω_* for short) (see [9]).

Let (X, τ) be an ordinary topological space, L a complete lattice. A mapping $f : X \to L$ is called lower semicontinuous, if f is continuous for the topology Ω_*.

Let (L^X, δ) be an L-fts. δ is called stratified, if for every $a \in L$, $\underline{a} \in \delta$. (L^X, δ) is called stratified, if δ is stratified.

δ is called weakly induced, if every $U \in \delta$ is a lower semicontinuous mapping from the background space $(X, [\delta])$ to L, (L^X, δ) is called weakly induced, if δ is weakly induced.

δ is called induced, if δ is exactly the family of all the lower semicontinuous mappings from the background space $(X, [\delta])$ to L. (L^X, δ) is called induced, if δ is induced (see [10]).

An L-fts (L^X, μ) is called the stratification of (L^X, δ) if μ is generated by $\delta \cup \{\underline{a} : a \in L\}$.

By an L-valued stratification space, we mean a stratified space or a weakly induced space or an induced space.

The following results and definitions are fundamental for the next sections.

Lemma 1.1. (cf. [5]) *If* $\alpha \in \mathrm{pr}(L)$ *and* $U = \bigvee_{j \in J} U_j$, $U(x) \nleq \alpha$, *then* $\exists\ j_\circ \in J$ *such that* $U_{j_\circ}(x) \nleq \alpha$.

Lemma 1.2. (cf. [5]) *If* $\alpha \in \mathrm{pr}(L)$ *and* $U \in L^X$, $V \in L^Y$ *such that* $(U \times V)(x,y) \nleq \alpha$, *then* $U(x) \nleq \alpha$ *and* $V(y) \nleq \alpha$.

Lemma 1.3. (cf. [5]) *Let* W *be fuzzy open of* L*-fuzzy product space* $(L^{X \times Y}, \delta \times \gamma)$ *such that* $W(x,y) \nleq \alpha$. *Then there exist* $U \in \delta, V \in \gamma$ *such that* $U(x) \nleq \alpha$ *and* $V(y) \nleq \alpha$ *where* $\alpha \in \mathrm{pr}(L)$.

Proof. By Lemmas 1.1 and 1.2.

Proposition 1.1. (cf. [9]) *Let (L^X, δ), (L^Y, μ) be L-fts's, $f^{\rightarrow} : (L^X, \delta) \rightarrow (L^Y, \mu)$ an L-fuzzy continuous mapping. Then $f : (X, [\delta]) \rightarrow (Y, [\mu])$ is continuous.*

Lemma 1.4. (cf. [9]) *Let (L^X, δ), (L^Y, γ) be L-fts's, $f : (X, [\delta]) \rightarrow (Y, [\mu])$ be continuous. If (L^X, δ) is stratified, (L^Y, γ) is weakly induced, then $f^{\rightarrow} : (L^X, \delta) \rightarrow (L^Y, \mu)$ is an L-fuzzy continuous mapping.*

Theorem 1.1. (cf. [9]) *Stratified, weakly induced and induced properties are hereditary and weakly induced property is strongly multiplicative.*

Theorem 1.2. (cf. [9]) *Let (L^X, δ), (L^Y, μ) be L-fts's, $f^{\rightarrow} : (L^X, \delta) \rightarrow (L^Y, \mu)$ an L-fuzzy continuous mapping, $\delta_\circ$ and $\mu_\circ$ be the stratifications of δ and μ respectively. Then $f^{\rightarrow} : (L^X, \delta_\circ) \rightarrow (L^Y, \mu_\circ)$ is continuous.*

Theorem 1.3. (cf. [9]) *Let (L^X, δ) be an L-fts, $Y \subset X$, $\delta_\circ$ the stratification of δ. Then $\delta_\circ \mid_Y$ is just the stratification of $\delta \mid_Y$.*

Theorem 1.4. (cf. [9]) *Let (L^X, δ) be an L-fts. Then (L^X, δ) is induced if and only if (L^X, δ) is both stratified and weakly induced.*

Theorem 1.5. (cf. [9]) *Let (L^X, δ) be an L-fts. Then the following are equivalent:*
(i) *(L^X, δ) is weakly induced;*
(ii) *For every $U \in \delta$ and every $a \in L$, $U_{(a)} \in [\delta]$;*
(iii) *For every $V \in \delta'$ and every $a \in L$, $V_{[a]} \in [\delta']$.*

Theorem 1.6. (cf. [9]) *Let (L^X, δ) be a weakly induced L-fts, $A \subset X$. Then for the interior A° and the closure A^- of A in $(X, [\delta])$, we have*
(i) $(1_A)^\circ = 1_{A^\circ}$, (ii) $(1_A)^- = 1_{A^-}$.

Lemma 1.5. *Let (L^X, δ) be an L-fts, $A \subset X$. If $A \in [\delta]$, then A is α-open.*

Proof. It is obvious.

Lemma 1.6. *Let (L^X, δ) be a weakly induced L-fts, $A \subset X$. Then A is α-open iff $A \in [\delta]$.*

Proof. $\Rightarrow$. Let $A \subset X$ be α-open. Then for each $x \in A$, there exists $U \in \delta$ with $U(x) \nleq \alpha$ and $U \wedge 1_{X-A} = \underline{0} \Rightarrow U \leq 1_A$. Since (L^X, δ) is weakly induced, it follows that for any $b \in L$, $b \leq \alpha$, $x \in U_{(b)} \in [\delta]$ and $x \in U_{(b)} \subset A$. So $A \in [\delta]$.
$\Leftarrow$. By Lemma 1.5.

Definition 1.1. *A system $(L^X, \delta_1, \delta_2)$ consisting of a non-empty set X with two L-fuzzy topologies δ_1 and δ_2 on L^X is called an L-fuzzy bitopological space (briefly L-fbts).*

Definition 1.2. *Let (L^X, δ, σ) be an L-fbts, $\alpha \in \mathrm{pr}(L)$. (L^X, δ, σ) is called α-PT_2 if $\forall x, y \in X$, $x \neq y$, there exist $U \in \delta$, $V \in \sigma$, such that $U(x) \nleq \alpha$, $V(y) \nleq \alpha$ and $U \wedge V = \underline{0}$, there exist $U' \in \sigma, V' \in \delta$, such that $U'(x) \nleq \alpha, V'(y) \nleq \alpha$ and $U' \wedge V' = \underline{0}$. In the case $L = I$, see* [8].

Definition 1.3. *Let $(L^X, \delta_1, \delta_2)$ be an L-fbts, $A \subset X$, $\alpha \in \mathrm{pr}(L)$. Then A is called α-pairwise closed (α-P-closed for short) iff A is α-closed in both (L^X, δ_1) and (L^X, δ_2).*

Definition 1.4. *An L-fuzzy mapping $f^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \sigma_1, \sigma_2)$ is called an L-fuzzy pairwise continuous (resp. α-pairwise continuous) mapping; briefly FPc (resp. α-Pc), if the induced mappings $f^{\rightarrow} : (L^X, \delta_k) \rightarrow (L^Y, \sigma_k)$ $(k = 1, 2)$ are L-fuzzy continuous (resp. α-continuous). In the case $L = I$, FPc mappings refer to* [14].

Definition 1.5. *An L-fbts* $(L^X, \delta_1, \delta_2)$ *is called pairwise stratified (resp. pairwise weakly induced, pairwise induced) L-fbts iff both* (L^X, δ_1) *and* (L^X, δ_2) *are stratified (resp. weakly induced, induced).*

For other definitions and results not explained in this paper, the reader may refer to [1, 2, 6, 7, 9].

§ 2. α-Fuzzy Pairwise Retracts

Definition 2.1. *Let* $(L^X, \delta_1, \delta_2)$ *be an L-fbts,* $Y \subset X$*. Then* $(L^Y, \delta_1 \mid_Y, \delta_2 \mid_Y)$ *is called an* α*-fuzzy pairwise retract (*α*-FPR for short) of* $(L^X, \delta_1, \delta_2)$ *if there exists an* α*-fuzzy pairwise continuous mapping* $r^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \delta_1 \mid_Y, \delta_2 \mid_Y)$ *with the identity mapping* $r \mid_Y = id_Y$*.*

Definition 2.2. *Let* $(L^X, \delta_1, \delta_2)$*,* (L^Y, μ_1, μ_2) *be L-fbts's,* $f^{\rightarrow} : L^X \rightarrow L^Y$ *an L-fuzzy mapping,* $\alpha \in \mathrm{pr}(L)$*. If for each* $x \in X$ *and* $V \in \mu_i$ *with* $V(f(x)) \nleq \alpha$*, there exists* $U \in \delta_i$ *with* $U(x) \nleq \alpha$*,* $i, j \in \{1, 2\}$*,* $i \neq j$*, such that*

(i) $f^{\rightarrow}(U) \leq V, \quad U \in \delta_j'$;
(ii) $f^{\rightarrow}(\delta_i\text{-int}\,(\delta_j\text{-cl}\,(U)) \leq V$;
(iii) $f^{\rightarrow}((\delta_j\text{-cl}\,(U)) \leq V$;
(iv) $f^{\rightarrow}(\delta_i\text{-int}(\delta_j\text{-cl}\,(U)) \leq \mu_i\text{-int}\,(\mu_j\text{-cl}\,(V))$;
(v) $f^{\rightarrow}(U) \leq \mu_i\text{-int}\,(\mu_j\text{-cl}\,(V))$,

then $f^{\rightarrow}$ *is called*

(i) α*-fuzzy pairwise clopen continuous (*α*-FPcoc, for short);*
(ii) α*-fuzzy pairwise super continuous (*α*-FPsc, for short);*
(iii) *strongly* α*-fuzzy pairwise continuous (s*α*-FPc, for short);*
(iv) α*-fuzzy pairwise* δ*-continuous (*α*-FP*δ*-c, for short);*
(v) α*-fuzzy pairwise almost continuous (*α*-FPac, for short).*

Definition 2.3. *Let* $(L^X, \delta_1, \delta_2)$ *be an L-fbts,* $Y \subset X$*. Then* $(L^Y, \delta_1 \mid_Y, \delta_2 \mid_Y)$ *is called an* α*-fuzzy pairwise clopen retract (resp. an* α*-fuzzy pairwise super retract, strongly* α*-fuzzy pairwise retract, an* α*-fuzzy pairwise* δ*-retract and an* α*-fuzzy pairwise almost retract);* α*-FPCOR (resp.* α*-FPSR, S*α*-FPR,* α*-FP*δ*-R and* α*-FPAR) for brevity; iff there exists an* α*-FPcoc (resp.* α*-FPsc, s*α*-FPc,* α*-FP*δ*-c and* α*-FPac)* $r^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \delta_1 \mid_Y, \delta_2 \mid_Y)$ *such that* $r \mid_Y = id_Y$*.*

Remark 2.1. Every L-fuzzy pairwise retract (L-FPR) is α-FPR.

The implications between these different notions of α-fuzzy pairwise retracts are given by the following diagram

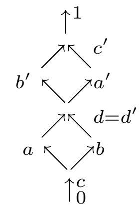
$$\begin{array}{ccccc} & & \text{S}\alpha\text{-FPR} & & L\text{-FPR} \\ & & \Downarrow & & \Downarrow \\ \alpha\text{-FPCOR} & \Longrightarrow & \alpha\text{-FPSR} & \Longrightarrow & \alpha\text{-FPR} \\ & & \Downarrow & & \Downarrow \\ & & \alpha\text{-FP}\delta\text{-R} & \Longrightarrow & \alpha\text{-FPAR} \end{array}$$

Example 2.1. Let $X = [0, 1]$, $Y = \{0, 1\}$, L be the lattice given by the following diagram. We define $r : X \rightarrow Y$ by

$$r(x) = \begin{cases} 0, & \text{if } \ x \in [0, 0.5], \\ 1, & \text{if } \ x \in (0.5, 1], \end{cases}$$

and let $\delta_1=\{\underline{0},U,V,\underline{1}\},\ \delta_2=\{\underline{0},W,\underline{1}\}$, where

$$U(x)=\begin{cases} c', & \text{if } \ x\in Y,\\ d, & \text{if } \ x\in X-Y;\end{cases}$$



$$V(x)=\begin{cases} d, & \text{if } \ x\in X-\{0.5\},\\ a, & \text{if } \ x=0.5;\end{cases}$$

$$W(x)=\begin{cases} a', & \text{if } \ x\in Y,\\ d, & \text{otherwise.}\end{cases}$$

One can easily show that $r^{\rightarrow}$ is α-FPc at $\alpha=b$ and hence $(L^Y,\delta_1\mid_Y,\delta_2\mid_Y)$ is an α-FPR of (L^X,δ_1,δ_2) but neither α-FPSR nor L-FPR.

And let $\delta_1=\{\underline{0},W,\underline{1}\},\ \delta_2=\{\underline{0},V^*,\underline{d},\underline{1}\}$, where

$$V^*(x)=\begin{cases} c, & \text{if } \ x\in Y,\\ c', & \text{if } \ x\in X-Y.\end{cases}$$

One can easily show that at $\alpha=b$, $(L^Y,\delta_1\mid_Y,\delta_2\mid_Y)$ is an α-FPAR of (L^X,δ_1,δ_2) but not α-FPδ-R.

Also, let $\delta_1=\{\underline{0},W,\underline{1}\}$ and $\delta_2=\{\underline{0},\underline{d},\underline{c},\underline{1}\}$. One can easily show that at $\alpha=b$, $(L^Y,\delta_1\mid_Y,\delta_2\mid_Y)$ is an α-FPSR of (L^X,δ_1,δ_2) but neither an α-FPCOR nor an Sα-FPR.

Example 2.2. Let $X=N=\{1,2,3,\cdots\}$ and $Y=\{5,10\}$, L be the same lattice given in Example 2.1. We define $r:X\rightarrow Y$ as follows:

$$r(x)=\begin{cases} 5, & \text{if } \ x \text{ is odd},\\ 10, & \text{if } \ x \text{ is even},\end{cases}$$

and let $\delta_1=\{\underline{0},U,\underline{1}\},\ \delta_2=\{\underline{0},W,\underline{1}\}$, where $U,\ W\in L^X$ defined as follows:

$$U(x)=\begin{cases} d, & 1\le x<5,\\ 1, & x\ge 5,\end{cases}$$

$$W(x)=\begin{cases} c', & (1\le x<5)\cup(x>10),\\ d, & 5\le x\le 10.\end{cases}$$

One can easily show that $r^{\rightarrow}$ is an α-FPac and hence $(L^Y,\delta_1\mid_Y,\delta_2\mid_Y)$ is an α-FPAR of (L^X,δ_1,δ_2) but not an α-FPR, at $\alpha=a$.

Also, let $\delta_1=\{\underline{0},\underline{a}',\underline{1}\}$ and $\delta_2=\{\underline{0},W,\underline{1}\}$. One can easily show that at $\alpha=a$, $(L^Y,\delta_1\mid_Y,\delta_2\mid_Y)$ is an α-FPδ-R of (L^X,δ_1,δ_2) but not an α-FPSR.

Definition 2.4. *Let (L^X,δ_1,δ_2) be an L-fbts. Then (L^X,δ_1,δ_2) is called*

(i) *α-pairwise regular space if for each $x\in X$ and each $U\in\delta_i$ with $U(x)\nleq\alpha$, there exists $V\in\delta_i$ with $V(x)\nleq\alpha$ such that δ_j-cl$(V)\le U$.*

(ii) *α-pairwise semiregular space if for each $x\in X$ and each $U\in\delta_i$ with $U(x)\nleq\alpha$, there exists $V\in\delta_i$ with $V(x)\nleq\alpha$ such that δ_i-int$(\delta_j$-cl$(V))\le U$.*

(iii) *α-pairwise almost regular space if for each $x\in X$ and each $U\in\delta_i$ with $U(x)\nleq\alpha$, there exists $V\in\delta_i$ with $V(x)\nleq\alpha$ such that δ_j-cl$(V)\le\delta_i$-int$(\delta_j$-cl$(U))$.*

Remark 2.2. From the preceding definition it is clear that every α-pairwise regular space is an α-pairwise semiregular space and also an α-pairwise almost regular space. Also an α-pairwise semiregular space and an α-pairwise almost regular space are independent notions.

Example 2.3. Let $X = \{x^1, x^2\}$, L be the same lattice given in Example 2.1.

Let $\delta_1 = \{\underline{0}, \underline{d}, \underline{1}\}$ and $\delta_2 = \{\underline{0}, \underline{a}, \underline{1}\}$. Then $(L^X, \delta_1, \delta_2)$ is α-pairwise semiregular but not α-pairwise almost regular at $\alpha = b$.

And let $\delta_1 = \{\underline{0}, x^1_a \vee x^2_{a'}, x^1_{a'} \vee x^2_{c'}, \underline{1}\}$ and $\delta_2 = \{\underline{0}, \underline{d}, \underline{1}\}$. Then $(L^X, \delta_1, \delta_2)$ is α-pairwise almost regular but not α-pairwise semiregular at $\alpha = b$.

Theorem 2.1. *Let $(L^X, \delta_1, \delta_2)$ be an α-pairwise semiregular L-fbts, $Y \subset X$. Then the following are equivalent:*

(i) *L^Y is an α-FPR of L^X;*
(ii) *L^Y is an α-FPSR of L^X.*

Proof. (ii)$\Rightarrow$(i). It follows from the definitions.

(i)$\Rightarrow$(ii). Let L^Y be an α-FPR of L^X. Then there exists an α-FPc $r^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \delta_1 \mid_Y, \delta_2 \mid_Y)$ with $r \mid_Y = id_Y$, so $r^{\rightarrow} : (L^X, \delta_1) \rightarrow (L^Y, \delta_1 \mid_Y)$, $r^{\rightarrow} : (L^X, \delta_2) \rightarrow (L^Y, \delta_2 \mid_Y)$ are α-Fc mappings. Then $\forall\, x \in X,\ V \in \delta_i \mid_Y$ with $V(r(x)) \nleq \alpha \Rightarrow \exists\, U \in \delta_i$ with $U(x) \nleq \alpha$ such that $r^{\rightarrow}(U) \leq V$. Since L^X is α-pairwise semiregular, $\exists\, W \in \delta_i$ with $W(x) \nleq \alpha$ such that $\delta_i\text{-int}\,(\delta_j\text{-cl}\,(W)) \leq U \Rightarrow r^{\rightarrow}(\delta_i\text{-int}\,(\delta_j\text{-cl}(W))) \leq r^{\rightarrow}(U) \leq V,\ i \neq j$. Then $r^{\rightarrow}$ is α-FPsc and hence L^Y is an α-FPSR of L^X.

Theorem 2.2. *Let $(L^X, \delta_1, \delta_2)$ be an L-fbts, $Y \subset X$ and $(L^Y, \delta_1 \mid_Y, \delta_2 \mid_Y)$ be α-pairwise semiregular. Consider the following properties*

(i) *L^Y is an α-FPR of L^X,*
(ii) *L^Y is an α-FPAR of L^X,*
(iii) *L^Y is an α-FPδ-R of L^X,*
(iv) *L^Y is an α-FPSR of L^X.*

Then, (iv)$\Longleftrightarrow$(iii)$\Longrightarrow$(i)$\Longleftrightarrow$(ii).

Proof. Clearly (iv)$\Rightarrow$(iii)$\Rightarrow$(ii), (iv)$\Rightarrow$(i)$\Rightarrow$(ii).

It suffices to show that (iii)$\Rightarrow$(iv) and (ii)$\Rightarrow$(i).

(iii)$\Rightarrow$(iv). Since L^Y is an α-FPδ-R of L^X, there exists an α-FPδ-c mapping $r^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \delta_1 \mid_Y, \delta_2 \mid_Y)$ such that $r \mid_Y = id_Y$.

Now we are going to prove that $r^{\rightarrow}$ is α-FPsc. Let $x \in X,\ W \in \delta_i \mid_Y$ with $W(r(x)) \nleq \alpha$, where L^Y is α-pairwise semiregular $\Rightarrow \exists\, V \in \delta_i \mid_Y$ with $V(r(x)) \nleq \alpha$ such that $\delta_i \mid_Y\text{-int}\,(\delta_j \mid_Y\text{-cl}\,(V)) \leq W$. Since L^Y is an α-FPδ-R of L^X $\Rightarrow \exists\, U \in \delta_i$ with $U(x) \nleq \alpha$ and $r^{\rightarrow}(\delta_i\text{-int}(\delta_j\text{-cl}\,(U))) \leq \delta_i \mid_Y\text{-int}(\delta_j \mid_Y\text{-cl}\,(V)) \leq W$, i.e., $r^{\rightarrow}$ is α-FPsc and hence L^Y is an α-FPSR of L^X.

(ii)$\Rightarrow$(i). Since L^Y is an α-FPAR of L^X, there exists an α-FPac mapping $r^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \delta_1 \mid_Y, \delta_2 \mid_Y)$ such that $r \mid_Y = id_Y$.

Now we are going to prove that $r^{\rightarrow}$ is α-FPc. Let $x \in X,\ W \in \delta_i \mid_Y$ with $W(r(x)) \nleq \alpha$ where L^Y is α-pairwise semiregular $\Rightarrow \exists\, V \in \delta_i \mid_Y$ with $V(r(x)) \nleq \alpha$ such that $\delta_i \mid_Y\text{-int}\,(\delta_j \mid_Y\text{-cl}\,(V)) \leq W$. Since L^Y is an α-FPAR of L^X $\Rightarrow \exists\, U \in \delta_i$ with $U(x) \nleq \alpha$ and $r^{\rightarrow}(U) \leq \delta_i \mid_Y\text{-int}\,(\delta_j \mid_Y\text{-cl}\,(V)) \leq W$, i.e., $r^{\rightarrow}$ is α-FPc and hence L^Y is an α-FPR of L^X.

Theorem 2.3. *Let $Y \subset X$, and $(L^X, \delta_1, \delta_2), (L^Y, \delta_1 \mid_Y, \delta_2 \mid_Y)$ are α-pairwise semiregular L-fbts's. Then the following are equivalent:*

(i) *L^Y is an α-FPR of L^X;*
(ii) *L^Y is an α-FPAR of L^X;*
(iii) *L^Y is an α-FPδ-R of L^X;*
(iv) *L^Y is an α-FPSR of L^X.*

Proof. It follows from Theorem 2.1 and Theorem 2.2.

Theorem 2.4. *Let $(L^X, \delta_1, \delta_2)$ be an α-pairwise almost regular L-fbts, $Y \subset X$. Then the following are equivalent:*
(i) L^Y *is an Sα-FPR of* L^X;
(ii) L^Y *is an α-FPSR of* L^X.

Proof. (i)$\Rightarrow$(ii). It is clear.

(ii)$\Rightarrow$(i). Since L^Y is an α-FPSR of L^X, there exists an α-FPsc mapping $r^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \delta_1 \mid_Y, \delta_2 \mid_Y)$ such that $r \mid_Y = id_Y$.

Now we are going to prove that $r^{\rightarrow}$ is sα-FPc. Let $x \in X,\ W \in \delta_i \mid_Y$ with $W(r(x)) \nleq \alpha$. Since L^Y is an α-FPSR of $L^X \Rightarrow \exists\ V \in \delta_i$ with $V(x) \nleq \alpha$ and $r^{\rightarrow}(\delta_i\text{-int}\,(\delta_j\text{-cl}\,(V))) \leq W$, but L^X is α-pairwise almost regular, $\exists\ U \in \delta_i$ with $U(x) \nleq \alpha$ and $\delta_j\text{-cl}\,(U) \leq \delta_i\text{-int}(\delta_j\text{-cl}\,(V)) \Rightarrow r^{\rightarrow}(\delta_j\text{-cl}\,(U)) \leq r^{\rightarrow}(\delta_i\text{-int}\,(\delta_j\text{-cl}\,(V)) \leq W$, i.e., $r^{\rightarrow}$ is sα-FPc and hence L^Y is an Sα-FPR of L^X.

Corollary 2.1. *Let $Y \subset X$, and $(L^X, \delta_1, \delta_2), (L^Y, \delta_1 \mid_Y, \delta_2 \mid_Y)$ are α-pairwise regular L-fbts's. Then the properties, α-FPR, α-FPSR, Sα-FPR, α-FPδ-R, α-FPAR are all equivalent.*

Theorem 2.5. *Let $f^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \sigma_1, \sigma_2)$ be an L-fuzzy mapping and $g : X \rightarrow X \times Y$ its ordinary graph. Then $g^{\rightarrow}$ is α-FPsc $\Longleftrightarrow$ $f^{\rightarrow}$ is α-FPsc and L^X is α-pairwise semiregular.*

Proof. $\Rightarrow$. Suppose $g^{\rightarrow}$ is α-FPsc. Let $x \in X,\ W \in \sigma_i$ with $W(f(x)) \nleq \alpha$. Then $U = \underline{1} \times W \in \delta_i \times \sigma_i$ such that $U(g(x)) \nleq \alpha$. Since $g^{\rightarrow}$ is α-FPsc $\Rightarrow\ \exists\ V \in \delta_i$ with $V(x) \nleq \alpha$ such that $g^{\rightarrow}(\delta_i\text{-int}\,(\delta_j\text{-cl}\,(V))) \leq U$, and $\delta_i\text{-int}\,(\delta_j\text{-cl}\,(V)) \leq g^{\leftarrow}(U) = \underline{1} \wedge f^{\leftarrow}(W) = f^{\leftarrow}(W) \Rightarrow f^{\rightarrow}(\delta_i\text{-int}\,(\delta_j\text{-cl}\,(V))) \leq f^{\rightarrow}f^{\leftarrow}(W) \leq W \Rightarrow f^{\rightarrow}$ is α-FPsc.

We show that L^X is α-pairwise semiregular. Let $x \in X,\ \theta \in \delta_i$ with $\theta(x) \nleq \alpha$. Then $\theta \times \underline{1} \in \delta_i \times \sigma_i$ such that $(\theta \times \underline{1})(g(x)) \nleq \alpha$. Since $g^{\rightarrow}$ is α-FPsc $\Rightarrow\ \exists\ \theta^* \in \delta_i$ with $\theta^*(x) \nleq \alpha$ such that

$$g^{\rightarrow}(\delta_i\text{-int}\,(\delta_j\text{-cl}\,(\theta^*))) \leq \theta \times \underline{1} \Rightarrow \delta_i\text{-int}\,(\delta_j\text{-cl}\,(\theta^*)) \leq g^{\leftarrow}(\theta \times \underline{1}) = \theta \wedge f^{\leftarrow}(\underline{1}) = \theta.$$

Then L^X is α-pairwise semiregular.

$\Leftarrow$. Assume $f^{\rightarrow}$ is α-FPsc and L^X is α-pairwise semiregular.

Let $x \in X,\ W \in \delta_i \times \sigma_i$ with $W(g(x)) \nleq \alpha$, by Lemma 1.3 $\Rightarrow\ \exists\ W_1 \in \delta_i$ with $W_1(x) \nleq \alpha,\ W_2 \in \sigma_i$ with $W_2(f(x)) \nleq \alpha$ such that $W_1 \times W_2 \leq W$. Since $f^{\rightarrow}$ is α-FPsc $\Rightarrow\ \exists\ \theta_2 \in \delta_i$ with $\theta_2(x) \nleq \alpha$ such that

$$f^{\rightarrow}(\delta_i\text{-int}\,(\delta_j\text{-cl}\,(\theta_2))) \leq W_2 \Rightarrow \delta_i\text{-int}\,(\delta_j\text{-cl}\,(\theta_2)) \leq f^{\leftarrow}(W_2),$$

and also L^X is α-pairwise semiregular $\Rightarrow\ \exists\ \theta_1 \in \delta_i$ with $\theta_1(x) \nleq \alpha$ such that $\delta_i\text{-int}\,(\delta_j\text{-cl}\,(\theta_1)) \leq W_1$. Clearly $\theta_1 \wedge \theta_2 = \theta \in \delta_i$ and $\theta(x) \nleq \alpha$,

$$\begin{aligned}
\delta_i\text{-int}\,(\delta_j\text{-cl}\,(\theta)) &= \delta_i\text{-int}\,(\delta_j\text{-cl}\,(\theta_1 \wedge \theta_2)) \\
&\leq (\delta_i\text{-int}\,(\delta_j\text{-cl}\,(\theta_1))) \wedge (\delta_i\text{-int}\,(\delta_j\text{-cl}\,(\theta_2))) \\
&\leq W_1 \wedge f^{\leftarrow}(W_2) = g^{\leftarrow}(W_1 \times W_2) \leq g^{\leftarrow}(W) \\
\Rightarrow g^{\rightarrow}(\delta_i\text{-int}\,(\delta_j\text{-cl}\,(\theta))) &\leq g^{\rightarrow}g^{\leftarrow}(W) \leq W.
\end{aligned}$$

Thus $g^{\rightarrow}$ is α-FPsc.

Corollary 2.2. *Let $(L^X, \delta_1, \delta_2)$ be an L-fbts, $Y \subset X$ and $r^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \delta_1 |_Y, \delta_2 |_Y)$ be an L-fuzzy mapping such that $r |_Y = id_Y$, $g : X \rightarrow X \times Y$ its ordinary graph. Then $g^{\rightarrow}$ is α-FPsc $\Longleftrightarrow L^Y$ is an α-FPSR of L^X and L^X is α-pairwise semiregular.*

Theorem 2.6. *Let $f^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \sigma_1, \sigma_2)$ be an L-fuzzy mapping and $g : X \rightarrow X \times Y$ its ordinary graph. If $g^{\rightarrow}$ is sα-FPc, then $f^{\rightarrow}$ is sα-FPc and L^X is α-pairwise almost regular.*

Proof. Suppose $f^{\rightarrow}$ is sα-FPc. Let $x \in X$, $W \in \sigma_i$ with $W(f(x)) \nleq \alpha$. Then $U = \underline{1} \times W \in \delta_i \times \sigma_i$ such that $U(g(x)) \nleq \alpha$. Since $g^{\rightarrow}$ is sα-FPc $\Rightarrow \exists\, V \in \delta_i$ with $V(x) \nleq \alpha$ such that $g^{\rightarrow}(\delta_j\text{-cl}\,(V)) \leq U$, and

$$\begin{aligned} &\delta_j\text{-cl}\,(V) \leq g^{\leftarrow}(U) = \underline{1} \wedge f^{\leftarrow}(W) = f^{\leftarrow}(W) \\ &\Rightarrow f^{\rightarrow}(\delta_j\text{-cl}\,(V)) \leq f^{\rightarrow} f^{\leftarrow}(W) \leq W \\ &\Rightarrow f^{\rightarrow} \quad \text{is s}\alpha\text{-FPc.} \end{aligned}$$

We show that L^X is α-pairwise almost regular.

Let $x \in X$, $\theta \in \delta_i$ with $\theta(x) \nleq \alpha$. Then $\theta \times \underline{1} \in \delta_i \times \sigma_i$ such that $(\theta \times \underline{1})(g(x)) \nleq \alpha$. Since $g^{\rightarrow}$ is sα-FPc $\Rightarrow \exists\, \theta^* \in \delta_i$ with $\theta^*(x) \nleq \alpha$ such that

$$\begin{aligned} &g^{\rightarrow}(\delta_j\text{-cl}\,(\theta^*)) \leq \theta \times \underline{1} \\ &\Rightarrow \delta_j\text{-cl}\,(\theta^*) \leq g^{\leftarrow}(\theta \times \underline{1}) = \theta \wedge f^{\leftarrow}(\underline{1}) = \theta \leq \delta_i\text{-int}\,(\delta_j\text{-cl}\,(\theta)). \end{aligned}$$

Then L^X is α-pairwise almost regular.

Corollary 2.3. *Let $(L^X, \delta_1, \delta_2)$ be an L-fbts, $Y \subset X$ and $r^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \delta_1 |_Y, \delta_2 |_Y)$ be an L-fuzzy mapping such that $r |_Y = id_Y$, $g : X \rightarrow X \times Y$ its ordinary graph. If $g^{\rightarrow}$ is sα-FPc, then L^Y is an Sα-FPR of L^X and L^X is α-pairwise almost regular.*

Theorem 2.7. *Let $f^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \sigma_1, \sigma_2)$ be an L-fuzzy mapping and $g : X \rightarrow X \times Y$ its ordinary graph. Then*

$$g^{\rightarrow} \textit{ is s}\alpha\textit{-FPc} \Longleftrightarrow f^{\rightarrow} \textit{ is s}\alpha\textit{-FPc and } L^X \textit{ is } \alpha\textit{-pairwise regular.}$$

Proof. $\Rightarrow$. Assume $f^{\rightarrow}$ is sα-FPc. Let $x \in X$, $W \in \sigma_i$ with $W(f(x)) \nleq \alpha$. Then $U = \underline{1} \times W \in \delta_i \times \sigma_i$ such that $U(g(x)) \nleq \alpha$. Since $g^{\rightarrow}$ is sα-FPc $\Rightarrow \exists\, V \in \delta_i$ with $V(x) \nleq \alpha$ such that $g^{\rightarrow}(\delta_j\text{-cl}\,(V)) \leq U$, and

$$\begin{aligned} &\delta_j\text{-cl}\,(V) \leq g^{\leftarrow}(U) = \underline{1} \wedge f^{\leftarrow}(W) = f^{\leftarrow}(W) \\ &\Rightarrow f^{\rightarrow}(\delta_j\text{-cl}\,(V)) \leq f^{\rightarrow} f^{\leftarrow}(W) \leq W \\ &\Rightarrow f^{\rightarrow} \quad \text{is s}\alpha\text{-FPc.} \end{aligned}$$

We show that L^X is α-pairwise regular. Let $x \in X$, $\theta \in \delta_i$ with $\theta(x) \nleq \alpha$. Then $\theta \times \underline{1} \in \delta_i \times \sigma_i$ such that $(\theta \times \underline{1})(g(x)) \nleq \alpha$. Since $g^{\rightarrow}$ is sα-FPc $\Rightarrow \exists\, \theta^* \in \delta_i$ with $\theta^*(x) \nleq \alpha$ such that

$$g^{\rightarrow}(\delta_j\text{-cl}\,(\theta^*)) \leq \theta \times \underline{1} \ \Rightarrow \delta_j\text{-cl}\,(\theta^*) \leq g^{\leftarrow}(\theta \times \underline{1}) = \ \theta \wedge f^{\leftarrow}(\underline{1}) = \theta.$$

Then L^X is α-pairwise regular.

$\Leftarrow$. Assume $f^{\rightarrow}$ is sα-FPc of L^X and L^X is α-pairwise semiregular.

Let $x \in X$, $W \in \delta_i \times \sigma_i$ with $W(g(x)) \not\leq \alpha$. By Lemma 1.3 $\Rightarrow$ $\exists\, W_1 \in \delta_i$ with $W_1(x) \not\leq \alpha$, $W_2 \in \sigma_i$ with $W_2(f(x)) \not\leq \alpha$ such that $W_1 \times W_2 \leq W$, where $f^{\rightarrow}$ is sα-FPc. $\Rightarrow$ $\exists\, \theta_2 \in \delta_i$ with $\theta_2(x) \not\leq \alpha$ such that $f^{\rightarrow}(\delta_j\text{-cl}\,(\theta_2)) \leq W_2 \Rightarrow \delta_j\text{-cl}\,(\theta_2) \leq f^{\leftarrow}(W_2)$, and also L^X is α-pairwise regular $\Rightarrow \exists\, \theta_1 \in \delta_i$ with $\theta_1(x) \not\leq \alpha$ such that $\delta_j\text{-cl}\,(\theta_1) \leq W_1$.

Clearly $\theta_1 \wedge \theta_2 = \theta \in \delta_i$ such that $\theta(x) \not\leq \alpha$,

$$\begin{aligned}\delta_j\text{-cl}\,(\theta) &= \delta_j\text{-cl}\,(\theta_1 \wedge \theta_2) \leq (\delta_j\text{-cl}\,(\theta_1)) \wedge (\delta_j\text{-cl}\,(\theta_2))\\ &\leq W_1 \wedge f^{\leftarrow}(W_2) = g^{\leftarrow}(W_1 \times W_2) \leq g^{\leftarrow}(W)\\ \Rightarrow g^{\rightarrow}(\delta_j\text{-cl}\,(\theta)) &\leq g^{\rightarrow} g^{\leftarrow}(W) \leq W.\end{aligned}$$

Thus $g^{\rightarrow}$ is sα-FPc.

Corollary 2.4. *Let $(L^X, \delta_1, \delta_2)$ be L-fbts, $Y \subset X$ and $r^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \delta_1 |_Y, \delta_2 |_Y)$ be an L-fuzzy mapping such that $r |_Y = id_Y$, $g : X \rightarrow X \times Y$ its ordinary graph. Then $g^{\rightarrow}$ is sα-FPc $\Longleftrightarrow L^Y$ is an Sα-FPR of L^X and L^X is α-pairwise regular.*

Proposition 2.1. *The composition of α-FPc (resp. α-FPcoc, α-FPsc, sα-FPc, α-FPδ-c) mappings is an α-FPc (resp. α-FPcoc, α-FPsc, sα-FPc, α-FPδ-c) mapping.*

Proof. It is obvious.

Theorem 2.8. *Let $(L^X, \delta_1, \delta_2)$ be an L-fbts, $Y \subset X$. Then L^Y is an α-FPR (resp. α-FPCOR, α-FPSR, Sα-FPR, α-FPδ-R) of L^X iff, for any $(L^Z, \gamma_1, \gamma_2)$ L-fbts, every α-FPc (resp. α-FPcoc, α-FPsc, sα-FPc, α-FPδ-c) mapping $g^{\rightarrow} : L^Y \rightarrow L^Z$, $g^{\rightarrow}$ has an extension over X.*

Proof. By Proposition 2.1.

Theorem 2.9. *Let $(L^X, \delta_1, \delta_2)$ be an L-fbts, $Z \subset Y \subset X$. If L^Z is an α-FPCR (resp. α-FPCOR, α-FPR, α-FPCOR, α-FPAR, α-FPAR, Sα-FPR, Sα-FPR) of L^Y, and L^Y is an α-FPSR (resp. α-FPδ-R, Sα-FPR, α-FPAR, Sα-FPR, α-FPCR, α-FPδ-R, α-FPR) of L^X, then L^Z is an α-FPSR (resp. α-FPSR, Sα-FPR, α-FPR, α-FPδ-R, α-FPAR, α-FPSR, α-FPR) of L^X.*

Proof. It is obvious.

Definition 2.5. *Let $(L^X, \delta_1, \delta_2)$, $(L^Y, \gamma_1, \gamma_2)$ be L-fbts's. Then the L-fuzzy pairwise mapping $f^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \gamma_1, \gamma_2)$ is called $\triangle$-pairwise continuous (Δ-Pc for short) mapping if both $f^{\rightarrow} : (L^X, \delta_1) \rightarrow (L^Y, \gamma_1)$ and $f^{\rightarrow} : (L^X, \delta_2) \rightarrow (L^Y, \gamma_2)$ are $\triangle$-continuous mappings.*

And if $Y \subset X$, then $(L^Y, \delta_1 |_Y, \delta_2 |_Y)$ is called a Δ-pairwise retract (Δ-PR for short) of $(L^X, \delta_1, \delta_2)$ if there exists a Δ- pairwise continuous mapping $r^{\rightarrow} : (L^X, \delta_1, \delta_2) \rightarrow (L^Y, \delta_1 |_Y, \delta_2 |_Y)$ such that $r |_Y = id_Y$.

Clearly every α-PR is a $\triangle$-PR but the converse is not true in general.

Example 2.4. Let $X = \{x^1, x^2, x^3\}$, $Y = \{x^1\}$, and L the same lattice given in Example 2.1. Consider δ_1, δ_2 on L^X defined by

$$\delta_1 = \{\underline{0}, x^1_d \vee x^2_{c'} \vee x^3_b, \underline{1}\}, \qquad \delta_2 = \{\underline{0}, x^1_d \vee x^2_{a'} \vee x^3_{b'}, \underline{1}\}.$$

Clearly L^Y is a $\triangle$-PR of L^X but not an α-PR of L^X at $\alpha = a$.

Theorem 2.10. *Let (L^X, δ, σ) be an L-fbts and α-PT$_2$. Then every α-FPR of (L^X, δ, σ) is α-P-closed.*

Proof. Let $(L^Y, \delta|_Y, \sigma|_Y)$ be an α-FPR of (L^X, δ, σ), where (L^X, δ, σ) is α-PT$_2$. Then there exists an α-fuzzy continuous mapping $r^{\rightarrow} : L^X \rightarrow L^Y$ such that $r(y) = y$, $\forall y \in Y$. Let $x \in X - Y \Rightarrow x \neq r(x)$, $r(x) \in Y$. But (L^X, δ, σ) is α-PT$_2$, then there exist $U \in \delta$, $V \in \sigma$, such that $U(x) \nleq \alpha$, $V(r(x)) \nleq \alpha$ and $U \wedge V = \underline{0}$; there is $V' \in \delta$, $U' \in \sigma$, such that $V'(r(x)) \nleq \alpha$, $U'(x) \nleq \alpha$ and $U' \wedge V' = \underline{0}$.

Therefore $V|_Y \in \sigma|_Y$ and $V|_Y (r(x)) \nleq \alpha$. Since $r^{\rightarrow}$ is α-pairwise continuous $\Rightarrow$ $\exists\, W_1 \in \sigma$ such that $W_1(x) \nleq \alpha$ and $r^{\rightarrow}(W_1) \leq V|_Y$. Put $W_1^* = W_1 \wedge U' \in \sigma$ such that $W_1^*(x) \nleq \alpha$ and $W_1^* \wedge 1_Y = \underline{0}$.

And also therefore $V'|_Y \in \delta|_Y$ and $V'|_Y (r(x)) \nleq \alpha$.

Since $r^{\rightarrow}$ is α-pairwise continuous $\Rightarrow \exists\, W_2 \in \delta$ such that $W_2(x) \nleq \alpha$ and $r^{\rightarrow}(W_2) \leq V'|_Y$. Put $W_2^* = W_2 \wedge U \in \delta$ such that $W_2^*(x) \nleq \alpha$ and $W_2^* \wedge 1_Y = \underline{0}$.

For, assume that $\exists\ z \in Y$, $a \in L$-$\{0\}$ such that $W_1^*(z) > 0$, $W_2^*(z) > a$, hence $W_1(z) \wedge U'(z) > 0$. But

$$W_1(z) \leq r^{\leftarrow}r^{\rightarrow}(W_1)(z) \leq r^{\leftarrow}(V|_Y)(z) = (V|_Y)(r(z)) = (V|_Y)(z) = V(z).$$

That is $W_1(z) \leq V(z)$, so, $V(z) \wedge U'(z) > 0$, and similarly $V'(z) \wedge U(z) > a \Rightarrow (V \wedge U \wedge V' \wedge U')(z) > 0$, a contradiction to $U \wedge V = \underline{0}$ and $U' \wedge V' = \underline{0}$. Hence $W_1^* \wedge 1_Y = \underline{0}$ and $W_2^* \wedge 1_Y = \underline{0}$, so Y is α-closed in both (L^X, δ) and (L^Y, σ). Hence Y is α-P-closed.

§ 3. α-Fuzzy Pairwise Retract of L-Valued Pairwise Stratification Spaces

Proposition 3.1. *Let $(L^Y, \delta_1|_Y, \delta_2|_Y)$ be an L-FPR of $(L^X, \delta_1, \delta_2)$. Then*

$$(L^X, \delta_1, \delta_2) \text{ is pairwise stratified} \Longleftrightarrow (L^Y, \delta_1|_Y, \delta_2|_Y) \text{ is pairwise stratified.}$$

Theorem 3.1. *Let (L^X, μ_1, μ_2) be the pairwise stratification of $(L^X, \delta_1, \delta_2)$, and $(L^Y, \delta_1|_Y, \delta_2|_Y)$ be an L-FPR of $(L^X, \delta_1, \delta_2)$. Then $(L^Y, \mu_1|_Y, \mu_2|_Y)$ is an L-FPR of (L^X, μ_1, μ_2).*

Proof. By Theorem 1.2 and Theorem 1.3.

Proposition 3.2. *If $(L^Y, \delta_1|_Y, \delta_2|_Y)$ is an L-FPR of $(L^X, \delta_1, \delta_2)$. Then $(Y, [\delta_1|_Y], [\delta_2|_Y])$ is an ordinary pairwise retract of $(X, [\delta_1], [\delta_2])$.*

Proof. By Proposition 1.1.

Theorem 3.2. *Let $(L^X, \delta_1, \delta_2)$ be a pairwise induced L-fbts, $Y \subset X$.*

$$(L^Y, \delta_1|_Y, \delta_2|_Y) \text{ is an L-FPR of } (L^X, \delta_1, \delta_2)$$
$$\Longleftrightarrow (Y, [\delta_1|_Y], [\delta_2|_Y]) \text{ is an ordinary pairwise retract of } (X, [\delta_1], [\delta_2]).$$

Proof. $\Rightarrow$. By Proposition 3.2.
$\Leftarrow$. By Theorems 1.1, 1.4 and Lemma 1.4.

Proposition 3.3. *Let $(L^X, \delta_1, \delta_2)$ be a weakly induced L-fts, $Y \subset X$. $(Y, [\delta_1|_Y], [\delta_2|_Y])$ is an ordinary pairwise retract of $(X, [\delta_1], [\delta_2])$, iff $(L^Y, \delta_1|_Y, \delta_2|_Y)$ is a Δ-FPR of $(L^X, \delta_1, \delta_2)$.*

Proof. By Lemma 1.6.

Theorem 3.3. *Let $(L^X, \delta_1, \delta_2), (L^Y, \gamma_1, \gamma_2)$ be pairwise weakly induced L-fbts's, $f^{\rightarrow} : L^X \rightarrow L^Y$ be an L-fuzzy mapping. Then the following hold:*

(i) *If $f^{\rightarrow}$ is an α-FPc, then the ordinary mapping $f : (X, [\delta_1], [\delta_2]) \rightarrow (Y, [\gamma_1], [\gamma_2])$ is Pc;*

(ii) *If $f^{\rightarrow}$ is an α-FPac, then the ordinary mapping $f : (X, [\delta_1], [\delta_2]) \rightarrow (Y, [\gamma_1], [\gamma_2])$ is Pac.*

Proof. (i) Let $x \in X,\ f(x) \in A \in [\gamma_i],\ \alpha \in \text{pr}(L) \Rightarrow 1_A \in \gamma_i$ and $1_A(f(x)) \nleq \alpha$. But $f^{\rightarrow}$ is an α-FPc $\Rightarrow \exists\, W \in \delta_i$ with $W(x) \nleq \alpha$ and $f^{\rightarrow}(W) \leq 1_A$. Let $a \in L$. Since L^X is pairwise weakly induced, we have $W_{(a)} \in [\delta_i],\ (W)_{(a)} \leq f^{\leftarrow}f^{\rightarrow}(W)_{(a)} \leq f^{\leftarrow}(1_A)_{(a)} = f^{-1}(A) \Rightarrow f((W)_{(a)}) \leq A,\ i = 1, 2$. Then f is ordinary Pc.

(ii) Let $x \in X,\ f(x) \in A \in [\gamma_i],\ \alpha \in \text{pr}(L) \Rightarrow 1_A \in \gamma_i$ and $1_A(f(x)) \nleq \alpha$. But $f^{\rightarrow}$ is α-FPac $\Rightarrow \exists\, W \in \delta_i$ with $W(x) \nleq \alpha$ by Theorem 1.6. $f^{\rightarrow}(W) \leq \gamma_i\text{-int}\,(\gamma_j\text{-cl}\,(1_A)) = \gamma_i\text{-int}\,(1_{[\gamma_j]}\text{-cl}\,(A)) = 1_{[\gamma_i]}\text{-int}\,([\gamma_j]\text{-cl}\,(A))$.

Let $a \in L \Rightarrow W_{(a)} \in [\delta_i],\ W_{(a)} \leq f^{\leftarrow}f^{\rightarrow}(W)_{(a)} \leq f^{\leftarrow}(1_{[\gamma_i]}\text{-int}\,([\gamma_j]\text{-cl}\,(A)))_{(a)} = f^{-1}([\gamma_i]\text{-int}\,([\gamma_j]\text{-cl}\,(A))) \Rightarrow f(W_{(a)}) \leq [\gamma_i]\text{-int}\,([\gamma_j]\text{-cl}\,(A))$. Then f is ordinary Pac.

Example 3.1. Let $X = R,\ Y = I,\ L = \left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}$. Consider $f : X \rightarrow Y$ defined as

$$f(x) = \begin{cases} 0, & x \leq 0, \\ 1 - x, & 0 < x < 1, \\ 1, & x \geq 1, \end{cases}$$

$\delta_1,\ \delta_2$ on L^X defined as $\delta_1 = \{\underline{0}, U, \underline{1}\},\ \delta_2 = \{\underline{0}, V, \underline{1}\}$, where

$$U(x) = \begin{cases} \frac{4}{5}, & \text{if } x \in (-\infty, -1), \\ \frac{3}{5}, & \text{if } x \in [-1, \infty), \end{cases}$$

$$V(x) = \begin{cases} 1, & \text{if } x \in (-\infty, -1), \\ \frac{2}{5}, & \text{if } x \in [-1, \infty). \end{cases}$$

Then $[\delta_1] = [\delta_2] = \{\emptyset, X\}$, clearly $f : (X, [\delta_1], [\delta_2]) \rightarrow (Y, [\delta_1\ |_Y], [\delta_2\ |_Y])$ is an ordinary pairwise continuous mapping (OPc) but $f^{\rightarrow}$ is not α-FPc at $\alpha = \frac{1}{5}$. And also f is an ordinary Pac mapping but $f^{\rightarrow}$ is not α-FPac.

Theorem 3.4. *If $(L^X, \delta_1, \delta_2)$ is a pairwise induced L-fbts, $Y \subset X$, then the following are equivalent:*

(i) *L^Y is an α-FPR of L^X;*

(ii) *Y is an ordinary α-PR of X;*

(iii) *L^Y is an $\triangle$-FPR of L^X;*

(iv) *Y is an ordinary PR of X.*

Proof. By Theorems 3.2, 3.3 and Proposition 3.3.

Remark 3.1. Let $(L^X, \delta_1, \delta_2)$ be a pairwise weakly induced L-fbts, $(X, [\delta_1], [\delta_2])$ be the pairwise background space of $(L^X, \delta_1, \delta_2)$ and $Y \subset X$. Then we have the following diagram

$$\begin{array}{ccccc}
Y \text{ is an OPR of } X & \Longleftrightarrow & L^Y \text{ is an } \Delta\text{-FPR of } L^X \\
\Uparrow & & \Uparrow \\
L^Y \text{ is an FPR of } L^X & \Longrightarrow & L^Y \text{ is an } \alpha\text{-FPR of } L^X \\
 & & \Uparrow \\
L^Y \text{ is an } \alpha\text{-FPCR of } L^X & \Longrightarrow & L^Y \text{ is an } \alpha\text{-FPSR of } L^X \\
\Downarrow & & \Downarrow \\
L^Y \text{ is an } \alpha\text{-FPAR of } L^X & \Longleftarrow & L^Y \text{ is an } \alpha\text{-F}\delta\text{-PR of } L^X \\
\Downarrow & & \\
Y \text{ is an OPAR of } X, & &
\end{array}$$

O=ordinary.

Theorem 3.5. *Let $(L^X, \delta_1, \delta_2)$ be a pairwise weakly induced L-fbts. Then the following are equivalent:*

(i) *$(L^X, \delta_1, \delta_2)$ is α-PT_2;*

(ii) *The set $A = \{(x, y) : (x, y) \in X \times X,\ x = y\}$ is closed in $(X \times X, [\delta_1 \times \delta_2])$.*

Proof. (i)⇒(ii). Let $(x, y) \in A' \Rightarrow x \neq y$, but L^X is α-PT$_2$ $\Rightarrow \exists\, U \in \delta_1,\ U(x) \nleq \alpha,\ V \in \delta_2, V(y) \nleq \alpha$ and $U \wedge V = \underline{0}$. Since L^X is pairwise weakly induced, we have $U_{(\alpha)} \in [\delta_1],\ V_{(\alpha)} \in [\delta_2]$ and $x \in U_{(\alpha)},\ y \in V_{(\alpha)}$. But $U \wedge V = \underline{0} \Rightarrow U_{(\alpha)} \cap V_{(\alpha)} = \emptyset \Rightarrow \forall\,(x, y) \in U_{(\alpha)} \times V_{(\alpha)} \Rightarrow x \neq y,\ (x, y) \in U_{(\alpha)} \times V_{(\alpha)} \subset A'$, then A' is open and hence A is closed in $(X \times X, [\delta_1 \times \delta_2])$.

(ii)⇒(i). Let $x, y \in X,\ x \neq y \Rightarrow (x, y) \in A'$, but A' is open $\Rightarrow \exists\, G \in [\delta_1],\ H \in [\delta_2]$ and $G \times H \subset A',\ G \cap H = \emptyset \ \Rightarrow 1_G \in \delta_1,\ 1_H \in \delta_2$, for every $\alpha \in \mathrm{pr}(L) \Rightarrow 1_G(x) \nleq \alpha,\ 1_H(y) \nleq \alpha$ and $1_G \wedge 1_H = \underline{0}$.

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