Chin. Ann. Math.
25B:4(2004),545–554.

ARC-TRANSITIVE CUBIC GRAPHS OF ORDER $4p$****

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Abstract

In this paper, a complete classification of arc-transitive cubic graphs of order $4p$ is given.

Keywords Arc-transitive graph, Cubic s-regular graph, Coverings of a graph
2000 MR Subject Classification 05C25, 20B25

§ 1. Introduction and Preliminaries

Throughout this paper graphs are finite, simple and undirected. For a graph X, let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}\,(X)$ denote the vertex set, edge set, arc set and full automorphism group of X, respectively. For $s \geq 1$, an s-arc of X is a sequence $(v_0, v_1, \cdots, v_s)$ of $s+1$ vertices such that $(v_i, v_{i+1}) \in A(X)$ and $v_i \neq v_{i+2}$. Then X is said to be s-arc-transitive if $\mathrm{Aut}\,(X)$ acts transitively on the set of s-arcs of X. In particular, if $\mathrm{Aut}\,(X)$ acts regularly on the s-arcs, then X is called s-regular.

Let G be a finite group and S a subset of G such that $S = S^{-1}$ and $1 \notin S$. Then we define the Cayley graph $X = \mathrm{Cay}\,(G, S)$ of G with respect to S to be the graph with the vertex set $V(X) = G$ and the edge set $E(X) = \{\{g, sg\} \mid g \in G, s \in S\}$.

This note deals with cubic graphs. One of significant work in this topic is due to Tutte who proved in 1947 that every finite cubic symmetric graph is s-regular for some $s \leq 5$ (see [14, 15]). Later, since 1970s, many authors have done a lot of work on arc-transitive cubic graphs. See [3, 7, 9, 12, 13, 16] for example.

This note is an attempt to determine arc-transitive cubic graphs of special orders. Let p be a prime. From [5] we may read off all arc-transitive cubic graphs of order $2p$. We write it as a theorem.

Theorem 1.1. *Let p be a prime. Then every connected arc-transitive cubic graph of order $2p$ is isomorphic to one of the following:*

(1) K_4 *where* $p = 2$;

Manuscript received August 15, 2003.
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****Project supported by the National Natural Science Foundation of China (No.19831050, No.10171086), the Shanxi Provincial Natural Science Foundation of China (No.20011004), the Key Program of the Ministry of Education of China (No.02023), the Returned Abroad-student Fund of Shanxi Province (No.[2002]16) and the Research Fund for the Doctoral Program of Higher Education (No.2000000102).

(2) $K_{3,3}$ *where* $p=3$;

(3) *The Petersen graph* O_3 *where* $p=5$;

(4) *When* $p\equiv 1 \pmod 3$, *the graph* $G(2p,3)$. *This graph has a vertex set* $\mathbb{Z}_p\cup\mathbb{Z}_p'$, *where* $\mathbb{Z}_p=\{0,1,\cdots,p-1\}$ *is a cyclic group of order* p *written additively, and* $\mathbb{Z}_p'=\{0',1',\cdots,(p-1)'\}$ *is another copy of cyclic group of order* p, *and the edge set of this graph is* $\{xy' \bigm| x-y\in H(p,3)\}$, *where* $H(p,3)$ *is the subgroup of order* 3 *of the multiplicative group* $\mathbb{Z}_p^*$ *of* $\mathbb{Z}_p$. *If* $p=7$, *then* $G(2\cdot 7,3)$ *is the Heawood graph.*

The main purpose of this note is to give a classification of arc-transitive cubic graphs of order $4p$, where p is a prime. For our purpose, we mainly use group-theoretical method and a theorem due to Lorimer [11]. Before stating it, we need the concept of block graph of a G-vertex-transitive graph X.

Assume that G acts on $V(X)$ transitively and imprimitively. Assume that $\Sigma=\{B_1,B_2,\cdots,\ B_n\}$ is a complete block system. The block graph $\overline{X}$ with respect to Σ is defined by

$$V(\overline{X})=\Sigma,$$
$$E(\overline{X})=\{\{B_i,B_j\} \bigm| \text{there exists } v_i\in B_i, v_j\in B_j, \text{ such that } \{v_i,v_j\}\in E(X)\}.$$

If N is an intransitive normal subgroup of G, then the orbits of N form a complete block system. This special case is often very useful.

Now we state Lorimer's Theorem.

Theorem 1.2. *Let* X *be a connected arc-transitive graph of valency* p, p *a prime. Let* $G\le \mathrm{Aut}\,(X)$ *act on* X *arc-transitively. And every proper subgroup of* G *does not act arc-transitively on* X. *Let* N *be a maximal semiregular normal subgroup of* G. *Then one of the following is true.*

(1) $N=1$, *and* G *is a nonabelian simple group;*

(2) N *is transitive on* $V(X)$ *and hence* $G=N:\mathbb{Z}_p$, *the semidirect product of* N *and* $\mathbb{Z}_p$;

(3) N *has two orbits on* $V(X)$ *and* X *is bipartite;*

(4) N *has more than two orbits on* $V(X)$ *and* X *is a covering graph of the block graph which has the orbits of* N *as vertices.*

Since we need to determine arc-transitive coverings of a special graph, a new linear criterion in [6] for lifting automorphisms of the base graph to the covering graphs is useful. Now we give a brief description of a special case (only for our application) of this criterion.

Given a graph X, called the base graph, and a group K, called the voltage group, a voltage assignment of X is a function $\phi: A(X)\to K$ with the property that $\phi(u,v)=\phi(v,u)^{-1}$ for each $(u,v)\in A(X)$. The graph $\widetilde{X}=X\times_\phi K$ derived from ϕ is defined by $V(\widetilde{X})=V(X)\times K$ and $E(\widetilde{X})=\{((u,g),(v,\phi(u,v)g)) \bigm| (u,v)\in E(X),\ g\in K\}$. If all the voltages $\phi(u,v)$ generate K, this graph is connected and called a regular covering of X with respect to the voltage assignment ϕ and the group K. Each $\{(u,g) \bigm| g\in A\}$ is called a fibre of $\widetilde{X}$. Moreover, by defining $(u,g')^g:=(u,g'g)$ for any $g\in K$ and $(u,g')\in V(X\times_\phi K)$, K can be identified with a fibre-preserving automorphism subgroup of $\mathrm{Aut}\,(X\times_\phi K)$ acting regularly on each fibre. Let α be an automorphism of X. α can be lifted to an automorphism $\tilde{\alpha}$ of $\widetilde{X}$ if $\pi\tilde{\alpha}=\alpha\pi$, where π is the the first coordinate projection from $\widetilde{X}$ to X. Our problem is to find a necessary and sufficient condition for α to be lifted. For this we should consider all possible voltage assignments.

Assume that X has $n+1$ vertices and $n+1+m$ edges. We name the vertices of X by $\{1,2,\cdots,n+1\}$. An arc (i,j) is called positive if $i<j$; otherwise, it is called negative. We

use $A^+(X)$ and $A^-(X)$ to denote the sets of positive and negative arcs of X, respectively.

Now assume that $K = \mathbb{Z}_p$. Choose a spanning tree T of X, whose edge set is E_0 with $|E_0| = n$. Next choose a co-tree edge e_1 arbitrarily. Let $E_2 = E(X) \setminus (E_0 \cup E_1)$, where $E_1 = \{e_1\}$. Write down the arcs in $A^+(E_0)$, $A^+(E_1)$ and $A^+(E_2)$ in a certain order. Assign the voltages to the above three sets so that $\Phi_0 = \mathbf{0}$, $\Phi_1 = 1$ and $\Phi_2 = M$, where $\{\Phi_i\}$, as n by 1, 1 by 1, and m by 1 matrices, are formed by the voltages assigned to the arcs in $A^+(E_i)$ according to the given order. We call M the generating matrix of the voltage assignment.

For a given spanning tree T, and a positive cotree arc (u, v), there is a unique path from v to u in T which is denoted by $[v, \cdots, u]$. We call the closed walk $(u, [v, \cdots, u])$ the fundamental cycle belonging to (u, v), and denote it by $C(u, v; T)$.

We may define the incidence matrix P for the fundamental cycles of the graph X with respect to the tree T as follows. For each positive cotree arc (u, v), let $\mathbf{p}^{u,v}$ be the n-dimensional row vector over $\mathrm{GF}(p)$ whose (i, j)-coordinate $\mathbf{p}^{u,v}_{i,j}$ indexed by the positive tree arc (i, j) of the given order is defined by

$$\mathbf{p}^{u,v}_{i,j} = \begin{cases} 1, & \text{if } (i,j) \text{ is in } C(u,v;T), \\ -1, & \text{if } (j,i) \text{ is in } C(u,v;T), \\ 0, & \text{otherwise.} \end{cases}$$

Then P is the $(m+1) \times n$ matrix whose row vectors are $\mathbf{p}^{u,v}$, indexed by the positive cotree arcs (u, v) of the given order.

Applying the matrix P, we let $\mathbf{D} = ((-M, I_{m\times m})P, -M, I_{m\times m})$, whose columns are indexed by the arcs in $A^+(E_0), A^+(E_1), A^+(E_2)$ according to the given order. We call the matrix $\mathbf{D}$ the discriminant matrix for a lift of α. For convenience, set $\mathbf{D}_0 = (-M, I_{m\times m})P$, $\mathbf{D}_1 = -M$ and $\mathbf{D}_2 = I_{m\times m}$, so that $\mathbf{D} = (\mathbf{D}_0, \mathbf{D}_1, \mathbf{D}_2)$, as a block matrix.

Let $\mathbf{D} = (\cdots, \mathbf{c}_{i,j}, \cdots)$, where $\mathbf{c}_{i,j}$ is the column indexed by $(i, j) \in A^+(X)$. For a given $\sigma \in \mathrm{Aut}\,(X)$, let $\mathbf{c}^{\sigma}_{i,j} = \mathbf{c}_{i^{\sigma^{-1}}, j^{\sigma^{-1}}}$, where we assume that $\mathbf{c}_{i,j} = -\mathbf{c}_{j,i}$ for any arc (i, j). Let $\mathbf{D}^{\sigma} = (\cdots, \mathbf{c}^{\sigma}_{i,j}, \cdots)$ for any $(i, j) \in A^+(X)$, and let $(\mathbf{D}^{\sigma})_0$, $(\mathbf{D}^{\sigma})_1$ and $(\mathbf{D}^{\sigma})_2$ denote the first, the second and the third blocks of the matrix $\mathbf{D}^{\sigma}$ respectively, as before. Then, by the main theorem in [6], one can say that

$$\alpha \text{ can be lifted} \iff (\mathbf{D}^{\alpha})_1 + (\mathbf{D}^{\alpha})_2 M = \mathbf{0}. \tag{1.1}$$

Now we are in the position to state the main result of this note.

Theorem 1.3. *Let X be a connected arc-transitive cubic graph of order $4p$, p a prime. Then X is one of the following: Q_3, the 3-dimensional cube; D_{20}, the dodecahedron; C_{28}, the Coxeter graph; and $\mathrm{GP}(10, 3)$, the generalized Peterson graph, which is also the standard double cover of Petersen graph.*

§ 2. Proof of Theorem 1.3

By Theorem 1.2, we have four cases. If Case 1 happens, we have

Lemma 2.1. *If $N = 1$, then $G \cong A_5$ or $\mathrm{PSL}(2, 7)$, and $X \cong D_{20}$ or C_{28}.*

Proof. Since G acts arc-transitively on X, by Tutte [14, 15] the stabilizer G_v has order dividing 48, and hence $12p \bigm| |G| \bigm| 2^6 3p$. Thus G is a three-prime simple group. By [10, pp.12–14], G is one of $A_5, A_6, \mathrm{PSL}(2, 7)$, $\mathrm{PSL}(2, 8)$, $\mathrm{PSL}(2, 17)$, $\mathrm{PSL}(3, 3)$, $\mathrm{PSU}(3, 3)$ and $\mathrm{PSU}(4, 2)$. Checking the orders, we have that $G \cong A_5$ for $p = 5$ and $G \cong \mathrm{PSL}(2, 7)$ for $p = 7$.

Case 1. $G \cong A_5$, $|V(X)| = 20$. In this case X is a Sabidussi coset graph $X \cong \mathrm{Sab}\,(G, H, HuH)$ where $|H| = 3$ and u is an involution. Since A_5 has one class of elements of order 3, we may assume that $H = \langle (123) \rangle$. Since X is connected, $G = \langle HuH \rangle = \langle H, u \rangle$. Hence u has the form $(a4)(b5)$; so $u = (14)(25), (24)(15), (14)(35), (34)(15), (24)(35)$, or $(34)(25)$. Since these six involutions are conjugate under $N_{S_5}(H) \cong S_3 \times S_2 = \langle (12), (13) \rangle \times \langle (45) \rangle$. The graph X is unique up to isomorphism. This graph is D_{20}, the Dodecahedron.

Case 2. $G \cong \mathrm{PSL}(2, 7)$, $|V(X)| = 28$. In this case X is a Sabidussi coset graph $X \cong \mathrm{Sab}\,(G, H, HuH)$ where $|H| = 6$ and u is a 2-element. Since $\mathrm{PSL}(2, 7)$ has one class of subgroups of order 3 whose normalizer is isomorphic to S_3. So $\mathrm{PSL}(2, 7)$ has one class of S_3 and $H \cong S_3$. We claim that every subgroup $K \cong S_3$ is contained in two subgroups isomorphic to S_4, whose intersection is K. To prove this we consider the action of $G = \mathrm{PSL}(2, 7) \cong \mathrm{GL}(3, 2)$ on the Fano plane $\mathcal{H}$. Assume that the points of $\mathcal{H}$ are $\{(x, y, z) \mid x, y, z \in \mathrm{GF}(2)\}$ and the lines of $\mathcal{H}$ are $\{[a, b, c] \mid a, b, c \in \mathrm{GF}(2)\}$, and that $(x, y, z) \in [a, b, c] \iff ax + by + cz = 0$. The point and the line stabilizers in G are all isomorphic to S_4. For example, the stabilizer of $(1, 0, 0)$ in G is

$$M = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ u & v & w \\ x & y & z \end{pmatrix} \middle| \, u, v, w, x, y, z \in \mathrm{GF}(2), vz - wy = 1 \right\} \cong S_4, \tag{2.1}$$

and the stabilizer of $[1, 0, 0]$ in G is

$$N = \left\{ \begin{pmatrix} 1 & s & t \\ 0 & v & w \\ 0 & y & z \end{pmatrix} \middle| \, s, t, v, w, y, z \in \mathrm{GF}(2), vz - wy = 1 \right\} \cong S_4. \tag{2.2}$$

It is easy to verify that the intersection of two point stabilizers or two line stabilizers has order 4, and that the intersection of a point stabilizers and a line stabilizer has order 8 if the point and the line are incident; has order 6 (and isomorphic to S_3) if the point and the line are not incident. The above examples M and N has intersection S_3. In fact, we have

$$M \cap N = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & v & w \\ 0 & y & z \end{pmatrix} \middle| \, v, w, y, z \in \mathrm{GF}(2), vz - wy = 1 \right\} \cong S_3. \tag{2.3}$$

Take $H = M \cap N$ and $u = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Let $X = \mathrm{Sab}\,(G, H, HuH)$. We shall show that X is the unique arc-transitive cubic Sabidussi coset graph with respect to G and its subgroup H. Hence it is isomorphic to the Coxeter graph. First, it is easy to check that $o(u) = 4$. Secondly, since M and N are the only maximal subgroups of G containing H, we have $\langle H, u \rangle = G$. Thirdly, it is easy to calculate that $H \cap H^u = \left\{ I, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\} := L$. Since $|L| = 2$, the valency of X is 3. Finally, we show that the only connected orbital graph of valency 3 of G on $\Omega = [G : H]$ is X, where $[G : H]$ is the set of right cosets of H in G. For this, it is easy to see that u has six choices, that is, $u^{\pm 1}$, $b^{-1}u^{\pm 1}b$ and $bu^{\pm 1}b^{-1}$, where b is an element of order 3 in H, and these 6 elements give the same double coset HuH.

If Theorem 1.2(2) happens, we have

Lemma 2.2. *If N is transitive on $V(X)$, then $G = N : \mathbb{Z}_3$ and $X \cong Q_3$.*

Proof. In this case $X = \mathrm{Cay}\,(N, S)$ and $G \leq N_A(N)$, where $A = \mathrm{Aut}\,(X)$. This implies $\mathbb{Z}_3 \leq \mathrm{Aut}\,(N, S)$. Let $\mathbb{Z}_3 = \langle \alpha \rangle$. Since $|N| = 4p$, N is one of the following:

(1) $N = D_{4p}=\langle a, b \mid a^{2p} = b^2 = 1, a^b = a^{-1}\rangle$;
(2) $N = Q_{4p} = \langle a, b \mid a^{2p} = 1, b^2 = a^p, a^b = a^{-1}\rangle$;
(3) $N = \langle a, b \mid a^p = b^4 = 1, a^b = a^k\rangle,\ p \equiv 1 \pmod 4,\ k^2 \equiv -1 \pmod p$;
(4) N is abelian.

Case (1). Since $S^{-1} = S$, S contains an involution. Since α cyclically permutes the three elements in S, so all the elements in S are involutions and $a^p \notin S$. We may assume that $S = \{b, ba^i, ba^j\}$. Since $(a^j)^\alpha = (bba^j)^\alpha = ba^ib = a^{-i}$, we have $o(a^i) = o(a^j)$, and $\langle a\rangle = \langle a^i, a^j\rangle$. It follows that both i and j are coprime to $2p$. Without loss of generality, we may assume $S = \{b, ba, ba^k\}$. On the other hand, $a^\alpha = (bba)^\alpha = baba^k = a^{k-1}$, $(ba^k)^\alpha = baa^{k(k-1)} = b$. So $a^{k^2-k+1} = 1, 2p \mid k^2 - k + 1$, a contradiction.

Case (2). Since the generalized quaternion group has only one involution, $N \neq \langle S\rangle$, and hence X is disconnected. So, no new graph occurs here.

Case (3). If $(b^ia^j)^2 = 1$, then $b^{2j}a^{(k^j+1)i} = 1$, which implies that $j = 2$. This shows that the elements in S has the form b^2a^j. We assume that $S = \{b^2a^i, b^2a^j, b^2a^k\}$. It is easy to check that $N \neq \langle S\rangle$, and hence X is disconnected. Again, no new graph occurs here.

Case (4). Since N is abelian, one has $N \cong \mathbb{Z}_2^3$, and $X \cong Q_3$.

Now assume that Theorem 1.2(3) happens. If $p = 2$, it is easy to see that $X \cong Q_3$, the 3-dimensional cube. If $p \neq 2$, taking the Sylow p-subgroup H of N, since N is semiregular, H is characteristic in N and hence $H \lhd G$. Consider the block graph $\overline{X}$ of X whose vertices are the orbits of H. It is easy to see that $\overline{X}$ is a cycle of size 4. However, X is a covering graph of $\overline{X}$, X should have valency 2, a contradiction.

Finally assume that Theorem 1.2(4) happens. In this case N has more than two orbits on $V(X)$ and X is a covering graph of the block graph $\overline{X}$ given by the orbits of N. Since N is semiregular, $|N| = 2$, 4 or p. If $|N| = 4$, $|V(\overline{X})| = p$ is odd, which implies that the valency of $\overline{X}$ is not 3, a contradiction. If $|N| = p$, $\overline{X} \cong K_4$, X is a covering graph of K_4 with covering transformation group $\mathbb{Z}_p$. Lemma 2.3 will prove that $X \cong Q_3$. If $|N| = 2$, $\overline{X}$ has order $2p$. By Theorem 1.1, $\overline{X} \cong K_4$, $K_{3,3}$, $G(2p, 3)$ or O_3. We should work out all arc-transitive 2-fold coverings of these graphs. For K_4, Lemma 2.3 has done. For $K_{3,3}$ and $G(2p, 3)$, Lemmas 2.4 and 2.5 will prove that there are no such coverings, respectively. Finally for O_3, [6] has determined the only coverings are D_{20}, the dodecahedron, and $P(10, 3)$, the generalized Petersen graph which is also the standard double covering of Petersen graph.

In the next three lemmas, for the unification of symbol usage with the graph covering theory, we use X to denote block graph $\overline{X}$, and use $\widetilde{X}$ to denote the covering graph of the block graph.

Lemma 2.3. *Let $X = K_4$, and the covering transformation group $K = \mathbb{Z}_p$, p a prime. Then the arc-transitive regular covering $\widetilde{X} \cong Q_3$ and $p = 2$.*

Proof. Take a spanning tree T of K_4, see Fig. 1.

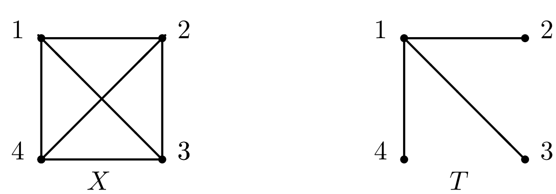

Fig. 1

Let $E_0 = E(T), E = E(X) \setminus E_0$. $A^+(E_0) = \{(1,2), (1,3), (1,4)\}$, $A^+(E) = \{(2,3),$

$(2,4),(3,4)\}$. Then the incidence matrix of fundamental cycles of X with respect to T is

$$P = \begin{array}{c} \\ (2,3) \\ (2,4) \\ (3,4) \end{array} \begin{array}{c} \begin{array}{ccc} (1,2) & (1,3) & (1,4) \end{array} \\ \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \end{array}.$$

Take $\alpha = (12)(34), \beta = (234) \in \mathrm{Aut}\,(X)$. By Equation (1.1) we let $M = (a,b)^t$, $E_1 = \{23\}$ and $E_2 = \{24, 34\}$. Then

$$\mathbf{D}_0 = (-M, I_{2\times 2})P = \begin{array}{c} \begin{array}{ccc} (1,2) & (1,3) & (1,4) \end{array} \\ \begin{pmatrix} 1-a & a, & -1 \\ -b & 1+b & -1 \end{pmatrix} \end{array},$$

$$\mathbf{D} = (\mathbf{D}_0, \mathbf{D}_1, \mathbf{D}_2) = \begin{array}{c} \begin{array}{cccccc} (1,2) & (1,3) & (1,4) & (2,3) & (2,4) & (3,4) \end{array} \\ \begin{pmatrix} 1-a & a, & -1 & -a & 1 & 0 \\ -b & 1+b & -1 & -b & 0 & 1 \end{pmatrix} \end{array}.$$

If arc-transitive coverings exist, then α, β can be lifted. By Equation (1.1),

$$\begin{aligned} \mathbf{0} &= \mathbf{D}_1^{\alpha} + \mathbf{D}_2^{\alpha} M = (c_{23})^{\alpha} + (c_{24}c_{34})^{\alpha} M \\ &= (c_{14}) + (c_{13}c_{43})M \\ &= \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 1+b & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} a^2 - 1 \\ a + ab - b - 1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \mathbf{0} &= \mathbf{D}_1^{\beta} + \mathbf{D}_2^{\beta} M = (c_{23})^{\beta} + (c_{24}c_{34})^{\beta} M \\ &= (c_{34}) + (c_{32}c_{42})M \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} a & -1 \\ b & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} a^2 - b \\ ab + 1 \end{pmatrix}. \end{aligned}$$

So we have $a^2 \equiv 1 \pmod p$, $a^2 \equiv b \pmod p$, and hence $b \equiv 1 \pmod p$; also we have $a + ab - b - 1 \equiv 0 \pmod p$ and $ab + 1 \equiv 0 \pmod p$, we get $a \equiv 3 \pmod p$. It follows from $ab + 1 \equiv 0 \pmod p$ that $4 \equiv 0 \pmod p$. Since p is a prime, we have $p = 2$ and $\widetilde{X} \cong Q_3$.

Lemma 2.4. *Let $X = K_{3,3}$, and the covering transformation group $K = \mathbb{Z}_2$. Then the arc-transitive regular covering $\widetilde{X}$ does not exist.*

Proof. Take a spanning tree T of X; see Fig. 2.

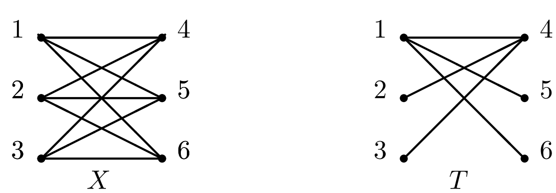

Fig. 2

Let $E_0 = E(T), E = E(X) \setminus E_0$ and let

$$A^+(E_0) = \{(1,4),(1,5),(1,6),(2,4),(3,4)\}, \quad A^+(E) = \{(2,5),(2,6),(3,5),(3,6)\}.$$

Then the incidence matrix of fundamental cycles of X with respect to T is

$$P = \begin{array}{c} (2,5) \\ (2,6) \\ (3,5) \\ (3,6) \end{array} \overset{\begin{array}{ccccc} (1,4) & (1,5) & (1,6) & (2,4) & (3,4) \end{array}}{\begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 \end{pmatrix}}.$$

Take $\alpha = (123) \in \mathrm{Aut}\,(X)$. By Equation (1.1), we let $M = (a,b,c)^t$, $E_1 = \{25\}$ and $E_2 = \{26, 35, 36\}$. Then

$$\mathbf{D}_0 = (-M, I_{3\times 3})P = \overset{\begin{array}{ccccc} (1,4) & (1,5) & (1,6) & (2,4) & (3,4) \end{array}}{\begin{pmatrix} -a+1 & a & -1 & a-1 & 0 \\ -b+1 & b-1 & 0 & b & -1 \\ -c+1 & -c & -1 & c & -1 \end{pmatrix}}.$$

$$\mathbf{D} = (\mathbf{D}_0, \mathbf{D}_1, \mathbf{D}_2) = \overset{\begin{array}{ccccccccc} (1,4) & (1,5) & (1,6) & (2,4) & (3,4) & (2,5) & (2,6) & (3,5) & (3,6) \end{array}}{\begin{pmatrix} -a+1 & a & -1 & a-1 & 0 & -a & 1 & 0 & 0 \\ -b+1 & b-1 & 0 & b & -1 & -b & 0 & 1 & 0 \\ -c+1 & -c & -1 & c & -1 & -c & 0 & 0 & 1 \end{pmatrix}}.$$

If the arc-transitive coverings of X exist, then α can be lifted. By Equation (1.1),

$$\begin{aligned} \mathbf{0} &= \mathbf{D}_1^\alpha + \mathbf{D}_2^\alpha M = (c_{25})^\alpha + (c_{26}c_{35}c_{36})^\alpha M \\ &= (c_{35}) + (c_{36}c_{15}c_{16})M \\ &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & a & -1 \\ 0 & b-1 & 0 \\ 1 & c & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= \begin{pmatrix} ab-c \\ b^2-b+1 \\ a+bc-c \end{pmatrix}. \end{aligned}$$

So $b^2 - b + 1 \equiv 0 \pmod 2$, which is a contradiction.

Lemma 2.5. *Let $X = G(2p,3)$, and the covering transformation group $K = \mathbb{Z}_2$. Then the arc-transitive regular covering $\widetilde{X}$ does not exist.*

Proof. First we assume that $p > 7$. In this case $\mathrm{Aut}\,(X) = (\mathbb{Z}_p : \mathbb{Z}_3) : \mathbb{Z}_2$. If X has an arc-transitive regular covering $\widetilde{X}$, then $\mathrm{Aut}\,(X)$ can be lifted. So the fibre-preserving group G has order $12p$. Since $p \equiv 1 \pmod 3$, G has normal Sylow p-subgroup P. Consider the block graph $\overline{X}$ of $\widetilde{X}$ relative to the orbits of P. Then $\overline{X} = K_4$. By Lemma 2.3, the only arc-transitive covering graph of K_4 with $K = \mathbb{Z}_p$ is Q_3, a contradiction.

Now we assume that $p = 7$. In this case X is the Heawood graph and $A = \mathrm{Aut}\,(X) = \mathrm{PSL}(2,7) : \mathbb{Z}_2$. It follows that A has arc-transitive automorphism group H of order 42, 168 and 336. We shall prove that these three kinds of groups cannot be lifted. If $|H| = 42$ and H can be lifted, then the covering graph $\widetilde{X}$ has an automorphism group of order 84. It follows that the Sylow p-subgroup P of H is normal in H. The argument in the above paragraph gives a contradiction. Now assume that $|H| \geq 168$. We shall use the linear criterion in the above two lemmas.

Let T be a spanning tree of the Heawood graph X; see Fig. 3.

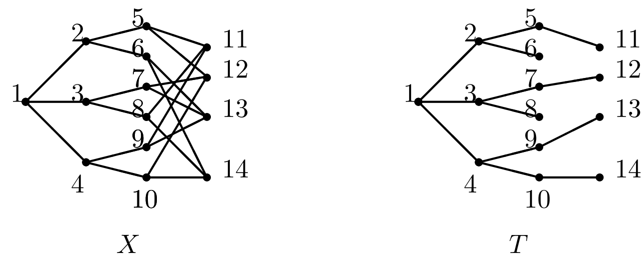

Fig. 3

Let $E_0 = E(T), E = E(X) \setminus E_0$, and let

$$\begin{aligned} A^+(E_0) = \{&(1,2),(1,3),(1,4),(2,5),(2,6),(3,7),(3,8),(4,9),\\ &(4,10),(5,11),(7,12),(9,13),(10,14)\},\\ A^+(E) = \{&(5,12),(6,13),(6,14),(7,13),(8,11),(8,14),(9,11),(10,12)\}. \end{aligned}$$

Then the incidence matrix of fundamental cycles of X with respect to T is

$$P = \begin{array}{c} \\ (5,12)\\ (6,13)\\ (6,14)\\ (7,13)\\ (8,11)\\ (8,14)\\ (9,11)\\ (10,12) \end{array} \begin{array}{ccccccccccccc} (1,2) & (1,3) & (1,4) & (2,5) & (2,6) & (3,7) & (3,8) & (4,9) & (4,10) & (5,11) & (7,12) & (9,13) & (10,14) \end{array} \atop \left(\begin{array}{ccccccccccccc} 1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0\\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1\\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0\\ -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1\\ -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{array}\right).$$

Take

$$\alpha = (2,3,4)(5,8,9)(6,7,10)(14,13,12),$$
$$\beta = (1,7,10,9,6,8,5)(2,3,12,4,13,14,11) \in \mathrm{Aut}\,(X).$$

By Equation (1.1), we let $M = (a,b,c,d,e,f,g)^t$, and

$$E_1 = \{(5,12)\},$$
$$E_2 = \{(6,13),(6,14),(7,13),(8,11),(8,14),(9,11),(10,12)\}.$$

Then

$$\mathbf{D}_0 = (-M, I_{7\times 7})P$$

$$= \begin{array}{ccccccccccccc} (1,2) & (1,3) & (1,4) & (2,5) & (2,6) & (3,7) & (3,8) & (4,9) & (4,10) & (5,11) & (7,12) & (9,13) & (10,14) \end{array} \atop \left(\begin{array}{ccccccccccccc} -a+1 & a & -1 & -a & 1 & a & 0 & -1 & 0 & 0 & a & -1 & 0\\ -b+1 & b & -1 & -b & 1 & b & 0 & 0 & -1 & 0 & b & 0 & -1\\ -c & 1+c & -1 & -c & 0 & 1+c & 0 & -1 & 0 & 0 & c & -1 & 0\\ -d-1 & 1+d & 0 & -d-1 & 0 & d & 1 & 0 & 0 & -1 & d & 0 & 0\\ -e & 1+e & -1 & -e & 0 & e & 1 & 0 & -1 & 0 & e & 0 & -1\\ -f-1 & f & 1 & -f-1 & 0 & f & 0 & 1 & 0 & -1 & f & 0 & 0\\ -g & g-1 & 1 & -g & 0 & g-1 & 0 & 0 & 1 & 0 & g-1 & 0 & 0 \end{array}\right).$$

$$\mathbf{D}_1 = (a,b,c,d,e,f,g)^t,$$

$$\mathbf{D}_2 = I_{7\times 7}.$$

If H can be lifted, then at least α, β can be lifted. By Equation (1.1),

$$\begin{aligned}\mathbf{0} &= \mathbf{D}_1^{\alpha} + \mathbf{D}_2^{\alpha} M \\ &= (c_{(5,12)})^{\alpha} + (c_{(6,13)}c_{(6,14)}c_{(7,13)}c_{(8,11)}c_{(8,14)}c_{(9,11)}c_{(10,12)})^{\alpha} M \\ &= \begin{pmatrix} a^2-e \\ g+ab \\ b-e+ac \\ -f+ad \\ ae+1 \\ d+af-f \\ c+ag-a \end{pmatrix}.\end{aligned}$$

From this equation we have $a = e \equiv 1 \pmod 2$, $d = f \equiv 0 \pmod 2$, and the value of b, c, g is 1 or 0 (mod 2). It follows that

$$\begin{aligned}\mathbf{0} &= \mathbf{D}_1^{\beta} + \mathbf{D}_2^{\beta} M \\ &= (c_{(5,12)})^{\beta} + (c_{(6,13)}c_{(6,14)}c_{(7,13)}c_{(8,11)}c_{(8,14)}c_{(9,11)}c_{(10,12)})^{\beta} M,\end{aligned}$$

$$\begin{pmatrix} -1+g \\ -1-c \\ -1+g \\ b-1 \\ -c \\ -g \\ 1 \end{pmatrix} = \mathbf{0},$$

which is also a contradiction.

§ 3 . Final Remarks

1. When we finished the work in this note, we were told that Feng et. al. [8, Theorem 4.6] also obtained the same result. However, their work heavily relied on a census of arc-transitive cubic graphs (see [2]) which used MAGMA and the aid of a computer. Our work is purely mathematical.

2. The same methods used in this note can also be used in other classification problems of arc-transitive cubic graphs. For example, we also get the following theorem, the proof of it is omitted.

Theorem 3.1. *Let X be a connected arc-transitive cubic graph of order $6p$, p a prime. Then X is one of the following:*

(1) *Two regular coverings of $K_{3,3}$ with covering transformation groups $\mathbb{Z}_3$ and $\mathbb{Z}_p$;*

(2) *A 5-regular cubic graph of order 30 (see [1, p.125]);*

(3) *A vertex-primitive 4-regular cubic graph of order 102 found by Wong (see [16], and also see [1, 18B]).*

Acknowledgement. The work in this paper was done when the first author visited Shanxi Teachers University.

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