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CODIMENSION 3 BIFURCATIONS OF HOMOCLINIC ORBITS WITH ORBIT FLIPS AND INCLINATION FLIPS

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Abstract

The homoclinic bifurcations in four dimensional vector fields are investigated by setting up a local coordinates near the homoclinic orbit. This homoclinic orbit is non-principal in the meanings that its positive semi-orbit takes orbit flip and its unstable foliation takes inclination flip. The existence, nonexistence, uniqueness and coexistence of the 1-homoclinic orbit and the 1-periodic orbit are studied. The existence of the two-fold periodic orbit and three-fold periodic orbit are also obtained.

Keywords Bifurcation, Homoclinic orbit, Orbit flip, Inclination flip, Periodic orbit
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§ 1. Introduction and Hypotheses

Recently, people have obtained many results on the bifurcations of principal homoclinic or heteroclinic orbits in higher dimensional vector fields (see, for example, [1–4, 6–10, 13, 16–18]). Few studies are concerned in the non-principal homoclinic orbits yet. For example, Sandstede [14] investigated codimension-two bifurcations of homoclinic orbits with an orbit flip. Kisaka, et al. [11] studied codimension-two bifurcations of homoclinic orbits with an inclination flip. Homburg and Krauskopf [5] studied codimension-three bifurcations in the case that the resonance and either an orbit or an inclination flip hold simultaneously, and put forward some conjectures. Oldeman, et al. [12] treated these conjectures on codimension-three resonant homoclinic flip bifurcations by numerical techniques. Because of the complexity, these non-principal homoclinic orbits and their associated bifurcations were mainly studied for 3-dimensional systems in the above mentioned references. In this paper, we study the codimension 3 bifurcations of homoclinic orbits with both an orbit flip and an inclination flip in 4-dimensional systems. Consider the following C^r system and its unperturbed system

$$\dot{z} = f(z) + g(z, \mu), \tag{1.1}$$

$$\dot{z} = f(z), \tag{1.2}$$

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where $r \geq 5,\ z \in \mathbf{R}^{\mathbf{4}},\ \mu \in \mathbf{R}^{\mathbf{3}},\ f(0) = 0,\ g(0,\mu) = g(z,0) = 0,\ f,g \in C^r$. Assume that system (1.2) has a homoclinic orbit $\Gamma = \{z = r(t) : t \in \mathbf{R}\},\ r(\pm\infty) = 0$ and the eigenvalues of $D_z f(0)$ are $-\rho_2, -\rho_1, \lambda_1, \lambda_2$, which satisfy

$$-\rho_2 < -\rho_1 < 0 < \lambda_1 < \lambda_2.$$

Denote by W^s and W^u the stable and unstable manifolds of the saddle $O(0,0)$, and $T_o W^{uu}$ and $T_o W^{s-}$ the eigenvectors associated with the eigenvalues λ_2 and $-\rho_1$, respectively. Let $e^{\pm} = \lim\limits_{t\to\pm\infty} \frac{\dot r(-t)}{|\dot r(-t)|}$, and $e^+ \in T_0 W^u, e^- \in T_0 W^s$ are unit eigenvectors corresponding the eigenvalues $\lambda_1, -\rho_2$ respectively. Here, e^- is a unit eigenvector corresponding to the eigenvalue $-\rho_2$ means that Γ enters the critical point O in positive time along the strong stable direction of $T_o W^s$, that is to say, Γ is a homoclinic orbit with orbit flip, and so it is non-principal. Further we need the following hypotheses

(H1) $\dim(T_{r(t)} W^u \cap T_{r(t)} W^s) = 1$.

(H2) $\operatorname{span}(T_{r(t)} W^u, T_{r(t)} W^s, T_0 W^{uu}) = \mathbf{R}^4, \qquad t \gg 1,$
$\operatorname{span}(T_{r(t)} W^u, T_{r(t)} W^s, T_0 W^{s-}) = \mathbf{R}^4, \qquad t \ll -1.$

Hypothesis (H2) is equivalent to

$T_{r(t)} W^u \to e^+ \oplus e^- \qquad \text{as } t \to +\infty,$
$T_{r(t)} W^s \to e^+ \oplus e^- \qquad \text{as } t \to -\infty.$

The latter implies that $T_{r(t)} W^s$ has the strong inclination property (as $t \to -\infty$), and consequently it is principal (see [2]); while the former implies that $T_{r(t)} W^u$ is inclination flip (as $t \to +\infty$), and consequently it is non-principal. With the above assumptions, the homoclinic orbit Γ is codimension-three. Clearly, this kind of homoclinic orbits can occur only in the systems with dimension larger than 3.

§ 2. Preliminary Results and Poincaré Map

Suppose that U is small enough. We can introduce a C^{r-1} change such that system (1.1) has the following normal form in U (see [15]):

$$\begin{cases} \dot x = x(\lambda_1(\mu) + o(1)) + O(u)(O(y) + O(v)), \\ \dot y = y(-\rho_1(\mu) + o(1)) + O(v)(O(x) + O(u)), \\ \dot u = u(\lambda_2(\mu) + o(1)) + O(x)(O(x) + O(y) + O(v)), \\ \dot v = v(-\rho_2(\mu) + o(1)) + O(y)(O(x) + O(y) + O(u)). \end{cases} \tag{2.1}$$

System (2.1) is C^{r-2}.

Denote $A(t) = D_z f(r(t))$. We consider the linear system and its adjoint system

$$\dot z = A(t) z, \tag{2.2}$$

$$\dot \psi = -A^*(t)\psi. \tag{2.3}$$

Let T be the moment such that

$$r(-T) = (\delta, 0, \delta_u, 0), \qquad r(T) = (0, 0, 0, \delta),$$

where $|\delta_u| \ll \delta$ and δ is small enough so that $\{z : |z| < 2\delta\} \subset U$.

Lemma 2.1. *System* (2.2) *has a fundamental solution matrix*

$$Z(t) = (z_1^*(t), z_0^*(t), z_2^*(t), z_3^*(t))$$

satisfying

$$\begin{aligned}
&z_1(t) \in (T_{r(t)}W^u)^c \cap (T_{r(t)}W^s)^c,\\
&z_0(t) = \tfrac{-\dot r(t)}{|\dot r(T)|} \in T_{r(t)}W^u \cap T_{r(t)}W^s,\\
&z_2(t) \in T_{r(t)}W^u \quad \textit{and it is linearly independent of } \ z_0(t),\\
&z_3(t) \in T_{r(t)}W^s \quad \textit{and it is also linearly independent of } \ z_0(t),
\end{aligned}$$

$$Z(-T) = \begin{pmatrix} \omega_{10} & \omega_{00} & 0 & \omega_{30} \\ \omega_{11} & 0 & 0 & \omega_{31} \\ \omega_{12} & \omega_{02} & 1 & \omega_{32} \\ 0 & 0 & 0 & \omega_{33} \end{pmatrix}, \qquad Z(T) = \begin{pmatrix} 0 & 0 & \omega_{20} & 0 \\ \overline{\omega}_{11} & 0 & \omega_{21} & 1 \\ 1 & 0 & \omega_{22} & 0 \\ 0 & 1 & \omega_{23} & 0 \end{pmatrix},$$

where $\omega_{00}, \omega_{11}, \omega_{33}, \omega_{20}$ *are all not equal to zero, and* $\omega_{00} < 0$, $|\overline{\omega}_{11}| \ll 1$, $|\omega_{00}^{-1}\omega_{02}| \ll 1$, $|\omega_{11}^{-1}\omega_{1i}| \ll 1,\ i = 0, 2,\ |\omega_{33}^{-1}\omega_{3i}| \ll 1,\ i = 0, 1, 2,\ |\omega_{20}^{-1}\omega_{2i}| \ll 1,\ i = 1, 3.$

Proof. By the expressions of the local invariant manifolds in U, the values of $z_0(t)$, $z_2(t), z_3(t)$ at $t = \pm T$ and $\omega_{00} < 0$ are clear. Owing to $\frac{\dot r(t)}{|\dot r(t)|} \to e^-$ (as $t \to +\infty$) and $T_{r(T)}W^u \to e^+ \oplus e^-$ (as $t \to +\infty$) in (H2), we know that the weak unstable component of $z_2(T)$ satisfies $\omega_{20} \neq 0$. Similarly, based on $\frac{\dot r(t)}{|\dot r(t)|} \to e^+ \in T_0W^u$ (as $t \to -\infty$) and the hypothesis that $T_{r(t)}W^s$ has the strong inclination property, we know that $z_3(t)$ with $z_3(T) = T_0W^{s-}$ approaches to T_0W^{ss} asymptotically (as $t \to -\infty$), and therefore, $\omega_{33} \neq 0$. Similarly to [18], we first take $\overline{z}_1(t) \in (T_{r(t)}W^u)^c \cap (T_{r(t)}W^s)^c$ such that $\overline{z}_1(T) = (0, 0, 1, 0)$, and $\overline{z}_1(-T) = (\overline{\omega}_{10}, \overline{\omega}_{11}, \overline{\omega}_{12}, \overline{\omega}_{13})$. If $\overline{\omega}_{13} = 0$, then we set $z_1 = \overline{z}_1(t)$. Otherwise, due to $\omega_{33} \neq 0$, we take

$$z_1(t) = \overline{z}_1(t) - \overline{\omega}_{13}\omega_{33}^{-1} z_3(t) \in (T_{r(t)}W^u)^c \cap (T_{r(t)}W^s)^c$$

with $\overline{\omega}_{11} = -\overline{\omega}_{13}\omega_{33}^{-1}$, and

$$z_1(-T) = (\overline{\omega}_{10} - \overline{\omega}_{13}\omega_{33}^{-1}\omega_{30}, \overline{\omega}_{11} - \overline{\omega}_{13}\omega_{33}^{-1}\omega_{31}, \overline{\omega}_{12} - \overline{\omega}_{13}\omega_{33}^{-1}\omega_{32}, 0).$$

According to Liouville's formula, $\det Z(T) \neq 0$ implies $\det Z(-T) \neq 0$, and so $\omega_{11} \neq 0$.

Now we show $|\omega_{33}^{-1}\omega_{3i}| \ll 1$ for $i = 0, 1, 2$. Let T increase to $T + T_1$, then

$$\begin{aligned}
z_3(T + T_1) &= e^{-\rho_1 T_1} z_3(T),\\
z_3(-T - T_1) &= (\omega_{30}e^{-\lambda_1 T_1}, \omega_{31}e^{\rho_1 T_1}, \omega_{32}e^{-\lambda_2 T_1}, \omega_{33}e^{\rho_2 T_1}).
\end{aligned}$$

Reset $z_3(T + T_1) = (0, 1, 0, 0)$. Then it is easy to see that ω_{33} becomes $\omega_{33}e^{(\rho_1+\rho_2)T_1}$ and the new components of $z_3(T + T_1)$ satisfy $|\omega_{33}^{-1}\omega_{3i}| \to 0$ as $T_1 \to +\infty$ for $i = 0, 1, 2$. The remainings can be proved in the same way. Thus the proof is complete.

Denote $\Psi^*(t) = Z^{-1}(t) = (\psi_1^*(t), \psi_0^*(t), \psi_2^*(t), \psi_3^*(t))^*$, which means that $\Psi(t)$ is a fundamental solution matrix of (2.3). Taking the following transformation in the neighborhood of Γ

$$z = r(t) + (z_1(t), z_2(t), z_3(t))(n_1, n_2, n_3) \overset{\text{def}}{=} S(t), \qquad t \in [-T, T],$$

system (1.1) becomes

$$\dot{n}_j(t) = \psi_j^*(t) g(r(t), \mu) + \text{h.o.t.}, \qquad j = 1, 2, 3. \tag{2.4}$$

Equation (2.4) produces a map $P_1 : S_1 \to S_0$, where $S_1 = \{z = S(-T) : |z| < \frac{3}{2}\delta\}$, $S_0 = \{z = S(T) : |z| < \frac{3}{2}\delta\}$. Integrating two sides of Equation (2.4) from $-T$ to T, we get

$$n_j(T) = n_j(-T) + M_j \mu + \text{h.o.t.}, \qquad j = 1, 2, 3, \tag{2.5}$$

where

$$M_j = \int_{-T}^{T} \psi_j^*(t) g_\mu(r(t), 0) dt, \qquad j = 1, 2, 3.$$

Lemma 2.2. $M_j = \displaystyle\int_{-\infty}^{\infty} \psi_j^*(t) g_\mu(r(t), 0) dt, \qquad j = 1, 3.$

Proof. We first have $r(t) = (0, 0, 0, r_4(t))$ as $t \geq T$, where $|r_4(t)| = o(\delta)$. Then Equation (2.1) tells us that $g_\mu(r(t), 0) = (0, 0, 0, g_4(t))$ as $t \geq T$. Further we have

$$A(t) = \begin{pmatrix} \lambda_1 + O(\delta) & 0 & O(\delta) & 0 \\ O(\delta) & -\rho_1 + O(\delta) & O(\delta) & 0 \\ O(\delta) & 0 & \lambda_2 + O(\delta) & 0 \\ O(\delta) & O(\delta) & O(\delta) & -\rho_2 + O(\delta) \end{pmatrix} \qquad as \;\; t \geq T.$$

Based on $\Psi^*(T) Z(T) = I$, we see that the fourth components of $\psi_1(t), \psi_2(t)$ and $\psi_3(t)$ are all zero at $t = T$. It turns out that they are always equal to zero for $t \geq T$ by solving Equation (2.3). Similarly, we have $r(t) = (r_1(t), 0, r_3(t), 0)$, $g_\mu(r(t), 0) = (g_1(t), 0, g_3(t), 0)$ as $t \leq -T$ and

$$A(t) = \begin{pmatrix} \lambda_1 + O(\delta) & O(\delta) & O(\delta) & O(\delta) \\ 0 & -\rho_1 + O(\delta) & 0 & O(\delta) \\ O(\delta) & O(\delta) & \lambda_2 + O(\delta) & O(\delta) \\ 0 & O(\delta) & 0 & -\rho_2 + O(\delta) \end{pmatrix} \qquad \text{as} \;\; t \leq -T.$$

So we can also show that the first and third components of $\psi_j(t)$ $(j = 1, 3)$ are equal to zero for $t \leq -T$. The proof is complete.

Now we consider the map $P_0 : S_0 \to S_1, \quad q_0 \stackrel{\text{def}}{=} (x_0, y_0, u_0, v_0) \mapsto q_1 \stackrel{\text{def}}{=} (x_1, y_1, u_1, v_1)$, which is induced by the flow of system (2.1) in the neighborhood U. Let τ be the time flying from q_0 to q_1 and $s = e^{-\lambda_1(\mu)\tau}$. Omitting all higher terms we get

$$x_0 = x_1 s, \qquad y_1 = s^{\frac{\rho_1}{\lambda_1}} y_0, \qquad u_0 = u_1 s^{\frac{\lambda_2}{\lambda_1}}, \qquad v_1 = s^{\frac{\rho_2}{\lambda_1}} v_0. \tag{2.6}$$

In order to obtain the expression of Poincaré map, we need the relationship between q_0, q_1 and their new coordinates $q_0(n_1^0, n_2^0, n_3^0)$, $q_1(n_1^1, n_2^1, n_3^1)$. Using the following formulas

$$(x_0, y_0, u_0, v_0) = r(T) + z_1(T) n_1^0 + z_2(T) n_2^0 + z_3(T) n_3^0,$$
$$(x_1, y_1, u_1, v_1) = r(-T) + z_1(-T) n_1^1 + z_2(-T) n_2^1 + z_3(-T) n_3^1,$$

and the expressions of $Z(T), Z(-T)$, we have

$$\begin{aligned}
n_1^0 &\approx u_1 s^{\frac{\lambda_2}{\lambda_1}} - \omega_{22}\omega_{20}^{-1}\delta s,\\
n_2^0 &\approx \omega_{20}^{-1}\delta s,\\
n_3^0 &\approx y_0 - \overline{\omega}_{11}u_1 s^{\frac{\lambda_2}{\lambda_1}} + (\omega_{22}\overline{\omega}_{11} - \omega_{21})\omega_{20}^{-1}\delta s,\\
v_0 &\approx \delta + \omega_{23}n_2^0 \approx \delta;
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
n_1^1 &\approx \omega_{11}^{-1} s^{\frac{\rho_1}{\lambda_1}} y_0 - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}},\\
n_2^1 &\approx u_1 - \delta_u - \omega_{12}\omega_{11}^{-1} s^{\frac{\rho_1}{\lambda_1}} y_0 + (\omega_{12}\omega_{11}^{-1}\omega_{31} - \omega_{32})\omega_{33}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}},\\
n_3^1 &\approx \omega_{33}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}},\\
x_1 &\approx \delta + \omega_{10}n_1^1 + \omega_{30}n_3^1 \approx \delta.
\end{aligned} \tag{2.8}$$

Combining Equalities (2.5) and (2.8), and using $n_i^1 = n_i(-T)$ for $i = 1, 2, 3$, we get the Poincaré map $P \overset{\text{def}}{=} P_1 \circ P_0$:

$$\begin{aligned}
n_1(T) &= \omega_{11}^{-1} s^{\frac{\rho_1}{\lambda_1}} y_0 - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}} + M_1\mu + \text{h.o.t.},\\
n_2(T) &= u_1 - \delta_u - \omega_{12}\omega_{11}^{-1} s^{\frac{\rho_1}{\lambda_1}} y_0 + (\omega_{12}\omega_{11}^{-1}\omega_{31} - \omega_{32})\omega_{33}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}} + M_2\mu + \text{h.o.t.},\\
n_3(T) &= \omega_{33}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}} + M_3\mu + \text{h.o.t.}
\end{aligned} \tag{2.9}$$

Now, Equalities (2.7) and (2.9) yield the bifurcation equations

$$G(s, y_0, u_1, \mu) \overset{\text{def}}{=} (G_1, G_2, G_3) \overset{\text{def}}{=} P(q_0) - q_0 = 0,$$

where $G_i = n_i(T) - n_i^0,\ \ i = 1,\ 2,\ 3,$

$$G_1 \overset{\text{def}}{=} \omega_{11}^{-1} s^{\frac{\rho_1}{\lambda_1}} y_0 - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}} - u_1 s^{\frac{\lambda_2}{\lambda_1}} + \omega_{22}\omega_{20}^{-1}\delta s + M_1\mu + \text{h.o.t.}, \tag{2.10}$$

$$G_2 \overset{\text{def}}{=} u_1 - \delta_u - \omega_{12}\omega_{11}^{-1} s^{\frac{\rho_1}{\lambda_1}} y_0 + (\omega_{12}\omega_{11}^{-1}\omega_{31} - \omega_{32})\omega_{33}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}} - \omega_{20}^{-1}\delta s + M_2\mu + \text{h.o.t.}, \tag{2.11}$$

$$G_3 \overset{\text{def}}{=} \omega_{33}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}} - y_0 + \overline{\omega}_{11}u_1 s^{\frac{\lambda_2}{\lambda_1}} - (\omega_{22}\overline{\omega}_{11} - \omega_{21})\omega_{20}^{-1}\delta s + M_3\mu + \text{h.o.t.} \tag{2.12}$$

§ 3. The Main Results and Their Proofs

Assume that all hypotheses in Section 1 are valid. We first consider the existence of the 1-homoclinic orbit. In this case we have $s = 0$ and the equation $G = 0$ becomes

$$\begin{aligned}
M_1\mu + \text{h.o.t.} &= 0,\\
u_1 - \delta_u + M_2\mu + \text{h.o.t.} &= 0,\\
-y_0 + M_3\mu + \text{h.o.t.} &= 0.
\end{aligned}$$

Thus, the following proposition is a direct consequence of the Implicit Function Theorem.

Theorem 3.1. *If $M_1 \neq 0$, then there exists a surface $\Sigma : M_1\mu + o(|\mu|) = 0$, such that there is a unique homoclinic orbit Γ_μ of system* (1.1) *in the neighborhood of Γ for $\mu \in \Sigma$.*

Remark 3.1. If $\mu \in \Sigma \cap \{\mu : M_3\mu \neq 0\}$, then the coordinate y_0 of $\Gamma_\mu \cap S_0$ is not equal to zero. This indicates that it is not along the strong stable direction T_oW^{ss} as Γ_μ entering the origin O. Therefore, Γ_μ is not an orbit-flip homoclinic orbit.

Then, we study the existence and nonexistence of the periodic orbits. By eliminating y_0, u_1 in the second and the third components of $G = 0$, we obtain the following equation

$$\begin{aligned} F(s) \overset{\text{def}}{=} & \big(M_1 + s^{\frac{\lambda_2}{\lambda_1}} M_2 + \omega_{11}^{-1} s^{\frac{\rho_1}{\lambda_1}} M_3\big)\mu + \big[\omega_{22}\omega_{20}^{-1} s - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1} s^{\frac{\rho_2}{\lambda_1}} \\ & + \omega_{11}^{-1}\omega_{33}^{-1} s^{\frac{\rho_1+\rho_2}{\lambda_1}} + (\omega_{21}\omega_{11}^{-1}\omega_{20}^{-1} - \omega_{11}^{-1}\overline{\omega}_{11}\omega_{22}\omega_{20}^{-1}) s^{\frac{\lambda_1+\rho_1}{\lambda_1}} \\ & - \omega_{20}^{-1} s^{\frac{\lambda_1+\lambda_2}{\lambda_1}}\big]\delta - \delta_u s^{\frac{\lambda_2}{\lambda_1}} + \text{h.o.t.} = 0. \end{aligned} \tag{3.1}$$

Theorem 3.2. *Suppose that $M_1 \neq 0$, and $\dfrac{\rho_2}{\lambda_1} < 1$. Then we have*

(1) *If $\omega_{31}\omega_{33}M_3\mu < 0$, $\omega_{11}M_1\mu M_3\mu < 0$ (resp. > 0) and $0 < |\mu| \ll 1$, then system* (1.1) *has a unique (resp. not any)* 1*-periodic orbit near Γ.*

(2) *If $\omega_{31}\omega_{33}M_3\mu > 0$, $\omega_{11}M_1\mu M_3\mu > 0$ and $0 < |\mu| \ll 1$, then system* (1.1) *has a unique* 1*-periodic orbit near Γ.*

(3) *If $\omega_{31}\omega_{33}M_3\mu > 0$, $\omega_{11}M_1\mu M_3\mu < 0$ and $0 < |\mu| \ll 1$, then we have*

system (1.1) *has not any 1-periodic orbits near Γ as $\Delta > 0$ (resp. < 0) and $M_1\mu > 0$ (resp. < 0);*

system (1.1) *has a unique two-fold* 1*-periodic orbit near Γ as $\Delta = 0$;*

system (1.1) *has exactly two* 1*-periodic orbits near Γ as $\Delta < 0$ (resp. > 0) and $M_1\mu < 0$ (resp. > 0),*

where

$$\Delta = M_1\mu + \frac{(\rho_2 - \rho_1)\omega_{31}\delta}{\rho_1\omega_{11}\omega_{33}}\Big(\frac{\rho_1\omega_{33}}{\rho_2\omega_{31}\delta}M_3\mu\Big)^{\frac{\rho_2}{\rho_2-\rho_1}} + h.o.t.$$

Proof. Under the hypotheses of the theorem, we may rewrite (3.1) into the following form

$$M_1\mu + \omega_{11}^{-1}M_3\mu s^{\frac{\rho_1}{\lambda_1}} - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}} + \text{h.o.t.} = 0.$$

Let $s^{\frac{\rho_1}{\lambda_1}} = t$. Then the above equation becomes

$$h(t) \overset{\text{def}}{=} M_1\mu + \omega_{11}^{-1}M_3\mu t - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1}\delta t^{\frac{\rho_2}{\rho_1}} + \text{h.o.t.} = 0. \tag{3.2}$$

For the equation

$$h'(t) = \omega_{11}^{-1}M_3\mu - \frac{\rho_2\delta\omega_{31}}{\rho_1\omega_{11}\omega_{33}} t^{\frac{\rho_2-\rho_1}{\rho_1}} + \text{h.o.t.} = 0,$$

we have a unique small positive solution

$$t = t_0 = \Big(\frac{\rho_1\omega_{33}M_3\mu}{\rho_2\omega_{31}\delta}\Big)^{\frac{\rho_1}{\rho_2-\rho_1}} + \text{h.o.t.}$$

for $\omega_{31}\omega_{33}M_3\mu > 0$, and no small positive solution for $\omega_{31}\omega_{33}M_3\mu < 0$.

(1) It is easy to see that $h(t) \neq 0$ for small $t \in \mathbf{R}^+$, as $\omega_{31}\omega_{33}M_3\mu < 0$ and $\omega_{11}M_1\mu M_3\mu > 0$. If $\omega_{31}\omega_{33}M_3\mu < 0$ and $\omega_{11}M_1\mu M_3\mu < 0$, then we have $h'(t) > 0$ (resp. < 0) for small

$t \in \mathbf{R}^+$, and $h(0) = M_1\mu < 0$ (resp. > 0), $h(\bar{t}) = \omega_{11}^{-1}M_3\mu\bar{t} + \text{h.o.t.} > 0$ (resp. < 0) as $\omega_{11}M_3\mu > 0$ (resp. < 0), where $\bar{t} = \left(\frac{\omega_{11}\omega_{33}M_1\mu}{\omega_{31}\delta}\right)^{\frac{\rho_1}{\rho_2}}$. Therefore, (1) holds.

(2) Without loss of generality, let $M_1\mu > 0$, $\omega_{11}M_3\mu > 0$, $\omega_{11}\omega_{31}\omega_{33} > 0$. Then we have

$$\begin{aligned} h(t_0) &= M_1\mu + \omega_{11}^{-1}M_3\mu t_0 - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1}\delta t_0^{\frac{\rho_2}{\rho_1}} + \text{h.o.t.} \\ &= M_1\mu + \omega_{11}^{-1}\rho_2^{-1}(\rho_2-\rho_1)M_3\mu t_0 + \text{h.o.t.} \\ &= \Delta > 0. \end{aligned}$$

Thus, the straight-line L: $h_1(t) = M_1\mu + \omega_{11}^{-1}M_3\mu t$ intersects the curve C: $h_2(t) = \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1}\delta t^{\frac{\rho_2}{\rho_1}} + \text{h.o.t.}$ at a unique point $t = t' > t_0$. Because $h_1(t') = h_2(t') \to 0$ as $\mu \to 0$, t' is small enough for $|\mu| \ll 1$. (2) holds.

(3) Similarly to (2), we may as well assume $M_1\mu > 0$, $\omega_{11}M_3\mu < 0$, $\omega_{11}\omega_{31}\omega_{33} < 0$. Then we have

$$\begin{aligned} &h'(t_0) = 0, \quad h''(t_0) > 0 \qquad \text{as } |\mu| \ll 1, \qquad h'(t) < 0 \qquad \text{as } t \in (0, t_0), \\ &h(0) > 0, \quad h(t_0) = \Delta. \end{aligned}$$

Hence, if $h(t_0) = 0$, then the straight-line L is tangent to the curve C at point $t = t_0$; If $h(t_0) > 0$, then the straight-line L does not intersect the curve C; If $h(t_0) < 0$, then the straight-line L intersects the curve C at exact two points $t = t_1, t_2$ and $0 < t_1 < t_0 < t_2$. The proof is complete.

Remark 3.2. It is easy to see that the inequality and equality conditions given in Theorem 3.2(1)–(3) are all well defined if $\omega_{31} \neq 0$ and Rank $(M_1, M_3) = 2$. The surface Σ defined by $h(t_0) = \Delta = 0$ is called the two-fold periodic orbit bifurcation surface.

Remark 3.3. If $\dfrac{\rho_2}{\lambda_1} = 1$, then we consider the equation

$$M_1\mu + M_3\mu\omega_{11}^{-1}s^{\frac{\rho_1}{\lambda_1}} + (\omega_{22}\omega_{20}^{-1} - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1})\delta s + \text{h.o.t.} = 0;$$

if $\dfrac{\rho_2}{\lambda_1} > 1 > \dfrac{\rho_1}{\lambda_1}$, then we consider the equation

$$M_1\mu + \omega_{11}^{-1}M_3\mu s^{\frac{\rho_1}{\lambda_1}} + \omega_{22}\omega_{20}^{-1}\delta s + \text{h.o.t.} = 0,$$

and can also obtain some similar results.

Now, we show that system (1.1) may have the three-fold 1-periodic orbit in the following theorem. Set

$$\begin{aligned} B(t) &= (M_1 + \omega_{11}^{-1}M_3 t + M_2 t^{\frac{\lambda_2}{\rho_1}})\mu - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1}\delta t^{\frac{\rho_2}{\rho_1}} + \text{h.o.t.}, \\ p &= \frac{6B'(t_3)}{B'''(t_3)}, \qquad q = \frac{6B(t_3)}{B'''(t_3)}, \qquad t_3 = \Big[\frac{\lambda_2(\lambda_2-\rho_1)\omega_{11}\omega_{33}M_2\mu}{\rho_2(\rho_2-\rho_1)\omega_{31}\delta}\Big]^{\frac{\rho_1}{\rho_2-\lambda_2}} + \text{h.o.t.}, \\ B(t_3) &= M_1\mu + \omega_{11}^{-1}M_3\mu t_3 + \frac{(\rho_2-\lambda_2)(\rho_2+\lambda_2-\rho_1)M_2\mu}{\rho_2(\rho_2-\rho_1)}t_3^{\frac{\lambda_2}{\rho_1}} + \text{h.o.t.}, \\ B'(t_3) &= \omega_{11}^{-1}M_3\mu + \frac{\lambda_2(\rho_2-\lambda_2)M_2\mu}{\rho_1(\rho_2-\rho_1)}t_3^{\frac{\lambda_2-\rho_1}{\rho_1}} + \text{h.o.t.} \\ &= \omega_{11}^{-1}M_3\mu + \frac{\rho_2(\rho_2-\lambda_2)\omega_{31}\delta}{\rho_1(\lambda_2-\rho_1)\omega_{11}\omega_{33}}t_3^{\frac{\rho_2-\rho_1}{\rho_1}} + \text{h.o.t.}, \\ B'''(t_3) &= \lambda_2(\lambda_2-\rho_1)(\lambda_2-\rho_2)\rho_1^{-3}M_2\mu t_3^{\frac{\lambda_2-3\rho_1}{\rho_1}} + \text{h.o.t.} = O\big(|M_2\mu|^{\frac{\rho_2-3\rho_1}{\rho_2-\lambda_2}}\big). \end{aligned}$$

Theorem 3.3. *Suppose that* $\min\{\lambda_1+\rho_1, 3\rho_1\} > \rho_2 > \lambda_2 > \lambda_1 > \rho_1,\ \delta_u = 0,\ \omega_{22} = 0.$

(1) *In the case* $\omega_{11}\omega_{31}\omega_{33}M_2\mu < 0$, *we have*

(1a) *If* $\omega_{11}M_2\mu M_3\mu > 0$, *then system* (1.1) *has a unique* (*resp. not any*) 1*-periodic orbits near* Γ *as* $\omega_{11}M_1\mu M_3\mu < 0$ (*resp.* > 0) *and* $0 < |\mu| \ll 1$.

(1b) *If* $\omega_{11}M_2\mu M_3\mu < 0$ *and* $\omega_{11}M_1\mu M_3\mu > 0$, *then system* (1.1) *has a unique* 1*-periodic orbits near* Γ *as* $0 < |\mu| \ll 1$.

(1c) *If* $\omega_{11}M_2\mu M_3\mu < 0$ *and* $\omega_{11}M_1\mu M_3\mu < 0$, *then*

system (1.1) *has not any* 1*-periodic orbits near* Γ *as* $B(t_0) > 0$ (*resp.* < 0), $M_1\mu > 0$ (*resp.* < 0) *and* $0 < |\mu| \ll 1$;

system (1.1) *has a unique two-fold* 1*-periodic orbits near* Γ *as* $B(t_0) = 0$ *and* $0 < |\mu| \ll 1$;

system (1.1) *has exactly two* 1*-periodic orbits near* Γ *as* $B(t_0) < 0$ (*resp.* > 0), $M_1\mu > 0$ (*resp.* < 0) *and* $0 < |\mu| \ll 1$,

where t_0 *is a unique small positive solution of equation* $B'(t) = 0$.

(2) *In the case* $\omega_{11}\omega_{31}\omega_{33}M_2\mu > 0$, *we have*

(2a) *If* $p > 0$, *then system* (1.1) *has a unique* (*not any*) 1*-periodic orbit near* Γ *as* $pt_3 - q + t_3^3 + h.o.t. > 0$ (*resp.* ≤ 0) *and* $0 < |\mu| \ll 1$.

(2b) *If* $p = 0$, *then*

system (1.1) *has a unique three-fold* 1*-periodic orbit near* Γ *as* $q = 0$ $\big($*that is,* μ *is situated in a codimension* 2 *bifurcation curve* Σ_1 *defined by* $\big[\frac{\lambda_2(\lambda_2-\rho_1)\omega_{11}\omega_{33}M_2\mu}{\rho_2(\rho_2-\rho_1)\omega_{31}\delta}\big]^{\frac{\rho_1}{\rho_2-\lambda_2}} + h.o.t. = \big[\frac{-\rho_1(\lambda_2-\rho_1)\omega_{33}M_3\mu}{\rho_2(\rho_2-\lambda_2)\omega_{31}\delta}\big]^{\frac{\rho_1}{\rho_2-\rho_1}} + h.o.t. = \big[\frac{\lambda_2\rho_1\omega_{11}\omega_{33}M_1\mu}{(\rho_2-\rho_1)(\rho_2-\lambda_2)\omega_{31}\delta}\big]^{\frac{\rho_1}{\rho_2}} + h.o.t.\big)$ *and* $0 < |\mu| \ll 1$;

system (1.1) *has a unique* 1*-periodic orbit near* Γ *as* $q < 0$, *or* $0 < q < t_3^3 + h.o.t.$ *and* $0 < |\mu| \ll 1$;

system (1.1) *has not any* 1*-periodic orbit near* Γ *as* $q \ge t_3^3 + h.o.t.$ *and* $0 < |\mu| \ll 1$, *and has a unique* 1*-homoclinic orbit near* Γ *as* $q = t_3^3 + h.o.t.$ *and* $0 < |\mu| \ll 1$.

(2c) *If* $p < 0$ *and* $t_3 - \sqrt{-\frac{p}{3}} + h.o.t. \le 0$, *then*

system (1.1) *has exactly one* 1*-periodic orbit near* Γ *as* $-t_3^3 + h.o.t. \le pt_3 - q$ *and* $0 < |\mu| \ll 1$;

system (1.1) *has exactly two* 1*-periodic orbits near* Γ *as* $p(t_3 + \sqrt{-\frac{p}{3}}) + \sqrt{-(\frac{p}{3})^3} + h.o.t. < pt_3 - q < -t_3^3 + h.o.t.$ *and* $0 < |\mu| \ll 1$;

system (1.1) *has exactly one two-fold* 1*-periodic orbit near* Γ *as* $pt_3 - q = p(t_3 + \sqrt{-\frac{p}{3}}) + \sqrt{-(\frac{p}{3})^3} + h.o.t.$ *and* $0 < |\mu| \ll 1$;

system (1.1) *has not any* 1*-periodic orbit near* Γ *as* $pt_3 - q < p(t_3 + \sqrt{-\frac{p}{3}}) + \sqrt{-(\frac{p}{3})^3} + h.o.t.$ *and* $0 < |\mu| \ll 1$.

(2d) *If* $p < 0$ *and* $t_3 - \sqrt{-\frac{p}{3}} + h.o.t. > 0$, *then*

system (1.1) *has exactly one* 1*-periodic orbit near* Γ *as* $p(t_3 - \sqrt{-\frac{p}{3}}) - \sqrt{-(\frac{p}{3})^3} + h.o.t. < pt_3 - q$ *and* $0 < |\mu| \ll 1$;

system (1.1) *has exactly one two-fold and one simple* 1*-periodic orbits near* Γ *as* $p(t_3 - \sqrt{-\frac{p}{3}}) - \sqrt{-(\frac{p}{3})^3} + h.o.t. = pt_3 - q$ *and* $0 < |\mu| \ll 1$;

system (1.1) *has exactly three* 1*-periodic orbits near* Γ *as* $-t_3^3 + h.o.t. < pt_3 - q < p(t_3 - \sqrt{-\frac{p}{3}}) - \sqrt{-(\frac{p}{3})^3} + h.o.t.$ *and* $0 < |\mu| \ll 1$;

system (1.1) *has two* 1*-periodic orbits near* Γ *as* $p(t_3 + \sqrt{-\frac{p}{3}}) + \sqrt{-(\frac{p}{3})^3} + h.o.t. < pt_3 - q \le -t_3^3 + h.o.t.$ *and* $0 < |\mu| \ll 1$;

system (1.1) *has one two-fold* 1*-periodic orbit near* Γ *as* $p(t_3 + \sqrt{-\frac{p}{3}}) + \sqrt{-(\frac{p}{3})^3} + h.o.t. = pt_3 - q$ *and* $0 < |\mu| \ll 1$;

system (1.1) *has not any* 1*-periodic orbit near* Γ *as* $pt_3 - q < p(t_3 + \sqrt{-\frac{p}{3}}) + \sqrt{-(\frac{p}{3})^3} +$ *h.o.t. and* $0 < |\mu| \ll 1$.

Proof. Under the hypotheses $\lambda_1 + \rho_1 > \rho_2 > \lambda_2 > \lambda_1 > \rho_1,\ \delta_u = \omega_{22} = 0$, we see that, to solve (3.1), it suffices to solve the following equation

$$\big(M_1 + M_2 s^{\frac{\lambda_2}{\lambda_1}} + \omega_{11}^{-1} M_3 s^{\frac{\rho_1}{\lambda_1}}\big)\mu - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}} + \text{h.o.t.} = 0.$$

Let $t = s^{\frac{\rho_1}{\lambda_1}}$. We obtain

$$B(t) = \big(M_1 + \omega_{11}^{-1} M_3 t + M_2 t^{\frac{\lambda_2}{\rho_1}}\big)\mu - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1}\delta t^{\frac{\rho_2}{\rho_1}} + \text{h.o.t.} = 0, \tag{3.3}$$

and

$$B'(t) = \Big(\frac{\lambda_2}{\rho_1} M_2 t^{\frac{\lambda_2-\rho_1}{\rho_1}} + \omega_{11}^{-1} M_3\Big)\mu - \frac{\rho_2\delta\omega_{31}}{\rho_1\omega_{11}\omega_{33}} t^{\frac{\rho_2-\rho_1}{\rho_1}} + \text{h.o.t.},$$
$$B''(t) = \frac{\lambda_2(\lambda_2-\rho_1)}{\rho_1^2} M_2\mu t^{\frac{\lambda_2-2\rho_1}{\rho_1}} - \frac{\rho_2\delta(\rho_2-\rho_1)\omega_{31}}{\rho_1^2\omega_{11}\omega_{33}} t^{\frac{\rho_2-2\rho_1}{\rho_1}} + \text{h.o.t.}$$

We first consider Case (1). When $M_1\mu, \omega_{11}M_3\mu, M_2\mu, -\omega_{11}\omega_{31}\omega_{33}\delta$ are all positive (or negative), we have $B(t) \neq 0$ for small $t \in \mathbf{R}^+$. When $\omega_{11}M_3\mu, M_2\mu, -\omega_{11}\omega_{31}\omega_{33}\delta$ are all positive (or negative), but $\omega_{11}M_1\mu M_3\mu < 0$, we have $B'(t) \neq 0$ for $t \in \mathbf{R}^+$, and $B(0)B(\hat{t}) = M_1\mu\big(\omega_{11}^{-1}M_3\hat{t} + M_2\hat{t}^{\frac{\lambda_2}{\rho_1}} + \text{h.o.t.}\big)\mu < 0$, where $\hat{t} = \big(\frac{\omega_{11}\omega_{33}M_1\mu}{\omega_{31}\delta}\big)^{\frac{\rho_1}{\rho_2}}$. Therefore (1a) holds.

To prove Subcase (1b), without loss of generality, we assume that $M_1\mu > 0, \omega_{11}M_3\mu > 0, M_2\mu < 0$ and $\omega_{11}\omega_{31}\omega_{33} > 0$. Because $B'(0)B'(\bar{t}) = \omega_{11}^{-1}M_3\mu\big(\frac{\lambda_2}{\rho_1}M_2\mu\bar{t}^{\frac{\lambda_2-\rho_1}{\rho_1}} + \text{h.o.t.}\big) < 0$ and $B''(t) < 0$ for small $t \in \mathbf{R}^+$, the equation $B'(t) = 0$ has a unique small positive solution $t = t_0 \in (0, \bar{t})$, where $\bar{t} = \big(\frac{\rho_1\omega_{33}M_3\mu}{\rho_2\omega_{31}\delta}\big)^{\frac{\rho_1}{\rho_2-\rho_1}}$. Hence, $B'(t) > 0$ for $t \in (0, t_0)$ and $B'(t) < 0$ for $t > t_0$. On the other hand, the equation $(M_1 + \omega_{11}^{-1}M_3 t)\mu - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1}\delta t^{\frac{\rho_2}{\rho_1}} = 0$ has a unique small positive solution $t = \tilde{t}$. (The argument is similar to Case (2) in Theorem 3.2). Thereby, $B(0)B(\tilde{t}) = M_1\mu\big(M_2\mu\tilde{t}^{\frac{\lambda_2}{\rho_1}} + \text{h.o.t.}\big) < 0$. By the continuity of function $B(t)$, Equation (3.3) has a unique small positive solution $t^* \in (t_0, \tilde{t})$. (1b) holds.

For Subcase (1c), we note that $t = t_0$ is a two-fold solution of Equation (3.3) as $B(t_0) = 0$. Thereby, (1c) also holds.

Next, we consider Subcases (2a)–(2d).

Solving the equation $B''(t) = 0$, we get its unique small positive solution

$$t = t_3 = \Big[\frac{\lambda_2(\lambda_2-\rho_1)\omega_{11}\omega_{33}M_2\mu}{\rho_2(\rho_2-\rho_1)\omega_{31}\delta}\Big]^{\frac{\rho_1}{\rho_2-\lambda_2}} + \text{h.o.t.} \qquad \text{as} \ \ \omega_{11}\omega_{31}\omega_{33}M_2\mu > 0.$$

Hence Equation (3.3) is equivalent to

$$\begin{aligned} B(t) &= B(t_3) + B'(t_3)(t-t_3) + \frac{1}{6}B'''(t_3)(t-t_3)^3 + \text{h.o.t.} \\ &= \frac{1}{6}B'''(t_3)[q + p(t-t_3) + (t-t_3)^3 + \text{h.o.t.}] \\ &= 0. \end{aligned} \tag{3.4}$$

Clearly, the zero points of $B(t)$ are corresponding to the intersections of the line L: $H_0(t) = -p(t-t_3) - q$ with the curve C: $H(t) = (t-t_3)^3 + \text{h.o.t.}$ Thus, it is easy to see that

Subcase (2a) is true. To show (2b), we need only to notice that if $B'(t_3) = p = 0$, then we have

$$t_3 = \Big[-\frac{\rho_1(\rho_2-\rho_1)M_3\mu}{\lambda_2(\rho_2-\lambda_2)\omega_{11}M_2\mu}\Big]^{\frac{\rho_1}{\lambda_2-\rho_1}} + \text{h.o.t.} = \Big[-\frac{\rho_1(\lambda_2-\rho_1)\omega_{33}M_3\mu}{\rho_2(\rho_2-\lambda_2)\omega_{31}\delta}\Big]^{\frac{\rho_1}{\rho_2-\rho_1}} + \text{h.o.t.} \stackrel{\text{def}}{=} t_4;$$

if $B(t_3) = B(t_4) = q = 0$, then

$$t_3 = t_4 = \Big[\frac{\lambda_2\rho_1\omega_{11}\omega_{33}M_1\mu}{(\rho_2-\rho_1)(\rho_2-\lambda_2)\omega_{31}\delta}\Big]^{\frac{\rho_1}{\rho_2}} + \text{h.o.t.} \stackrel{\text{def}}{=} t_5.$$

Now we show Subcases (2c) and (2d). Owing to

$$B'''(t_3) = \lambda_2(\lambda_2-\rho_1)(\lambda_2-\rho_2)\rho_1^{-3}M_2\mu t_3^{\frac{\lambda_2-3\rho_1}{\rho_1}} + \text{h.o.t.} = O\big(|M_2\mu|^{\frac{\rho_2-3\rho_1}{\rho_2-\lambda_2}}\big),$$

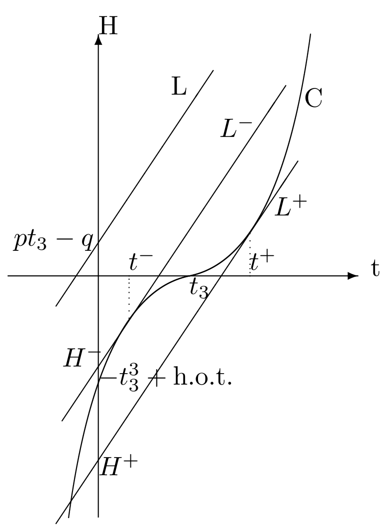
we see that the condition $3\rho_1 > \rho_2$ ensures $|p|, |q| \ll 1$ as $|\mu| \ll 1$. If $p < 0$, then (3.4) implies that $B'(t) = 0$ has exactly two small solutions $t^\pm \approx t_3 \pm\sqrt{-\frac{p}{3}}$ as $|\mu| \ll 1$. It means that the curve C has two tangent lines $\mathrm{L}^\pm$: $H_0^\pm(t) = -p(t-t^\pm) \pm\sqrt{-(\frac{p}{3})^3}$, which are parallel to the line L. The lines $\mathrm{L}^\pm$ are intersect the vertical axis at points $H^\pm(0, pt^\pm \pm\sqrt{-(\frac{p}{3})^3}\,)$, respectively. Moreover, we can show that the point $C_0(0, -t_3^3 + \text{h.o.t.})$ is situated between points H^- and H^+ as $t^- = t_3 -\sqrt{-\frac{p}{3}} > 0$. In fact, if $t_3 > \sqrt{-\frac{p}{3}}$, then $pt^+ +\sqrt{-(\frac{p}{3})^3} = pt_3 - 2\sqrt{-(\frac{p}{3})^3} < pt_3 - 2t_3^3 < -2t_3^3$. Therefore, conclusions of Subcases (2c) and (2d) hold (see the following figure). The proof is complete.

Remark 3.4. If $\delta_u^2 + \omega_{22}^2 \neq 0$, then the bifurcation pattern is the same as in Theorem 3.2.

Remark 3.5. If $\omega_{31} = 0$, then we eventually can find out the lowest order term in Equation (3.1) (under some appropriate assumption, for example, $\omega_{11}^{-1}\omega_{33}^{-1}s^{\frac{\rho_1+\rho_2}{\lambda_1}}$ may be the nonzero lowest order term) and do some similar discussion.

Remark 3.6. The inequality conditions and the bifurcation surfaces given in Theorem 3.3 (1) and (2) are well defined if Rank $(M_1, M_2, M_3) = 3$.

If $M_1 = 0$ or confined on the surface $M_1\mu = 0$, we can obtain the following results concerned with the existence of periodic orbits.



Theorem 3.4. *Suppose that* $M_1\mu = 0$, $\delta_u = 0$ *and* $\rho_1 < \lambda_1 < \lambda_2 < \rho_2 < \lambda_1 + \rho_1$ *hold.*

(1) *In the case* $\omega_{22} \neq 0$, *system* (1.1) *has a unique* (*not any*) 1-*periodic orbit near* Γ *as* $\omega_{11}\omega_{22}\omega_{20}M_3\mu < 0$ (*resp.* > 0) *and* $0 < |\mu| \ll 1$.

(2) *In the case* $\omega_{22} = 0$, *the followings are true.*

(2a) *If* $\omega_{11}\omega_{33}\omega_{31}M_2\mu < 0$, *then system* (1.1) *has a unique* (*not any*) 1-*periodic orbit near* Γ *as* $\omega_{11}M_2\mu M_3\mu < 0$ (*resp.* > 0) *and* $0 < |\mu| \ll 1$.

(2b) *If* $\omega_{11}\omega_{33}\omega_{31}M_2\mu > 0$ *and* $\omega_{11}M_2\mu M_3\mu > 0$, *then system* (1.1) *has a unique* 1-*periodic orbit near* Γ *as* $0 < |\mu| \ll 1$.

(2c) *If* $\omega_{11}\omega_{33}\omega_{31}M_2\mu > 0, \omega_{11}M_2\mu M_3\mu < 0$ *and put*

$$\Delta_1 = \omega_{11}^{-1}M_3\mu + \frac{\omega_{31}\delta(\rho_2 - \lambda_2)}{\omega_{11}\omega_{33}(\lambda_2 - \rho_1)}\Big[\frac{(\lambda_2 - \rho_1)\omega_{11}\omega_{33}M_2\mu}{(\rho_2 - \rho_1)\omega_{31}\delta}\Big]^{\frac{\rho_2 - \rho_1}{\rho_2 - \lambda_2}} + h.o.t.,$$

then we have

system (1.1) *has not any* 1-*periodic orbit near* Γ *as* $\Delta_1 > 0$ (*resp.* < 0), $\omega_{11}M_3\mu > 0$ (*resp.* < 0) *and* $0 < |\mu| \ll 1$;

system (1.1) *has a unique two-fold* 1-*periodic orbit near* Γ *as* $\Delta_1 = 0$ *and* $0 < |\mu| \ll 1$;

system (1.1) *has exactly two* 1-*periodic orbit near* Γ *as* $\Delta_1 < 0$ (*resp.* > 0), $\omega_{11}M_3\mu > 0$ (*resp.* < 0) *and* $0 < |\mu| \ll 1$.

The proof is similar to Theorem 3.2.

Remark 3.7. If $\rho_1 < \rho_2 < \lambda_1 < \lambda_2$ (resp. $\rho_1 < \lambda_1 < \rho_2 < \lambda_2$), then we need study the equation

$$\big(M_2 s^{\frac{\lambda_2}{\lambda_1}} + \omega_{11}^{-1}M_3 s^{\frac{\rho_1}{\lambda_1}}\big)\mu - \omega_{11}^{-1}\omega_{31}\omega_{33}^{-1}s^{\frac{\rho_2}{\lambda_1}}\delta + \text{h.o.t.} = 0$$

$$\big(\text{resp. } \big(M_2 s^{\frac{\lambda_2}{\lambda_1}} + \omega_{11}^{-1}M_3 s^{\frac{\rho_1}{\lambda_1}}\big)\mu + \omega_{22}\omega_{20}^{-1}\delta s + \text{h.o.t.} = 0\big)$$

and can obtain some similar conclusion on the existence and nonexistence of periodic orbit for system (1.1).

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