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WEIERSTRASS REPRESENTATION FOR SURFACES WITH PRESCRIBED NORMAL GAUSS MAP AND GAUSS CURVATURE IN H^3**

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Abstract

The author obtains a Weierstrass representation for surfaces with prescribed normal Gauss map and Gauss curvature in H^3. A differential equation about the hyperbolic Gauss map is also obtained, which characterizes the relation among the hyperbolic Gauss map, the normal Gauss map and Gauss curvature. The author discusses the harmonicity of the normal Gauss map and the hyperbolic Gauss map from surface with constant Gauss curvature in H^3 to S^2 with certain altered conformal metric. Finally, the author considers the surface whose normal Gauss map is conformal and derives a completely nonlinear differential equation of second order which graph must satisfy.

Keywords Hyperbolic space, Hyperbolic Gauss map, Normal Gauss map, Weierstrass representation, Harmonic map
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§ 1 . Introduction

K. Kenmotsu [7] obtained a Weierstrass representation for surfaces of R^3 with prescribed mean curvature and Gauss map, which generalized the classical Weierstrass representation of minimal surfaces in R^3. D. A. Hoffman and R. Osserman [6] gave the Weierstrass representation for surfaces of R^n with prescribed Gauss map.

R. L. Bryant [3] defined the hyperbolic Gauss map for surfaces of H^3 and using the hyperbolic Gauss map he obtained the Weierstrass representation for surfaces of H^3 with constant mean curvature one. Under this framework, many properties for surfaces of H^3 with constant mean curvature one are studied (see [14–16]). M. Kokubu [9] defined the normal Gauss map for surfaces of H^n and obtained Weierstrass representation for minimal surfaces of H^n. Using Kokubu's normal Gauss map, R. Aiyama and K. Akutagawa [1] and the author [10] obtained the Kenmotsu-type Weierstrass representation for surfaces of H^3 with prescribed mean curvature and normal Gauss map independently.

J. A. Gálvez and A. Martínez [4] studied the properties of Gauss map for surfaces of R^3. Particularly, the conformal structure on surfaces is induced by the second fundamental form. Motivated by their method, we give a Weierstrass representation for surfaces with prescribed normal Gauss map and Gauss curvature in H^3 in this paper. Moreover, the

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conformal structure on surfaces is induced by its second fundamental form. We obtain a differential equation about hyperbolic Gauss map which characterizes the relation among the hyperbolic Gauss map, the normal Gauss map and Gauss curvature. We also discuss the case of constant Gauss curvature in which the normal Gauss map and the hyperbolic Gauss map are harmonic map from surfaces to S^2 with certain altered conformal metric. Finally, we discuss the surface whose normal Gauss map is conformal and derive a completely nonlinear differential equation of second order whose graph must satisfy.

§ 2 . Surface Theory of H^3

Take upper half-space model of hyperbolic 3-space $H^3 = \{(x_1, x_2, x_3) \in R^3 : x_3 > 0\}$ with the Riemannian metric $g = \frac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2)$ and constant sectional curvature -1.

Let Σ be a connected 2-dimensional smooth surface and $x : \Sigma \to H^3$ be an immersion of Σ into H^3 with local coordinates u_1, u_2. In the sequel, we agree the following ranges of indices: $1 \leq A, B, \cdots \leq 3, 1 \leq i, j, \cdots \leq 2$. The first and the second fundamental forms of the immersion are written, respectively, as $ds^2 = g_{ij}du_i du_j$ and $h = h_{ij}du_i du_j$. The unit normal vector field of $x(\Sigma)$ in H^3 is

$$\vec{n} = x_3 e_{31}\frac{\partial}{\partial x_1} + x_3 e_{32}\frac{\partial}{\partial x_2} + x_3 e_{33}\frac{\partial}{\partial x_3},$$

where

$$e_{31} = \frac{1}{x_3^2(g_{11}g_{22} - g_{12}^2)^{\frac{1}{2}}}\Big(\frac{\partial x_2}{\partial u_1}\frac{\partial x_3}{\partial u_2} - \frac{\partial x_2}{\partial u_2}\frac{\partial x_3}{\partial u_1}\Big),$$
$$e_{32} = \frac{1}{x_3^2(g_{11}g_{22} - g_{12}^2)^{\frac{1}{2}}}\Big(\frac{\partial x_3}{\partial u_1}\frac{\partial x_1}{\partial u_2} - \frac{\partial x_1}{\partial u_1}\frac{\partial x_3}{\partial u_2}\Big),$$
$$e_{33} = \frac{1}{x_3^2(g_{11}g_{22} - g_{12}^2)^{\frac{1}{2}}}\Big(\frac{\partial x_1}{\partial u_1}\frac{\partial x_2}{\partial u_2} - \frac{\partial x_1}{\partial u_2}\frac{\partial x_2}{\partial u_1}\Big),$$

$$e_{31}^2 + e_{32}^2 + e_{33}^2 = 1.$$

We have the Weingarten formula

$$\frac{\partial e_{3A}}{\partial u_k} = \frac{1}{x_3}\Big(e_{33}\frac{\partial x_A}{\partial u_k} - e_{3B}\frac{\partial x_B}{\partial u_k}\delta_{3A} - h_{jk}g^{jl}\frac{\partial x_A}{\partial u_l}\Big)$$

and the Gauss equation

$$K = -1 + \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}.$$

Identifying H^3 with Lie group

$$H^3 = \left\{ \begin{pmatrix} 1 & 0 & 0 & \log x_3 \\ 0 & x_3 & 0 & x_1 \\ 0 & 0 & x_3 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} : (x_1, x_2, x_3) \in H^3 \right\},$$

the multiplication is defined as matrix multiplication and the unit element is $e = (0, 0, 1)$. It is easy to know that the Riemannian metric is left-invariant and $X_1 = x_3\frac{\partial}{\partial x_1}, X_2 = x_3\frac{\partial}{\partial x_2}, X_3 = x_3\frac{\partial}{\partial x_3}$ are the left-invariant unit orthonormal vector fields. The unit normal vector field of $x(\Sigma)$ can be written as $\vec{n} = e_{31}X_1 + e_{32}X_2 + e_{33}X_3$. Left translating $\vec{n}$ to $T_e(H^3)$, we obtain

$$\tilde{\vec{n}} = L_{x^{-1}*}(\vec{n}) = e_{31}\frac{\partial}{\partial x_1}(e) + e_{32}\frac{\partial}{\partial x_2}(e) + e_{33}\frac{\partial}{\partial x_3}(e) \in S^2(1) \subset T_e(H^3).$$

By the stereographic projection, we get the map $\Sigma \to C\bigcup\{\infty\}$,

$$g_1(x) = \frac{e_{31} + ie_{32}}{1 - e_{33}}, \qquad \tilde{\vec{n}} \in U_1 = S^2(1)\backslash\{N\},$$
$$g_2(x) = \frac{e_{31} - ie_{32}}{1 + e_{33}}, \qquad \tilde{\vec{n}} \in U_2 = S^2(1)\backslash\{S\}.$$

Call g_1 (or g_2) the normal Gauss map of surface $x(\Sigma)$ (see [9, 10]). On $U_1 \bigcap U_2$, $g_1 g_2 = 1$. Throughout the paper, we only consider g_1 and write the normal Gauss map as $g : \Sigma \to C\bigcup\{\infty\}$. Then we have

$$e_{31} = \frac{g + \bar{g}}{1 + |g|^2}, \qquad e_{32} = -i\frac{g - \bar{g}}{1 + |g|^2}, \qquad e_{33} = \frac{|g|^2 - 1}{1 + |g|^2}. \tag{2.1}$$

Next, we define the hyperbolic Gauss map $\widetilde{G}$ (see [2, 11]). The space H^3 is not compact, but can be compactified by adding on an ideal boundary—2-sphere at infinity. We get a compactified space $\overline{H}^3 = H^3 \bigcup S^2_\infty$, where S^2_∞ can be viewed as the set of all geodesic rays with the initial point e. It is easy to know that $S^2_\infty \simeq \{(x_1, x_2, 0) \in R^3\}\bigcup\{\infty\} \simeq H^3(-1)/R^+$. There is a natural complex structure on S^2_∞ with x_1 and x_2, the local conformal parameter, and $x_1 + ix_2$, the local complex coordinate. For the immersion $x : \Sigma \to H^3$, at each point x, the oriented geodesic in H^3 passing through x with tangent vector $\vec{n}$ meets S^2_∞ two points. Since the geodesic is oriented, we may speak of one of the two points as the initial point and the other one as the final point. Call the final point the image of the hyperbolic Gauss map for $x(\Sigma)$ at the point x. Denote the hyperbolic Gauss map by $\widetilde{G}$.

Remark 2.1. By Proposition 34 in Chapter 7 of [13], the surfaces of H^3 with Gauss curvature $K = -1$ is either a totally geodesic or a local ruled surface. In the sequel of the paper, we consider the case of $K \neq -1$.

§ 3 . Surfaces with $K > -1$

Consider an immersion $x : \Sigma \to H^3$ with Gauss curvature $K > -1$. By the Gauss equation, we can choose suitable local coordinates on $x(\Sigma)$ such that the second fundamental form h becomes a positive definite metric on Σ. Σ will be considered as a Riemannian surface with the conformal structure induced by h. Let $z = u + iv$ be a complex coordinate, the first and the second fundamental form may be written as $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ and $h = e(du^2 + dv^2)$, where $e > 0$ and

$$E = \frac{1}{x_3^2}\Big(\Big(\frac{\partial x_1}{\partial u}\Big)^2 + \Big(\frac{\partial x_2}{\partial u}\Big)^2 + \Big(\frac{\partial x_3}{\partial u}\Big)^2\Big),$$
$$F = \frac{1}{x_3^2}\Big(\frac{\partial x_1}{\partial u}\frac{\partial x_1}{\partial v} + \frac{\partial x_2}{\partial u}\frac{\partial x_2}{\partial v} + \frac{\partial x_3}{\partial u}\frac{\partial x_3}{\partial v}\Big),$$
$$G = \frac{1}{x_3^2}\Big(\Big(\frac{\partial x_1}{\partial v}\Big)^2 + \Big(\frac{\partial x_2}{\partial v}\Big)^2 + \Big(\frac{\partial x_3}{\partial v}\Big)^2\Big).$$

Theorem 3.1. *Let $x:\Sigma\to H^3$ be an immersion with the normal Gauss map $g:\Sigma\to C\bigcup\{\infty\}$ and Gauss curvature K satisfying $K>-1$ and $K\neq-\frac{4|g|^2}{(1+|g|^2)^2}$. Then, we have*

$$\frac{\partial x_1}{\partial z}=x_3\Big\{\frac{(1-\bar g^2)g_z}{\big(\sqrt{K+1}+\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}-\frac{(1-g^2)\bar g_z}{\big(\sqrt{K+1}-\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\Big\}, \tag{3.1}$$

$$\frac{\partial x_2}{\partial z}=-ix_3\Big\{\frac{(1+\bar g^2)g_z}{\big(\sqrt{K+1}+\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}+\frac{(1+g^2)\bar g_z}{\big(\sqrt{K+1}-\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\Big\}, \tag{3.2}$$

$$\frac{\partial x_3}{\partial z}=x_3\Big\{\frac{2\bar g g_z}{\big(\sqrt{K+1}+\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}-\frac{2g\bar g_z}{\big(\sqrt{K+1}-\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\Big\}. \tag{3.3}$$

Proof. By the definition of the normal Gauss map g, we get

$$g_z=\frac{1}{(1-e_{33})^2}\Big(\frac{\partial(e_{31}+ie_{32})}{\partial z}-e_{33}\frac{\partial(e_{31}+ie_{32})}{\partial z}+\frac{\partial e_{33}}{\partial z}(e_{31}+ie_{32})\Big), \tag{3.4}$$

$$g_{\bar z}=\frac{1}{(1-e_{33})^2}\Big(\frac{\partial(e_{31}+ie_{32})}{\partial \bar z}-e_{33}\frac{\partial(e_{31}+ie_{32})}{\partial \bar z}+\frac{\partial e_{33}}{\partial \bar z}(e_{31}+ie_{32})\Big). \tag{3.5}$$

By the Weingarten formulas, the Gauss equation and the representation formulas of E, F and G, after a straight computation, we get

$$\begin{aligned}&-e_{33}\frac{\partial(e_{31}+ie_{32})}{\partial z}+\frac{\partial e_{33}}{\partial z}(e_{31}+ie_{32})\\ &=-\frac{1}{x_3}(e_{31}+ie_{32})\Big(e_{31}\frac{\partial x_1}{\partial z}+e_{32}\frac{\partial x_2}{\partial z}\Big)-\frac{e_{33}^2}{x_3}\Big(\frac{\partial x_1}{\partial z}+i\frac{\partial x_2}{\partial z}\Big)+\frac{\sqrt{K+1}}{x_3}\Big(\frac{\partial x_1}{\partial z}+i\frac{\partial x_2}{\partial z}\Big),\end{aligned}$$

$$\begin{aligned}&-e_{33}\frac{\partial(e_{31}+ie_{32})}{\partial \bar z}+\frac{\partial e_{33}}{\partial \bar z}(e_{31}+ie_{32})\\ &=-\frac{1}{x_3}(e_{31}+ie_{32})\Big(e_{31}\frac{\partial x_1}{\partial \bar z}+e_{32}\frac{\partial x_2}{\partial \bar z}\Big)-\frac{e_{33}^2}{x_3}\Big(\frac{\partial x_1}{\partial \bar z}+i\frac{\partial x_2}{\partial \bar z}\Big)-\frac{\sqrt{K+1}}{x_3}\Big(\frac{\partial x_1}{\partial \bar z}+i\frac{\partial x_2}{\partial \bar z}\Big).\end{aligned}$$

Using (2.1) and the above two formulas, from (3.4) and (3.5), we have

$$\begin{aligned}&\Big(\sqrt{K+1}-\frac{2g^2+|g|^4+1}{(1+|g|^2)^2}\Big)\frac{\partial x_1}{\partial z}+i\Big(\sqrt{K+1}+\frac{2g^2-|g|^4-1}{(1+|g|^2)^2}\Big)\frac{\partial x_2}{\partial z}\\ &=\frac{2(g_z+g^2\bar g_z)x_3}{(1+|g|^2)^2},\end{aligned} \tag{3.6}$$

$$\begin{aligned}&-\Big(\sqrt{K+1}+\frac{2\bar g^2+|g|^4+1}{(1+|g|^2)^2}\Big)\frac{\partial x_1}{\partial z}+i\Big(\sqrt{K+1}-\frac{2\bar g^2-|g|^4-1}{(1+|g|^2)^2}\Big)\frac{\partial x_2}{\partial z}\\ &=\frac{2(\bar g_z+\bar g^2 g_z)x_3}{(1+|g|^2)^2}.\end{aligned} \tag{3.7}$$

Solving the above system of equations about $\frac{\partial x_1}{\partial z}$ and $\frac{\partial x_2}{\partial z}$, we obtain (3.1) and (3.2). By $e_{31}\frac{\partial x_1}{\partial z}+e_{32}\frac{\partial x_2}{\partial z}+e_{33}\frac{\partial x_3}{\partial z}=0$, we get

$$\frac{\partial x_3}{\partial z}=\frac{g+\bar g}{1-|g|^2}\frac{\partial x_1}{\partial z}-\frac{i(g-\bar g)}{1-|g|^2}\frac{\partial x_2}{\partial z}. \tag{3.8}$$

Using (3.1), (3.2) and (3.8), we get (3.3).

By using (3.1), (3.2) and (3.3), a straight computation gives

Corollary 3.1. *Under the same conditions as in Theorem 3.1, the first, the second fundamental forms and the mean curvature of the immersion $x:\Sigma\to H^3$ are given, respectively, by*

$$
\begin{aligned}
ds^2 &= \Big(-\frac{4g_z\bar g_z}{K(1+|g|^2)^2+4|g|^2}\Big)dz^2+\Big(-\frac{4\bar g_{\bar z}g_{\bar z}}{K(1+|g|^2)^2+4|g|^2}\Big)d\bar z^2\\
&\quad+\Big(\frac{4|g_z|^2}{[\sqrt{K+1}\,(1+|g|^2)+(|g|^2-1)]^2}\\
&\quad+\frac{4|g_{\bar z}|^2}{[\sqrt{K+1}\,(1+|g|^2)-(|g|^2-1)]^2}\Big)|dz|^2, \qquad (3.9)\\
h &= \sqrt{K+1}\Big(\frac{4|g_{\bar z}|^2}{[\sqrt{K+1}\,(1+|g|^2)-(|g|^2-1)]^2}\\
&\quad-\frac{4|g_z|^2}{[\sqrt{K+1}\,(1+|g|^2)+(|g|^2-1)]^2}\Big)|dz|^2>0, \qquad (3.10)\\
H &= \sqrt{K+1}\,\frac{\frac{|g_{\bar z}|^2}{[\sqrt{K+1}\,(1+|g|^2)-(|g|^2-1)]^2}+\frac{|g_z|^2}{[\sqrt{K+1}\,(1+|g|^2)+(|g|^2-1)]^2}}{\frac{|g_{\bar z}|^2}{[\sqrt{K+1}\,(1+|g|^2)-(|g|^2-1)]^2}-\frac{|g_z|^2}{[\sqrt{K+1}\,(1+|g|^2)+(|g|^2-1)]^2}}. \qquad (3.11)
\end{aligned}
$$

Theorem 3.2. *Let $x:\Sigma\to H^3$ be an immersion with normal Gauss map $g:\Sigma\to C\bigcup\{\infty\}$ and Gauss curvature K satisfy $K>-1$ and $K\neq-\frac{4|g|^2}{(1+|g|^2)^2}$. Then, normal Gauss map g must satisfy*

$$
\begin{aligned}
&4(K+1)\Big[g_{z\bar z}-\frac{2K(1+|g|^2)+4}{K(1+|g|^2)^2+4|g|^2}\bar g g_z g_{\bar z}\Big]\\
&=\frac{\sqrt{K+1}\,(1+|g|^2)+(|g|^2-1)}{\sqrt{K+1}\,(1+|g|^2)-(|g|^2-1)}K_z g_{\bar z}+\frac{\sqrt{K+1}\,(1+|g|^2)-(|g|^2-1)}{\sqrt{K+1}\,(1+|g|^2)+(|g|^2-1)}K_{\bar z} g_z. \qquad (3.12)
\end{aligned}
$$

Proof. From Theorem 3.1, the equation $\frac{\partial}{\partial\bar z}\Big(\frac{\partial x_A}{\partial z}\Big)=\frac{\partial}{\partial z}\Big(\frac{\partial x_A}{\partial\bar z}\Big)$ can be written as

$$
\begin{aligned}
&\frac{2\sqrt{K+1}\,(1-\bar g^2)(1+|g|^2)^2g_{z\bar z}}{K+\frac{4|g|^2}{(1+|g|^2)^2}}-\frac{2\sqrt{K+1}\,(1-g^2)(1+|g|^2)^2\bar g_{z\bar z}}{K+\frac{4|g|^2}{(1+|g|^2)^2}}\\
&+K_z\Big\{\frac{(1-g^2)(1+|g|^2)^2\bar g_{\bar z}}{2\big(\sqrt{K+1}+\frac{|g|^2-1}{1+|g|^2}\big)^2\sqrt{K+1}}-\frac{(1-\bar g^2)(1+|g|^2)^2g_{\bar z}}{2\big(\sqrt{K+1}-\frac{|g|^2-1}{1+|g|^2}\big)^2\sqrt{K+1}}\Big\}\\
&+K_{\bar z}\Big\{\frac{(1-g^2)(1+|g|^2)^2\bar g_{z}}{2\big(\sqrt{K+1}-\frac{|g|^2-1}{1+|g|^2}\big)^2\sqrt{K+1}}-\frac{(1-\bar g^2)(1+|g|^2)^2g_{z}}{2\big(\sqrt{K+1}+\frac{|g|^2-1}{1+|g|^2}\big)^2\sqrt{K+1}}\Big\}\\
&+\frac{2\sqrt{K+1}\,[2K(1+|g|^2)+4]}{\big(K+\frac{4|g|^2}{(1+|g|^2)^2}\big)^2}((g-g^3)\bar g_z\bar g_{\bar z}-(\bar g-\bar g^3)g_zg_{\bar z})=0, \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2\sqrt{K+1}\,(1+\bar g^2)(1+|g|^2)^2g_{z\bar z}}{K+\frac{4|g|^2}{(1+|g|^2)^2}}+\frac{2\sqrt{K+1}\,(1+g^2)(1+|g|^2)^2\bar g_{z\bar z}}{K+\frac{4|g|^2}{(1+|g|^2)^2}}\\
&-K_z\Big\{\frac{(1+g^2)(1+|g|^2)^2\bar g_{\bar z}}{2\big(\sqrt{K+1}+\frac{|g|^2-1}{1+|g|^2}\big)^2\sqrt{K+1}}+\frac{(1+\bar g^2)(1+|g|^2)^2g_{\bar z}}{2\big(\sqrt{K+1}-\frac{|g|^2-1}{1+|g|^2}\big)^2\sqrt{K+1}}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&- K_{\bar z}\Big\{\frac{(1+g^2)(1+|g|^2)^2\bar g_z}{2\big(\sqrt{K+1}-\frac{|g|^2-1}{1+|g|^2}\big)^2\sqrt{K+1}}+\frac{(1+\bar g^2)(1+|g|^2)^2 g_z}{2\big(\sqrt{K+1}+\frac{|g|^2-1}{1+|g|^2}\big)^2\sqrt{K+1}}\Big\}\\
&- \frac{2\sqrt{K+1}\,[2K(1+|g|^2)+4]}{\big(K+\frac{4|g|^2}{(1+|g|^2)^2}\big)^2}((g+g^3)\bar g_z\bar g_{\bar z}+(\bar g+\bar g^3)g_z g_{\bar z})=0, \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{4\sqrt{K+1}\;\bar g(1+|g|^2)^2 g_{z\bar z}}{K+\frac{4|g|^2}{(1+|g|^2)^2}}-\frac{4\sqrt{K+1}\;g(1+|g|^2)^2\bar g_{z\bar z}}{K+\frac{4|g|^2}{(1+|g|^2)^2}}\\
&+ K_z\Big\{\frac{g(1+|g|^2)^2\bar g_{\bar z}}{\big(\sqrt{K+1}+\frac{|g|^2-1}{1+|g|^2}\big)^2\sqrt{K+1}}-\frac{\bar g(1+|g|^2)^2 g_{\bar z}}{\big(\sqrt{K+1}-\frac{|g|^2-1}{1+|g|^2}\big)^2\sqrt{K+1}}\Big\}\\
&+ K_{\bar z}\Big\{\frac{g(1+|g|^2)^2\bar g_z}{\big(\sqrt{K+1}-\frac{|g|^2-1}{1+|g|^2}\big)^2\sqrt{K+1}}-\frac{\bar g(1+|g|^2)^2 g_z}{\big(\sqrt{K+1}+\frac{|g|^2-1}{1+|g|^2}\big)^2\sqrt{K+1}}\Big\}\\
&+ \frac{4\sqrt{K+1}\,[2K(1+|g|^2)+4]}{\big(K+\frac{4|g|^2}{(1+|g|^2)^2}\big)^2}(g^2\bar g_z\bar g_{\bar z}-\bar g^2 g_z g_{\bar z})=0. \qquad (3.15)
\end{aligned}
$$

Adding (3.13), (3.14) and g times (3.15), we obtain (3.12).

By a straight computation, we know that (3.12) is the complete integrability condition for the system (3.1)–(3.3). So we have

Theorem 3.3. *Let Σ be a simply connected Riemannian surface with a reference point z_0 and K be a C^1 real-valued function on Σ satisfying $K > -1$. Let $g : \Sigma \to C\bigcup\{\infty\}$ be a smooth map satisfying* (3.12)*, $K \neq -\frac{4|g|^2}{(1+|g|^2)^2}$ and*

$$
\frac{4|g_{\bar z}|^2}{[\sqrt{K+1}\,(1+|g|^2)-(|g|^2-1)]^2}-\frac{4|g_z|^2}{[\sqrt{K+1}\,(1+|g|^2)+(|g|^2-1)]^2}>0. \qquad (3.16)
$$

Put

$$
\begin{aligned}
x_1 = 2\int_{z_0}^{z} x_3\mathrm{Re}\Big\{\Big(&\frac{(1-\bar g^2)g_z}{\big(\sqrt{K+1}+\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\\
&-\frac{(1-g^2)\bar g_z}{\big(\sqrt{K+1}-\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\Big)dz\Big\}, \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
x_2 = -2\int_{z_0}^{z} x_3\mathrm{Re}\Big\{i\Big(&\frac{(1+\bar g^2)g_z}{\big(\sqrt{K+1}+\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\\
&+\frac{(1+g^2)\bar g_z}{\big(\sqrt{K+1}-\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\Big)dz\Big\}, \qquad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
x_3 = 4\int_{z_0}^{z} x_3\mathrm{Re}\Big\{\Big(&\frac{\bar g g_z}{\big(\sqrt{K+1}+\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\\
&-\frac{g\bar g_z}{\big(\sqrt{K+1}-\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\Big)dz\Big\}. \qquad (3.19)
\end{aligned}
$$

Then $x = (x_1, x_2, x_3)$ is an immersed surface of H^3 with prescribed Gauss curvature K and normal Gauss map g. Moreover, the conformal structure on Σ is induced by the second fundamental form.

Proof. Direct computation may prove that g is the normal Gauss map of $x(\Sigma)$. By (3.9), (3.10) and the Gauss equation, we may prove that K is the Gauss curvature of $x(\Sigma)$.

From Theorems 3.2 and 3.3, we immediately get

Corollary 3.2. *Let Σ be a connected Riemannian surface and $x : \Sigma \to H^3$ be an immersion with the normal Gauss map $g : \Sigma \to C\bigcup\{\infty\}$ and the conformal structure induced by the second fundamental form. If Gauss curvature K is constant satisfying $K > -1$ and $K \neq -\frac{4|g|^2}{(1+|g|^2)^2}$, then the normal Gauss map must satisfy*

$$g_{z\bar{z}} - \frac{2K(1+|g|^2)+4}{K(1+|g|^2)^2+4|g|^2}\bar{g}g_z g_{\bar{z}} = 0. \tag{3.20}$$

That is, g is a harmonic map from Riemannian surface Σ to

$$(S^2,\, c|K(1+|\omega|^2)^2+4|\omega|^2|^{-1}d\omega d\overline{\omega}),$$

where c is an arbitrary positive constant. Conversely, let Σ be a simply connected Riemannian surface with a reference point z_0 and K be a constant on Σ satisfying $K > -1$. Let $g : \Sigma \to C\bigcup\{\infty\}$ be a smooth map satisfying (3.16), (3.20) *and $K \neq -\frac{4|g|^2}{(1+|g|^2)^2}$. Then the immersion $x = (x_1, x_2, x_3) : \Sigma \to H^3$ given by Theorem* 3.3 *is a surface with constant Gauss curvature K and normal Gauss map g. Moreover, the conformal structure on Σ is induced by the second fundamental form.*

Remark 3.1. Strictly speaking, when $-1 < K \leq 0$, $c|K(1+|\omega|^2)^2+4|\omega|^2|^{-1}d\omega d\overline{\omega}$ is not a Riemannian metric on S^2, because it diverges at the points satisfying $K = -\frac{4|\omega|^2}{(1+|\omega|^2)^2}$.

Similarly to the Hopf differential, we can define a quadratic differential on surface with Gauss curvature $K > -1$ (see [8] for the case of R^3):

$$\Phi = \phi dz^2 = \Big(\frac{E-G}{2} - iF\Big)dz^2 = -\frac{8g_z\bar{g}_z}{K(1+|g|^2)^2+4|g|^2}dz^2.$$

From Corollary 3.1, we have $ds^2 = \frac{1}{2}\Phi + \frac{1}{2}\overline{\Phi} + \frac{1}{2}(E+G)|dz|^2$. Φ is a global quadratic differential on surface. The following theorem gives another characterization of the surfaces with constant Gauss curvature $K > -1$.

Theorem 3.4. *Let Σ be a connected Riemannian surface and $x : \Sigma \to H^3$ be an immersion with the normal Gauss map $g : \Sigma \to C\bigcup\{\infty\}$ and Gauss curvature $K > -1$. The conformal structure is induced by the second fundamental form. Then Gauss curvature K is constant if and only if Φ is a holomorphic quadratic differential on Σ.*

Proof. (1) By (3.12), we have, on open set $U = \big\{z \in \Sigma \mid K \neq -\frac{4|g|^2}{(1+|g|^2)^2}\big\}$, that

$$\begin{aligned}&\frac{K+1}{2}\phi_{\bar{z}} - \frac{2g_z\bar{g}_z}{K(1+|g|^2)^2+4|g|^2}K_{\bar{z}}\\ &+\Big\{\frac{|g_{\bar{z}}|^2}{[\sqrt{K+1}\,(1+|g|^2)-(|g|^2-1)]^2} + \frac{|g_z|^2}{[\sqrt{K+1}\,(1+|g|^2)+(|g|^2-1)]^2}\Big\}K_z = 0,\end{aligned}$$

$$\begin{aligned}&\frac{K+1}{2}\bar{\phi}_z - \frac{2\bar{g}_{\bar{z}}g_{\bar{z}}}{K(1+|g|^2)^2+4|g|^2}K_z\\ &+\Big\{\frac{|g_{\bar{z}}|^2}{[\sqrt{K+1}\,(1+|g|^2)-(|g|^2-1)]^2} + \frac{|g_z|^2}{[\sqrt{K+1}\,(1+|g|^2)+(|g|^2-1)]^2}\Big\}K_{\bar{z}} = 0.\end{aligned}$$

On U, the necessity is obvious. Conversely, the above system of linear equations with respect to K_z and $K_{\bar{z}}$ has unique solution $K_z = K_{\bar{z}} = 0$ when Φ is a holomorphic differential.

(2) Assume that K is constant on Σ. Let $V = \Sigma \backslash U$. V is a closed set. If $z_0 \in V$ is a interior point of V, then by the following Section 6, $\Phi \equiv 0$ is holomorphic in an open neighbourhood of z_0. Or else, there exists a sequence of points $z_n \in U$ satisfying $z_n \to z_0$ as $n \to +\infty$. By (1), we know that $\phi_{\bar{z}}(z_n) = 0$. So $\phi_{\bar{z}}(z_0) = 0$. Combining (1) , we get $\phi_{\bar{z}} = 0$ on Σ . So Φ is holomorphic. Similarly, using (1) and the following Theorem 6.4, we can prove the sufficiency.

By Theorem 3.4, we may prove a known result (see [13]) which is similar to the Liebmann's Theorem in R^3.

Theorem 3.5. *A compact connected surface with positive constant Gauss curvature K in H^3 must be a geodesic sphere.*

Proof. By [3], we know that $x(\Sigma)$ is orientable. By Gauss-Bonnet Theorem, we know that Σ has genus zero, i.e., Σ is a topological sphere. So the holomorphic quadratic differential $\Phi \equiv 0$ on Σ. This implies that the first and the second fundamental forms induce the same conformal structure on Σ. Hence $x(\Sigma)$ is totally umbilics and must be a geodesic sphere.

From (3.9) and (3.10), we know that when the normal Gauss map g is anti-holomorphic the first and the second fundamental form induce the same conformal structure on Σ. Then, by the Corollary 3.1 in [10], $x(\Sigma)$ is totally umbilic and by (3.11), its mean curvature $H = \sqrt{K+1}$. By Theorem 29 of Chapter 7 in [12], $x(\Sigma)$ is a geodesic sphere, or a horosphere, or an equidistant surface but not ordinary Euclidean plane.

From Theorem 3.1, we have

Theorem 3.6. (Uniqueness) *Let Σ be a simply connected surface. Two immersions $x = (x_1, x_2, x_3) : \Sigma \to H^3$ and $y = (y_1, y_2, y_3) : \Sigma \to H^3$ have same normal Gauss map $g : \Sigma \to C \bigcup \{\infty\}$ and same Gauss curvature K satisfying $K > -1$ and $K \neq -\frac{4|g|^2}{(1+|g|^2)^2}$. If $x(\Sigma)$ and $y(\Sigma)$ have same conformal structure induced by the second fundamental form, then*

$$x_1 = cy_1 + d_1, \qquad x_2 = cy_2 + d_2, \qquad x_3 = cy_3,$$

where $c > 0$, d_1 and d_2 are arbitrary real constants. In other words, $x(\Sigma)$ differs $y(\Sigma)$ by a vertical hyperbolic translations of H^3.

Remark 3.2. The case of $K = -\frac{4|g|^2}{(1+|g|^2)^2} > -1$ will be discussed in Section 6 below.

§ 4 . Surfaces with $K < -1$

Consider an immersion $x : \Sigma \to H^3$ with Gauss curvature $K < -1$. By the Gauss equation, we see that the second fundamental form can be considered as a Lorentz metric on Σ and Σ can be considered as a Lorentz surface. The first and the second fundamental forms can be written as $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ and $h = 2fdudv$, where $f > 0$ is a function on Σ (see [12]).

As similar as done in Theorem 3.1, we have

Theorem 4.1. *Let $x : \Sigma \to H^3$ be an immersion with normal Gauss map $g : \Sigma \to C\bigcup\{\infty\}$ and Gauss curvature $K < -1$. Then, we have*

$$\frac{\partial x_1}{\partial u} = 2\mathrm{Im}\Big\{\frac{x_3(g^2-1)\bar{g}_u}{\big(\sqrt{-K-1}+i\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\Big\}, \tag{4.1}$$

$$\frac{\partial x_2}{\partial u} = -2\mathrm{Re}\Big\{\frac{x_3(g^2+1)\bar{g}_u}{\big(\sqrt{-K-1}+i\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\Big\}, \tag{4.2}$$

$$\frac{\partial x_3}{\partial u} = -4\mathrm{Im}\Big\{\frac{x_3 g\bar{g}_u}{\big(\sqrt{-K-1}+i\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\Big\}, \tag{4.3}$$

$$\frac{\partial x_1}{\partial v} = 2\mathrm{Im}\Big\{\frac{x_3(\bar{g}^2-1)g_v}{\big(\sqrt{-K-1}+i\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\Big\}, \tag{4.4}$$

$$\frac{\partial x_2}{\partial v} = 2\mathrm{Re}\Big\{\frac{x_3(\bar{g}^2+1)g_v}{\big(\sqrt{-K-1}+i\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\Big\}, \tag{4.5}$$

$$\frac{\partial x_3}{\partial v} = -4\mathrm{Im}\Big\{\frac{x_3\bar{g}g_v}{\big(\sqrt{-K-1}+i\frac{|g|^2-1}{1+|g|^2}\big)(1+|g|^2)^2}\Big\}. \tag{4.6}$$

By using (4.1)–(4.6), a straight computation gives us

Corollary 4.1. *Under the same conditions as in Theorem 4.1, the first, the second fundamental forms and the mean curvature of the immersion $x : \Sigma \to H^3$ are given, respectively, by*

$$\begin{aligned} ds^2 = {} & \Big(-\frac{4|g_u|^2}{K(1+|g|^2)^2+4|g|^2}\Big)du^2 + \Big(-\frac{4|g_v|^2}{K(1+|g|^2)^2+4|g|^2}\Big)dv^2 \\ & -\Big(\frac{4g_u\bar{g}_v}{[\sqrt{-K-1}\,(1+|g|^2)-i(|g|^2-1)]^2} \\ & + \frac{4\bar{g}_u g_v}{[\sqrt{-K-1}\,(1+|g|^2)+i(|g|^2-1)]^2}\Big)dudv, \end{aligned} \tag{4.7}$$

$$\begin{aligned} h = {} & 4i\sqrt{-K-1}\Big(\frac{g_u\bar{g}_v}{[\sqrt{-K-1}\,(1+|g|^2)-i(|g|^2-1)]^2} \\ & - \frac{\bar{g}_u g_v}{[\sqrt{-K-1}\,(1+|g|^2)+i(|g|^2-1)]^2}\Big)dudv, \end{aligned} \tag{4.8}$$

$$H = i\sqrt{-K-1}\,\frac{\frac{\bar{g}_u g_v}{[\sqrt{-K-1}\,(1+|g|^2)+i(|g|^2-1)]^2}+\frac{g_u\bar{g}_v}{[\sqrt{-K-1}\,(1+|g|^2)-i(|g|^2-1)]^2}}{\frac{\bar{g}_u g_v}{[\sqrt{-K-1}\,(1+|g|^2)+i(|g|^2-1)]^2}-\frac{g_u\bar{g}_v}{[\sqrt{-K-1}\,(1+|g|^2)-i(|g|^2-1)]^2}}. \tag{4.9}$$

Using Theorem 4.1, by a similar computation as Theorem 3.2, we get

Theorem 4.2. *Let $x : \Sigma \to H^3$ be an immersion with normal Gauss map $g : \Sigma \to C\bigcup\{\infty\}$ and Gauss curvature $K < -1$. Then the normal Gauss map g must satisfy*

$$\begin{aligned} & 4(K+1)\Big[g_{uv} - \frac{2K(1+|g|^2)+4}{K(1+|g|^2)^2+4|g|^2}\bar{g}g_u g_v\Big] \\ & = \frac{\sqrt{-K-1}\,(1+|g|^2)+i(|g|^2-1)}{\sqrt{-K-1}\,(1+|g|^2)-i(|g|^2-1)}K_v g_u + \frac{\sqrt{-K-1}\,(1+|g|^2)-i(|g|^2-1)}{\sqrt{-K-1}\,(1+|g|^2)+i(|g|^2-1)}K_u g_v. \end{aligned} \tag{4.10}$$

By a straight computation, we know that (4.10) is the complete integrability condition for the system (4.1)–(4.6). So we have

Theorem 4.3. *Let Σ be a simply connected Lorentz surface with a reference point (u_0, v_0) and K be a C^1 real-valued function on Σ satisfying $K < -1$. Let $g : \Sigma \to C \bigcup \{\infty\}$ be a smooth map satisfying* (4.10) *and*

$$i\Big(\frac{g_u\bar{g}_v}{[\sqrt{-K-1}\,(1+|g|^2) - i(|g|^2-1)]^2} - \frac{\bar{g}_u g_v}{[\sqrt{-K-1}\,(1+|g|^2) + i(|g|^2-1)]^2}\Big) > 0. \quad (4.11)$$

Put

$$x_1 = 2\int_{(u_0,v_0)}^{(u,v)} x_3 \mathrm{Im}\Big\{\frac{(g^2-1)\bar{g}_u du + (\bar{g}^2-1)g_v dv}{(\sqrt{-K-1} + i\frac{|g|^2-1}{1+|g|^2})(1+|g|^2)^2}\Big\}, \quad (4.12)$$

$$x_2 = -2\int_{(u_0,v_0)}^{(u,v)} x_3 \mathrm{Re}\Big\{\frac{(g^2+1)\bar{g}_u du - (\bar{g}^2+1)g_v dv}{(\sqrt{-K-1} + i\frac{|g|^2-1}{1+|g|^2})(1+|g|^2)^2}\Big\}, \quad (4.13)$$

$$x_3 = -4\int_{(u_0,v_0)}^{(u,v)} x_3 \mathrm{Im}\Big\{\frac{g\bar{g}_u du + \bar{g}g_v dv}{(\sqrt{-K-1} + i\frac{|g|^2-1}{1+|g|^2})(1+|g|^2)^2}\Big\}. \quad (4.14)$$

Then $x = (x_1, x_2, x_3)$ is an immersed surface in H^3 with prescribed Gauss curvature K and the normal Gauss map g. Moreover, the conformal structure on Σ is induced by the second fundamental form.

From Theorems 4.2 and 4.3, we immediately get

Corollary 4.2. *Let Σ be a connected Lorentz surface and $x : \Sigma \to H^3$ be an immersion with the normal Gauss map $g : \Sigma \to C \bigcup \{\infty\}$ and the conformal structure induced by the second fundamental form. If Gauss curvature K is constant satisfying $K < -1$, then the normal Gauss map g must satisfy*

$$g_{uv} - \frac{2K(1+|g|^2)+4}{K(1+|g|^2)^2 + 4|g|^2}\bar{g}g_u g_v = 0. \quad (4.15)$$

That is, g is a harmonic map from Lorentz surface to

$$(S^2, c|K(1+|\omega|^2)^2 + 4|\omega|^2|^{-1} d\omega d\overline{\omega}),$$

where c is an arbitrary positive constant. Conversely, let Σ be a simply connected Lorentz surface with a reference point (u_0, v_0) and K be a constant on Σ satisfying $K < -1$. Let $g : \Sigma \to C \bigcup \{\infty\}$ be a smooth map satisfying (4.15) *and* (4.11)*, then the immersion $x = (x_1, x_2, x_3) : \Sigma \to H^3$ given by Theorem 4.3 is a surface with constant Gauss curvature K and normal Gauss map g. Moreover, the conformal structure on Σ is induced by the second fundamental form.*

Similarly to the case of $K > -1$, we may define two quadratic differential on surface with Gauss curvature $K < -1$:

$$\Phi_1 = -\frac{4|g_u|^2}{K(1+|g|^2)^2 + 4|g|^2} du^2, \qquad \Phi_2 = -\frac{4|g_v|^2}{K(1+|g|^2)^2 + 4|g|^2} dv^2.$$

Φ_1 and Φ_2 are two global quadratic differential on Σ. Similarly to Theorem 3.4, we have

Theorem 4.4. *Let Σ be a connected Lorentz surface and $x : \Sigma \to H^3$ be an immersion with the normal Gauss map $g : \Sigma \to C \bigcup \{\infty\}$ and Gauss curvature $K < -1$. The conformal structure is induced by the second fundamental form. Then Gauss curvature K is constant if and only if Φ_1 and Φ_2 depend only on u and v respectively.*

From Theorem 4.1, we have

Theorem 4.5. (Uniqueness) *Let Σ be a simply connected surface. Two immersions $x = (x_1, x_2, x_3) : \Sigma \to H^3$ and $y = (y_1, y_2, y_3) : \Sigma \to H^3$ have same normal Gauss map $g : \Sigma \to C \bigcup \{\infty\}$ and same Gauss curvature K satisfying $K < -1$. If $x(\Sigma)$ and $y(\Sigma)$ have same conformal structure induced by the second fundamental form, then*

$$x_1 = cy_1 + d_1, \qquad x_2 = cy_2 + d_2, \qquad x_3 = cy_3,$$

where $c > 0$, d_1 and d_2 are arbitrary real constants. In other words, $x(\Sigma)$ differs $y(\Sigma)$ by a vertical hyperbolic translations of H^3.

§ 5. The Relation of the Hyperbolic Gauss Map, the Normal Gauss Map and Gauss Curvature

For an immersion $x : \Sigma \to H^3$, any oriented geodesic in H^3, passing through x with tangent vector $\vec{n}$ is either the Euclidean half-circle which is orthonormal to the coordinate plane $\{(x_1, x_2, 0) : (x_1, x_2) \in R^2\}$ or the Euclidean straight lines which is orthonormal to the above coordinate plane. By the Euclidean geometry, the coordinate of the center and the radius of the above half-circle are given, respectively, by $\big(x_1 - \frac{x_3(g+\bar{g})(1-|g|^2)}{4|g|^2}, x_2 - \frac{ix_3(\bar{g}-g)(1-|g|^2)}{4|g|^2}, 0\big)$ and $\frac{x_3(1+|g|^2)}{2|g|}$. So, we get the coordinate of hyperbolic Gauss map image

$$\widetilde{G}(u,v) = \Big(x_1(u,v) + \frac{x_3(u,v)(g(u,v) + \bar{g}(u,v))}{2}, x_2(u,v) + \frac{ix_3(u,v)(\bar{g}(u,v) - g(u,v))}{2}, 0\Big).$$

Write $\widetilde{G}(u,v)$ in complex coordinate on $\{(x_1, x_2, 0) : (x_1, x_2) \in R^2\}$, we get

Theorem 5.1. (cf. [1, 11]) *Let Σ be a connected surface and $x = (x_1, x_2, x_3) : \Sigma \to H^3$ be an immersion. Then the normal Gauss map g and the hyperbolic Gauss map $\widetilde{G}$ must satisfy*

$$\widetilde{G} = x_1 + ix_2 + x_3 g.$$

Obviously, $g = \infty$ if and only if $\widetilde{G} = \infty$.

Theorem 5.1 gives a relation between the hyperbolic Gauss map $\widetilde{G}$ and the normal Gauss map g, involving the position function of the surface $x(\Sigma)$. The following result further characterizes the relation among the hyperbolic Gauss map up to its third-order derivative, normal Gauss map g and Gauss curvature K.

Theorem 5.2. *Let Σ be a connected surface and $x : \Sigma \to H^3$ be an immersion with hyperbolic Gauss map $\widetilde{G} : \Sigma \to C \bigcup \{\infty\}$ and normal Gauss map $g : \Sigma \to C \bigcup \{\infty\}$ satisfying $g \neq 0$. Let K be Gauss curvature satisfying $K \neq 0$ and $K \neq -\frac{4|g|^2}{(1+|g|^2)^2}$. Assume that $x(\Sigma)$ has no umbilic. Then*

(1) *In the case of $K > -1$, we have*

$$\left(\frac{8\widetilde{G}_z \widetilde{G}_{\bar{z}}}{4\widetilde{G}_{z\bar{z}} - \frac{KK_z \widetilde{G}_{\bar{z}}}{(K+1)(\sqrt{K+1}-1)^2} - \frac{KK_{\bar{z}} \widetilde{G}_z}{(K+1)(\sqrt{K+1}+1)^2}}\right)_z = \widetilde{G}_z - \frac{(\sqrt{K+1}+1)\overline{\widetilde{G}}_z}{(\sqrt{K+1}-1)\bar{g}^2}, \tag{5.1}$$

$$\left(\frac{8\widetilde{G}_z \widetilde{G}_{\bar{z}}}{4\widetilde{G}_{z\bar{z}} - \frac{KK_z \widetilde{G}_{\bar{z}}}{(K+1)(\sqrt{K+1}-1)^2} - \frac{KK_{\bar{z}} \widetilde{G}_z}{(K+1)(\sqrt{K+1}+1)^2}}\right)_{\bar{z}} = \widetilde{G}_{\bar{z}} - \frac{(\sqrt{K+1}-1)\overline{\widetilde{G}}_{\bar{z}}}{(\sqrt{K+1}+1)\bar{g}^2}. \tag{5.2}$$

(2) *In the case of $K<-1$, we have*

$$\left(\frac{8\widetilde{G}_u\widetilde{G}_v}{4\widetilde{G}_{uv}+\frac{KK_v\widetilde{G}_u}{(K+1)(\sqrt{-K-1}-i)^2}+\frac{KK_u\widetilde{G}_v}{(K+1)(\sqrt{-K-1}+i)^2}}\right)_u=\widetilde{G}_u-\frac{(\sqrt{-K-1}-i)\overline{\widetilde{G}}_u}{(\sqrt{-K-1}+i)\bar{g}^2}, \tag{5.3}$$

$$\left(\frac{8\widetilde{G}_u\widetilde{G}_v}{4\widetilde{G}_{uv}+\frac{KK_v\widetilde{G}_u}{(K+1)(\sqrt{-K-1}-i)^2}+\frac{KK_u\widetilde{G}_v}{(K+1)(\sqrt{-K-1}+i)^2}}\right)_v=\widetilde{G}_v-\frac{(\sqrt{-K-1}+i)\overline{\widetilde{G}}_v}{(\sqrt{-K-1}-i)\bar{g}^2}. \tag{5.4}$$

Proof. (1) Using Theorem 5.1 and Theorem 3.1, we get

$$g_z=\frac{\sqrt{K+1}\,(1+|g|^2)+(|g|^2-1)}{x_3(\sqrt{K+1}+1)(1+|g|^2)}\widetilde{G}_z, \tag{5.5}$$

$$g_{\bar{z}}=\frac{\sqrt{K+1}\,(1+|g|^2)-(|g|^2-1)}{x_3(\sqrt{K+1}-1)(1+|g|^2)}\widetilde{G}_{\bar{z}}. \tag{5.6}$$

Taking derivatives on the two sides of (5.5) with respect to $\bar{z}$ and using (3.3), (5.5) and (5.6), we get

$$\begin{aligned}g_{z\bar{z}}&=\frac{\sqrt{K+1}\,(1+|g|^2)+(|g|^2-1)}{x_3(\sqrt{K+1}+1)(1+|g|^2)}\widetilde{G}_{z\bar{z}}\\&\quad+\frac{4\sqrt{K+1}\,\bar{g}\widetilde{G}_z\widetilde{G}_{\bar{z}}}{x_3^2K(1+|g|^2)^2}+\frac{K_{\bar{z}}\widetilde{G}_z}{x_3\sqrt{K+1}\,(\sqrt{K+1}+1)^2(1+|g|^2)}.\end{aligned}$$

Using (5.5), (5.6) and the above formula, from (3.12), we have

$$\frac{8\widetilde{G}_z\widetilde{G}_{\bar{z}}}{4\widetilde{G}_{z\bar{z}}-\frac{KK_z\widetilde{G}_{\bar{z}}}{(K+1)(\sqrt{K+1}-1)^2}-\frac{KK_{\bar{z}}\widetilde{G}_z}{(K+1)(\sqrt{K+1}+1)^2}}=\frac{x_3(1+|g|^2)}{\bar{g}}. \tag{5.7}$$

Taking derivatives on the two sides of (5.7) with respect to z (resp. $\bar{z}$), using (3.3), (5.5) and (5.6), we obtain (5.1) (resp. (5.2)).

(2) Similarly to (1), we have

$$g_u=\frac{\sqrt{-K-1}\,(1+|g|^2)-i(|g|^2-1)}{x_3(\sqrt{-K-1}-i)(1+|g|^2)}\widetilde{G}_u, \tag{5.8}$$

$$g_v=\frac{\sqrt{-K-1}\,(1+|g|^2)+i(|g|^2-1)}{x_3(\sqrt{-K-1}+i)(1+|g|^2)}\widetilde{G}_v, \tag{5.9}$$

$$\frac{8\widetilde{G}_u\widetilde{G}_v}{4\widetilde{G}_{uv}+\frac{KK_v\widetilde{G}_u}{(K+1)(\sqrt{-K-1}-i)^2}+\frac{KK_u\widetilde{G}_v}{(K+1)(\sqrt{-K-1}+i)^2}}=\frac{x_3(1+|g|^2)}{\bar{g}}. \tag{5.10}$$

Taking derivatives on the two sides of (5.10) with respect to u (resp. v), using (4.3), (5.8) (resp. (4.6), (5.9)), we obtain (5.3) (resp. (5.4)).

By Theorem 5.2, it is easy to get the necessary conditions about hyperbolic Gauss map $\widetilde{G}$ and Gauss curvature K of $x(\Sigma)$.

Corollary 5.1. *Under the same conditions as in Theorem* 5.2*, we have*

(1) *In the case of $K > -1$, we have*

$$\frac{\sqrt{K+1}-1}{(\sqrt{K+1}+1)\overline{\widetilde{G}}_z}\Bigg[\Bigg(\frac{8\widetilde{G}_z\widetilde{G}_{\bar z}}{4\widetilde{G}_{z\bar z}-\frac{KK_z\widetilde{G}_{\bar z}}{(K+1)(\sqrt{K+1}-1)^2}-\frac{KK_{\bar z}\widetilde{G}_z}{(K+1)(\sqrt{K+1}+1)^2}}\Bigg)_z-\widetilde{G}_z\Bigg]$$
$$=\frac{\sqrt{K+1}+1}{(\sqrt{K+1}-1)\overline{\widetilde{G}}_{\bar z}}\Bigg[\Bigg(\frac{8\widetilde{G}_z\widetilde{G}_{\bar z}}{4\widetilde{G}_{z\bar z}-\frac{KK_z\widetilde{G}_{\bar z}}{(K+1)(\sqrt{K+1}-1)^2}-\frac{KK_{\bar z}\widetilde{G}_z}{(K+1)(\sqrt{K+1}+1)^2}}\Bigg)_{\bar z}-\widetilde{G}_{\bar z}\Bigg]. \tag{5.11}$$

(2) *In the case of $K < -1$, we have*

$$\frac{\sqrt{-K-1}+i}{(\sqrt{-K-1}-i)\overline{\widetilde{G}}_u}\Bigg[\Bigg(\frac{8\widetilde{G}_u\widetilde{G}_v}{4\widetilde{G}_{uv}+\frac{KK_v\widetilde{G}_u}{(K+1)(\sqrt{-K-1}-i)^2}+\frac{KK_u\widetilde{G}_v}{(K+1)(\sqrt{-K-1}+i)^2}}\Bigg)_u-\widetilde{G}_u\Bigg]$$
$$=\frac{\sqrt{-K-1}-i}{(\sqrt{-K-1}+i)\overline{\widetilde{G}}_v}\Bigg[\Bigg(\frac{8\widetilde{G}_u\widetilde{G}_v}{4\widetilde{G}_{uv}+\frac{KK_v\widetilde{G}_u}{(K+1)(\sqrt{-K-1}-i)^2}+\frac{KK_u\widetilde{G}_v}{(K+1)(\sqrt{-K-1}+i)^2}}\Bigg)_v-\widetilde{G}_v\Bigg]. \tag{5.12}$$

Corollary 5.2. *Under the same conditions as in Theorem* 5.2*, we have*

(1) *In the case of $K > -1$, $\widetilde{G}$ satisfies the following linear elliptic differential equation of second order,*

$$\widetilde{G}_{z\bar z}-\left[\frac{Kg^2}{4\sqrt{K+1}}\left(\frac{\sqrt{K+1}-1}{(\sqrt{K+1}+1)g^2}\right)_{\bar z}\right]\widetilde{G}_z$$
$$+\left[\frac{Kg^2}{4\sqrt{K+1}}\left(\frac{\sqrt{K+1}+1}{(\sqrt{K+1}-1)g^2}\right)_z\right]\widetilde{G}_{\bar z}=0, \tag{5.13}$$

(2) *In the case of $K < -1$, $\widetilde{G}$ satisfies the following linear hyperbolic differential equation of second order,*

$$\widetilde{G}_{uv}+\left[\frac{iKg^2}{4\sqrt{-K-1}}\left(\frac{\sqrt{-K-1}+i}{(\sqrt{-K-1}-i)g^2}\right)_v\right]\widetilde{G}_u$$
$$-\left[\frac{iKg^2}{4\sqrt{-K-1}}\left(\frac{\sqrt{-K-1}-i}{(\sqrt{-K-1}+i)g^2}\right)_u\right]\widetilde{G}_v=0. \tag{5.14}$$

Corollary 5.3. (Uniqueness) *Let Σ be a simply connected surface. Given two umbilic-free immersions $x=(x_1,x_2,x_3):\Sigma\to H^3$ and $y=(y_1,y_2,y_3):\Sigma\to H^3$. The hyperbolic Gauss map, the normal Gauss map and the Gauss curvature of $x(\Sigma)$ and $y(\Sigma)$ are, respectively, $\widetilde{G}_1$, $\widetilde{G}_2$, $g_1(\neq 0)$, $g_2(\neq 0)$, $K_1\big(\neq 0,\ -\frac{4|g_1|^2}{(1+|g_1|^2)^2}\big)$ and $K_2\big(\neq 0,\ -\frac{4|g_2|^2}{(1+|g_2|^2)^2}\big)$. Assume that $x(\Sigma)$ and $y(\Sigma)$ have same conformal structure induced by the second fundamental form. If $\widetilde{G}_1=\widetilde{G}_2$ and $K_1=K_2$ on Σ, then $x\equiv y$ on Σ.*

Proof. From Theorem 5.2 and (5.7) (or (5.10)), we get $g_1=g_2$ on Σ. So, by Theorem 3.6 and Theorem 4.5, we have

$$x_1=cy_1+d_1,\qquad x_2=cy_2+d_2,\qquad x_3=cy_3,$$

where $c>0$, d_1 and d_2 are arbitrary real constants. By Theorem 5.1, we get, on Σ, that

$$\begin{aligned}\widetilde{G}_1&=x_1+ix_2+x_3g_1=(cy_1+d_1)+i(cy_2+d_2)+cy_3g_2\\&=c(y_1+iy_2+y_3g_2)+(d_1+id_2)=c\widetilde{G}_2+(d_1+id_2)\\&\equiv\widetilde{G}_2.\end{aligned}$$

By (5.5) and (5.6) (or (5.8) and (5.9)), we know that $\widetilde{G}_2$ is non-constant. So, we get $c = 1$ and $d_1 = d_2 = 0$. We have proved $x \equiv y$ on Σ.

Corollary 5.4. *Let Σ be a simply connected surface. Given smooth map $g : \Sigma \to C\bigcup\{\infty\}$ satisfying $g \neq 0$ and a C^1 real-valued function K on Σ satisfying $K \neq 0$, $K \neq -1$ and $K \neq -\frac{4|g|^2}{(1+|g|^2)^2}$. If $K > -1$ (resp. $K < -1$) and g satisfies (3.12) and (3.16) (resp. (4.10) and (4.11)), then $\widetilde{G} = x_1 + ix_2 + x_3 g$ is a solution of system of nonlinear partial differential equation (5.1) and (5.2) (resp. (5.3) and (5.4)), where x_1, x_2, x_3 are given by (3.17)–(3.19) (resp. (4.12)–(4.14)). And $c\widetilde{G} + (d_1 + id_2)$ is still a solution of (5.1) and (5.2) (resp. (5.3) and (5.4)), where $c \neq 0$, d_1 and d_2 are arbitrary real constants. When K is a C^2-function on Σ, $c\widetilde{G} + (d_1 + id_2)$ is also a solution of (5.13) (resp. (5.14)). Finally, $c\widetilde{G} + (d_1 + id_2)$ is the hyperbolic Gauss map of an immersion $(cx_1 + d_1, cx_2 + d_2, |c|x_3) : \Sigma \to H^3$.*

Remark 5.1. Theorem 5.2 implies that the normal Gauss map can be determined by the hyperbolic Gauss map and Gauss curvature. Conversely, can the hyperbolic Gauss map be determined by normal Gauss map and Gauss curvature? Or, is the solution of system of Equations (5.1) and (5.2) (or (5.3) and (5.4)) about the hyperbolic Gauss map $\widetilde{G}$ unique in the sense of $c\widetilde{G} + (d_1 + id_2)$? where $c \neq 0$, d_1 and d_2 are arbitrary real constants. An affirmative answer to the problem would give the Weierstrass representation for surfaces with prescribed hyperbolic Gauss map and Gauss curvature in H^3.

Theorem 5.3. *Let Σ be a simply connected surface. Let $x : \Sigma \to H^3$ be an immersion without umbilic. The normal Gauss map and Gauss curvature K satisfy $g \neq 0$, $K \neq 0$, and $K \neq -\frac{4|g|^2}{(1+|g|^2)^2}$. Assume that the hyperbolic Gauss map $\widetilde{G} : \Sigma \to C\bigcup\{\infty\}$ is a diffeomorphism from Σ to $\widetilde{G}(\Sigma)$ and the conformal structure on Σ is induced by the second fundamental form. Then*

(1) *If Gauss curvature K is constant and $K > -1$, then*

$$\widetilde{G}_{z\bar{z}} - \frac{2\bar{g}}{x_3(1+|g|^2)}\widetilde{G}_z\widetilde{G}_{\bar{z}} = 0, \tag{5.15}$$

i.e. $\widetilde{G}$ is a harmonic diffeomorphism from Riemannian surface Σ to

$$\Big(S^2, c\cdot\exp\Big\{-2\int\frac{\mathrm{Re}(\bar{g}(\widetilde{G}^{-1}(\omega))d\omega)}{x_3(\widetilde{G}^{-1}(\omega))(1+|g(\widetilde{G}^{-1}(\omega))|^2)}\Big\}d\omega d\overline{\omega}\Big),$$

where c is an arbitrary positive constant and the integrals are taken along a path from a fixed point to a variable point belonging to $\widetilde{G}(\Sigma)$.

(2) *If Gauss curvature K is constant and $K < -1$, then*

$$\widetilde{G}_{uv} - \frac{2\bar{g}}{x_3(1+|g|^2)}\widetilde{G}_u\widetilde{G}_v = 0, \tag{5.16}$$

i.e. $\widetilde{G}$ is a harmonic diffeomorphism from Lorentz surface Σ to

$$\Big(S^2, c\cdot\exp\Big\{-2\int\frac{\mathrm{Re}(\bar{g}(\widetilde{G}^{-1}(\omega))d\omega)}{x_3(\widetilde{G}^{-1}(\omega))(1+|g(\widetilde{G}^{-1}(\omega))|^2)}\Big\}d\omega d\overline{\omega}\Big),$$

where c and integral path are the same as in (1).

Proof. (1) By (5.7), we get (5.15). It is well known that $\widetilde{G}:(\Sigma,|dz|^2)\to(S^2,\mu^2 d\omega d\overline{\omega})$ is harmonic if and only if $\widetilde{G}$ satisfies

$$\widetilde{G}_{z\bar{z}}+\frac{2}{\mu}\frac{\partial\mu}{\partial\omega}\widetilde{G}_z\widetilde{G}_{\bar{z}}=0.$$

After checking the integrability of the equation

$$\frac{2}{\mu}\frac{\partial\mu}{\partial\omega}=-\frac{2\bar{g}(\widetilde{G}^{-1}(\omega))}{x_3(\widetilde{G}^{-1}(\omega))(1+|g(\widetilde{G}^{-1}(\omega))|^2)},$$

we have proved (1).

(2) Note that $\widetilde{G}:(\Sigma,2f\,dudv)\to(S^2,\mu^2 d\omega d\overline{\omega})$ is harmonic if and only if $\widetilde{G}$ satisfies

$$\widetilde{G}_{uv}+\frac{2}{\mu}\frac{\partial\mu}{\partial\omega}\widetilde{G}_u\widetilde{G}_v=0.$$

The proof of (2) is similar to that of (1).

Remark 5.2. Is this condition about diffeomorphism necessary for hyperbolic Gauss map to be a harmonic map from surfaces with constant Gauss curvature to S^2 with an altered conformal metric?

§ 6. Surfaces with $K=-\frac{4|g|^2}{(1+|g|^2)^2}>-1$

The Weierstrass representation formula in Section 3 is unadaptable to the surfaces with $K=-\frac{4|g|^2}{(1+|g|^2)^2}>-1$. In this section, by Theorem 5.1, using hyperbolic Gauss map $\widetilde{G}$ and normal Gauss map g, we give the Weierstrass representation formula for surfaces with $K=-\frac{4|g|^2}{(1+|g|^2)^2}>-1$.

Let Σ be a connected Riemannian surface and $x:\Sigma\to H^3$ be an immersion with local complex coordinates $z=u+iv$. The first and the second fundamental forms are given, respectively, by $ds^2=Edu^2+2Fdudv+Gdv^2$ and $h=e(du^2+dv^2)$, where $e>0$.

If normal Gauss map $g:\Sigma\to C\cup\{\infty\}$ is constant, then $x(\Sigma)$ is either an equidistant surface or a horosphere $x_3=\text{const.}>0$ which are ordinary Euclidean planes and satisfy $K=-\frac{4|g|^2}{(1+|g|^2)^2}>-1$. And they are only these surfaces with constant Gauss curvature $K=-\frac{4|g|^2}{(1+|g|^2)^2}>-1$. From now on, assume that normal Gauss map g is nonconstant. It is easy to see that when $|g|>1$, $\sqrt{K+1}=\frac{|g|^2-1}{1+|g|^2}$ and when $|g|<1$, $\sqrt{K+1}=\frac{1-|g|^2}{1+|g|^2}$.

Theorem 6.1. *Let Σ be connected Riemannian surface and $x:\Sigma\to H^3$ be an immersion with Gauss curvature K and normal Gauss map $g:\Sigma\to C\cup\{\infty\}$ satisfying $K=-\frac{4|g|^2}{(1+|g|^2)^2}>-1$. Then*

(1) *When $|g|>1$, we have $g_{\bar{z}}=0$, i.e. $g:\Sigma\to C\cup\{\infty\}$ is holomorphic map, and*

$$\frac{\partial x_1}{\partial z}=\frac{(1-\bar{g}^2)\widetilde{G}_z}{2(1+|g|^2)|g|^2}-\frac{(g^2-1)\overline{\widetilde{G}}_z}{2(1+|g|^2)},\tag{6.1}$$

$$\frac{\partial x_2}{\partial z}=-i\Big\{\frac{(1+\bar{g}^2)\widetilde{G}_z}{2(1+|g|^2)|g|^2}-\frac{(g^2+1)\overline{\widetilde{G}}_z}{2(1+|g|^2)}\Big\},\tag{6.2}$$

$$\frac{\partial x_3}{\partial z}=\frac{\bar{g}\widetilde{G}_z}{(1+|g|^2)|g|^2}+\frac{g\overline{\widetilde{G}}_z}{(1+|g|^2)}.\tag{6.3}$$

(2) *When $|g| < 1$, we have $g_z = 0$, i.e. $g : \Sigma \to C \cup \{\infty\}$ is an antiholomorphic map, and*

$$\frac{\partial x_1}{\partial z} = \frac{(1-g^2)\overline{\widetilde{G}}_z}{2(1+|g|^2)|g|^2} - \frac{(\bar{g}^2-1)\widetilde{G}_z}{2(1+|g|^2)}, \tag{6.4}$$

$$\frac{\partial x_2}{\partial z} = i\Big\{\frac{(1+g^2)\overline{\widetilde{G}}_z}{2(1+|g|^2)|g|^2} - \frac{(\bar{g}^2+1)\widetilde{G}_z}{2(1+|g|^2)}\Big\}, \tag{6.5}$$

$$\frac{\partial x_3}{\partial z} = \frac{g\overline{\widetilde{G}}_z}{(1+|g|^2)|g|^2} + \frac{\bar{g}\widetilde{G}_z}{(1+|g|^2)}. \tag{6.6}$$

Proof. (1) When $|g| > 1$, (3.6) and (3.7) become

$$-(1+g^2)\frac{\partial x_1}{\partial z} + i(g^2-1)\frac{\partial x_2}{\partial z} = x_3(g_z + g^2\bar{g}_z),$$
$$-(1+g^2)\frac{\partial x_1}{\partial z} + i(g^2-1)\frac{\partial x_2}{\partial z} = x_3\Big(g_z + \frac{\bar{g}_z}{\bar{g}^2}\Big).$$

By the above two formula, we get $g_{\bar{z}} = 0$ and

$$-(1+g^2)\frac{\partial x_1}{\partial z} + i(g^2-1)\frac{\partial x_2}{\partial z} = x_3 g_z. \tag{6.7}$$

By Theorem 5.1, (3.8) and (6.7), we get

$$\widetilde{G}_z = \frac{\partial x_1}{\partial z} + i\frac{\partial x_2}{\partial z} + g\frac{\partial x_3}{\partial z} + x_3 g_z = \frac{x_3|g|^2 g_z}{|g|^2-1}. \tag{6.8}$$

Similarly, by Theorem 5.1 and (3.8), we get

$$(1+\bar{g}^2)\frac{\partial x_1}{\partial z} + i(\bar{g}^2-1)\frac{\partial x_2}{\partial z} = -(|g|^2-1)\overline{\widetilde{G}}_z. \tag{6.9}$$

Solving the system of the linear equations (6.7) and (6.9) with respect to $\frac{\partial x_1}{\partial z}$ and $\frac{\partial x_2}{\partial z}$ and using (6.8), we get (6.1) and (6.2). Again by (3.8), we get (6.3). (1) is proved.

(2) When $g_z = 0$, still using (3.6) and (3.7), we get $g_z = 0$. Similarly to (6.8), we have

$$\widetilde{G}_{\bar{z}} = \frac{x_3|g|^2 g_{\bar{z}}}{|g|^2-1}. \tag{6.10}$$

The proof of (2) is similar to that of (1).

By using (6.1)–(6.6), a straight computation gives

Corollary 6.1. *Under the same conditions as in Theorem* 6.1*, the first, the second fundamental forms and the mean curvature of the immersion $x : \Sigma \to H^3$ are given, respectively, by*

(1) *When $|g| > 1$,*

$$ds^2 = \frac{|g|^2|g_z|^2\overline{\widetilde{G}}_z}{(|g|^2-1)^2\overline{\widetilde{G}}_{\bar{z}}}dz^2 + \frac{|g|^2|g_z|^2\widetilde{G}_{\bar{z}}}{(|g|^2-1)^2\widetilde{G}_z}d\bar{z}^2 + \frac{|g_z|^2(|g|^4|\widetilde{G}_{\bar{z}}|^2 + |\widetilde{G}_z|^2)}{(|g|^2-1)^2|\widetilde{G}_z|^2}|dz|^2,$$
$$h = \frac{|g_z|^2(|g|^4|\widetilde{G}_{\bar{z}}|^2 - |\widetilde{G}_z|^2)}{(|g|^4-1)|\widetilde{G}_z|^2}|dz|^2 > 0,$$
$$H = \frac{(|g|^2-1)(|g|^4|\widetilde{G}_{\bar{z}}|^2 + |\widetilde{G}_z|^2)}{(1+|g|^2)(|g|^4|\widetilde{G}_{\bar{z}}|^2 - |\widetilde{G}_z|^2)}.$$

(2) *When* $|g| < 1$,

$$ds^2 = \frac{|g|^2|g_{\bar z}|^2\widetilde{G}_z}{(|g|^2-1)^2\widetilde{G}_{\bar z}}dz^2 + \frac{|g|^2|g_{\bar z}|^2\overline{\widetilde{G}_{\bar z}}}{(|g|^2-1)^2\overline{\widetilde{G}_z}}d\bar z^2 + \frac{|g_{\bar z}|^2(|g|^4|\widetilde{G}_z|^2 + |\widetilde{G}_{\bar z}|^2)}{(|g|^2-1)^2|\widetilde{G}_{\bar z}|^2}|dz|^2,$$

$$h = \frac{|g_{\bar z}|^2(|g|^4|\widetilde{G}_z|^2 - |\widetilde{G}_{\bar z}|^2)}{(|g|^4-1)|\widetilde{G}_{\bar z}|^2}|dz|^2 > 0,$$

$$H = \frac{(|g|^2-1)(|g|^4|\widetilde{G}_z|^2 + |\widetilde{G}_{\bar z}|^2)}{(1+|g|^2)(|g|^4|\widetilde{G}_z|^2 - |\widetilde{G}_{\bar z}|^2)}.$$

Theorem 6.1 implies that the normal Gauss map of a surface satisfying $K = -\frac{4|g|^2}{(1+|g|^2)^2} > -1$ is a conformal map. We can also check the normal Gauss map of totally umbilics surfaces is an antiholomorphic map. Conversely, we have

Theorem 6.2. *Let Σ be connected Riemanian surface and $x : \Sigma \to H^3$ be an immersion with Gauss curvature $K > -1$ and conformal structure induced by the second fundamental form. Assume that the set of umbilics has no interior point. If the normal Gauss map $g : \Sigma \to C \cup \{\infty\}$ is conformal, then the Gauss curvature K and the normal Gauss map g must satisfy $K = -\frac{4|g|^2}{(1+|g|^2)^2} > -1$ $\big($when $\sqrt{K+1} = \frac{|g|^2-1}{1+|g|^2}$, $g : \Sigma \to C \cup \{\infty\}$ is an holomorphic map and when $\sqrt{K+1} = \frac{1-|g|^2}{1+|g|^2}$, $g : \Sigma \to C \cup \{\infty\}$ is an antiholomorphic map$\big)$.*

Proof. By Corollaries 3.1 and 6.1, we have

$$|g_{\bar z}| = \frac{1}{4}(E + G + 2\sqrt{EG - F^2}\ )^{\frac{1}{2}}|\sqrt{K+1}\ (1 + |g|^2) - (|g|^2 - 1)|,$$

$$|g_z| = \frac{1}{4}(E + G - 2\sqrt{EG - F^2}\ )^{\frac{1}{2}}|\sqrt{K+1}\ (1 + |g|^2) + (|g|^2 - 1)|.$$

Note that $E + G - 2\sqrt{EG - F^2} \geq 0$ and equality holds at points satisfying $E = G$ and $F = 0$, i.e. at umbilics. $g : \Sigma \to C \cup \{\infty\}$ is conformal if and only if either every point of $x(\Sigma)$ satisfies $K = -\frac{4|g|^2}{(1+|g|^2)^2}$ or is a umbilic. Since the set of umbilics has no interior point, we have proved Theorem 6.2.

For hyperbolic Gaus map $\widetilde{G} : \Sigma \to C \cup \{\infty\}$, J. A. Gálvez and A. Martĺnez and F. Milán [5] proved that $\widetilde{G}$ is conformal if and only if $x(\Sigma)$ is either flat or totally umbilics.

Remark 6.1. By (5.6) and Corollary 6.1, we know that when $|g| < 1$, $|\widetilde{G}_{\bar z}| = \frac{1}{2}x_3|g|^2(E + G + 2\sqrt{EG - F^2})^{\frac{1}{2}}$. By (6.10), we know that the normal Gauss map g has no holomorphic points.

Theorem 6.3. *Let Σ be connected Riemanian surface and $x : \Sigma \to H^3$ be an immersion satisfying $K = -\frac{4|g|^2}{(1+|g|^2)^2} > -1$. The conformal structure on Σ is induced by the second fundamental form. Then, the hyperbolic Gauss map $\widetilde{G}$ and the normal Gauss map g must satisfy*

(1) *When* $|g| > 1$,

$$\frac{\widetilde{G}_z}{g_z} > 0, \tag{6.11}$$

$$|g|^2|\widetilde{G}_{\bar z}| > |\widetilde{G}_z|, \tag{6.12}$$

$$\widetilde{G}_{z\bar z} + \frac{\bar g_{\bar z}}{(|g|^4-1)\bar g}\widetilde{G}_z - \frac{|g|^2\bar g g_z}{|g|^4-1}\widetilde{G}_{\bar z} = 0. \tag{6.13}$$

(2) *When* $|g|<1$, $g:\Sigma\to C\cup\{\infty\}$ *has no holomorphic points and*

$$\frac{\widetilde{G}_{\bar z}}{|g|^2 g_{\bar z}}<0, \tag{6.14}$$

$$\frac{|g|^2|\widetilde{G}_z|}{|\widetilde{G}_{\bar z}|}<1, \tag{6.15}$$

$$\widetilde{G}_{z\bar z}+\frac{\bar g_z}{(|g|^4-1)\bar g}\widetilde{G}_{\bar z}-\frac{|g|^2\bar g g_{\bar z}}{|g|^4-1}\widetilde{G}_z=0. \tag{6.16}$$

Proof. (1) By (6.8), we get (6.11). By $h>0$, we have (6.12). Taking derivatives on the two sides of (6.8) with respect to $\bar z$ and using (6.3) and $g_{\bar z}=0$, we may prove (6.13).

The proof of (2) is similar to that of (1).

For the global quadratic differential $\Phi=\phi dz^2=\left(\frac{E-G}{2}-iF\right)dz^2$ on Σ, we have

Theorem 6.4. *Under the same conditions as in Theorem* 6.3, Φ *is not a holomorphic quadratic differential on* Σ *unless* $x(\Sigma)$ *is an equidistant surface* (*i.e., ordinary Euclidean plane*).

Proof. When $|g|>1$, by Corollary 6.1, (6.11) and (6.13), we have

$$\phi_{\bar z}=2\left(\frac{|g|^2 g_z^2\overline{\widetilde{G}}_z}{(|g|^2-1)^2\widetilde{G}_z}\right)_{\bar z}=-\frac{2|g_z|^2\bar g g_z}{(|g|^2-1)^3(1+|g|^2)|\widetilde{G}_z|^2}\{|\widetilde{G}_z|^2+|g|^4|\widetilde{G}_{\bar z}|^2+2g^2\overline{\widetilde{G}}_z\overline{\widetilde{G}}_{\bar z}\}.$$

By (6.12), we know that $|G_z|^2+|g|^4|\widetilde{G}_{\bar z}|^2>2|g|^2|G_z||\widetilde{G}_{\bar z}|$ and hence $\phi_{\bar z}\neq 0$. So Φ is not holomorphic.

When $|g|<1$, similarly we may prove Φ is not holomorphic.

For immersion $x:\Sigma\to H^3$, by $\widetilde{G}=x_1+ix_2+x_3 g$, we get

$$x_1=\mathrm{Re}(\widetilde{G}-x_3 g),\qquad x_2=\mathrm{Im}(\widetilde{G}-x_3 g),$$

where x_3 is given by either (6.8) or (6.10). Next, we give the Weierstrass representation formula.

Theorem 6.5. *Let* Σ *be simply connected Riemannian surface. Suppose map* $\widetilde{G}:\Sigma\to C\cup\{\infty\}$ *and nonconstant conformal map* $g:\Sigma\to C\cup\{\infty\}$ *are given.*

(1) *If holomorphic map* $g:\Sigma\to C\cup\{\infty\}$ *satisfies* $|g|>1$ *and* (6.11)–(6.13), *then the system of linear equations* (6.1)–(6.3) *is integrable and a solution is given by*

$$x_1=\mathrm{Re}\left\{\widetilde{G}-\frac{|g|^2-1}{\bar g g_z}\widetilde{G}_z\right\}+C_1, \tag{6.17}$$

$$x_2=\mathrm{Im}\left\{\widetilde{G}-\frac{|g|^2-1}{\bar g g_z}\widetilde{G}_z\right\}+C_2, \tag{6.18}$$

$$x_3=\frac{|g|^2-1}{|g|^2 g_z}\widetilde{G}_z, \tag{6.19}$$

where C_1 *and* C_2 *are arbitrary real constant. Moreover,* $x=(x_1,x_2,x_3):\Sigma\to H^3$ *is an immersion with hyperbolic Gauss map* $\widetilde{G}+(C_1+iC_2):\Sigma\to C\cup\{\infty\}$, *normal Gauss map* $g:\Sigma\to C\cup\{\infty\}$ *and Gauss curvature* K *satisfying* $\sqrt{K+1}=\frac{|g|^2-1}{1+|g|^2}$. *And the conformal structure on* Σ *is induced by the second fundamental form. Conversely, any* *surface* $x : \Sigma \to H^3$ *satisfying* $\sqrt{K+1} = \frac{|g|^2-1}{1+|g|^2} > 0$ *can be given by* (6.17)–(6.19) *by means of hyperbolic Gauss map* $\widetilde{G}$ *and normal Gauss map* g $(C_1 = C_2 = 0)$, *where the conformal structure on* Σ *is induced by the second fundamental form.*

(2) *If antiholomorphic map* $g : \Sigma \to C \cup \{\infty\}$ *without holomorphic points satisfies* $|g| < 1$ *and* (6.14)–(6.16), *then the system of linear equations* (6.4)–(6.6) *is integrable and a solution is given by*

$$x_1 = \mathrm{Re}\Big\{\widetilde{G} - \frac{|g|^2-1}{\bar{g}g_{\bar{z}}}\widetilde{G}_{\bar{z}}\Big\} + C_1, \tag{6.20}$$

$$x_2 = \mathrm{Im}\Big\{\widetilde{G} - \frac{|g|^2-1}{\bar{g}g_{\bar{z}}}\widetilde{G}_{\bar{z}}\Big\} + C_2, \tag{6.21}$$

$$x_3 = \frac{|g|^2-1}{|g|^2 g_{\bar{z}}}\widetilde{G}_{\bar{z}}, \tag{6.22}$$

where C_1 *and* C_2 *are arbitrary real constants. Moreover,* $x = (x_1, x_2, x_3) : \Sigma \to H^3$ *is an immersion with hyperbolic Gauss map* $\widetilde{G} + (C_1 + iC_2) : \Sigma \to C \cup \{\infty\}$, *normal Gauss map* $g : \Sigma \to C \cup \{\infty\}$ *and Gauss curvature* K *satisfying* $\sqrt{K+1} = \frac{1-|g|^2}{1+|g|^2}$. *And the conformal structure on* Σ *is induced by the second fundamental form. Conversely, any surface* $x : \Sigma \to H^3$ *satisfying* $\sqrt{K+1} = \frac{1-|g|^2}{1+|g|^2} > 0$ *can be given by* (6.20)–(6.22) *by means of hyperbolic Gauss map* $\widetilde{G}$ *and normal Gauss map* g $(C_1 = C_2 = 0)$, *where the conformal structure on* Σ *is induced by the second fundamental form.*

Proof. Using Theorem 5.1 and Corollary 6.1, we may check this theorem.

In the following, we look for the graph $(u, v, f(u, v))$ with conformal normal Gauss map with respect to the conformal structure induced by the second fundamental form (see Theorem 6.2). Its Gauss curvature is given by

$$K = -1 + \frac{f^2(f_{uu}f_{vv} - f_{uv}^2) + f[(1+f_v^2)f_{uu} - 2f_uf_vf_{uv} + (1+f_u^2)f_{vv}] + (1+f_u^2+f_v^2)}{(1+f_u^2+f_v^2)^2}.$$

$K = -\frac{4|g|^2}{(1+|g|^2)^2} > -1$ is equivalent to

$$f(f_{uu}f_{vv} - f_{uv}^2) + [(1+f_v^2)f_{uu} - 2f_uf_vf_{uv} + (1+f_u^2)f_{vv}] = 0. \tag{6.23}$$

(6.23) is the equation of graph satisfying $K = -\frac{4|g|^2}{(1+|g|^2)^2} > -1$. Obviously, linear functions are solutions of the equation (6.23). Now, we look for a nonlinear special solution of the equation (6.23).

Let $f(u, v) = \phi(u) + \psi(v)$. (6.23) is equivalent to

$$\frac{\phi\phi'' + 1 + (\phi')^2}{\phi''} = -\frac{\psi\psi'' + 1 + (\psi')^2}{\psi''}.$$

The above formula must equal the constant A. We get

$$(\phi - A)\phi'' + 1 + (\phi')^2 = 0, \qquad (\psi + A)\psi'' + 1 + (\psi')^2 = 0.$$

Solving the above two ordinary differential equations, we get

$$\phi = \pm\sqrt{-u^2 + au + b} + A, \qquad \psi = \pm\sqrt{-v^2 + cv + d} - A.$$

Making a linear transformation of variable, we may obtain

$$f = \sqrt{a^2 - u^2} + \sqrt{b^2 - v^2}, \quad \text{or} \quad f = \sqrt{a^2 - u^2} - \sqrt{b^2 - v^2} > 0,$$

where $a > 0$ and $b > 0$ are constants.

Remark 6.2. Does there exist nontrivial solution of the equation (6.23) defined globally on R^2? Furthermore, how about $f > 0$?

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