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HILBERT-DIRAC OPERATORS IN CLIFFORD ANALYSIS

F. BRACKX* H. De SCHEPPER**

Abstract

Around the central theme of "square root" of the Laplace operator it is shown that the classical Riesz potentials of the first and of the second kind allow for an explicit expression of so-called Hilbert-Dirac convolution operators involving natural and complex powers of the Dirac operator.

Keywords Clifford analysis, Riesz potentials, Hilbert transformation, Distributions
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§ 1 . Introduction

This paper is at the crosspoint of the following concepts: the multi–dimensional Hilbert transform, the Dirac operator, Riesz potentials and Clifford distributions, linked together by the concept of "square root of the Laplace operator".

The Riesz potential (see e.g. [?]) I^γ defined by

$$I^\gamma[f](\underline{x}) = \frac{1}{H_m(\gamma)} \int_{\mathbb{R}^m} f(\underline{y}) |\underline{x} - \underline{y}|^{\gamma - m} dV(\underline{y}) \, ,$$

with

$$H_m(\gamma) = 2^\gamma \pi^{\frac{m}{2}} \frac{\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{m-\gamma}{2}\right)}$$

is a scalar convolution operator; for $\gamma = -1$ one obtains, up to a constant, the so-called "square root of $(-\Delta)$", Δ being the Laplace operator. On the other hand, the Dirac operator $\underline{\partial}$, which is at the heart of Clifford analysis, satisfies $\underline{\partial}^2 = -\Delta$, and thus may be considered as a Clifford vector valued "square root of $(-\Delta)$".

Apparently both "square roots of $(-\Delta)$" have a distinct nature. Nevertheless they are linked to each other by means of the Hilbert transform of Clifford analysis: it is well known that the Hilbert and the Dirac operator are commuting and that their composition is a scalar valued convolution operator which is precisely the "square root of $(-\Delta)$" in the sense of Riesz potentials. We call this composition of operators the Hilbert-Dirac operator. In Section 4 we consider the composition of the Hilbert operator with natural and complex powers of the Dirac operator. We construct explicit formulae for those compositions in terms of scalar and Clifford vector valued convolution operators, the properties of which are recalled in Section

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*Clifford Research Group, Department of Mathematical Analysis, Faculty of Engineering, Ghent University, Galglaan 2, B-9000 Gent, Belgium. **E-mail:** Freddy.Brackx@UGent.be
Clifford Research Group, Department of Mathematical Analysis, Faculty of Engineering, Ghent University, Galglaan 2, B-9000 Gent, Belgium. **E-mail: Hennie.DeSchepper@UGent.be

3. In order to keep the paper self-contained, an introduction to Clifford analysis is given in Section 2. In Section 5 we briefly reflect upon the nature of operators such as the Dirac and the Hilbert-Dirac operator and the connections between them. Finally in Section 6 we revert to the historical background of the Riesz potentials and the Hilbert operator and show how the convolution kernels in our Clifford setting are linked to the traditional ones of classical potential theory.

§ 2. Clifford Analysis

For the reader who is not familiar with Clifford analysis, we give a quick overview of the basic notions and results of this function theory which may be regarded as a generalization in a higher dimensional setting of the theory of holomorphic functions in the complex plane. For more details, we refer the reader to [2, 9].

Let $\mathbb{R}^{0,m}$ be the real vector space $\mathbb{R}^m$, endowed with a non-degenerate quadratic form of signature $(0, m)$, let $(e_1, \cdots, e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m}$, and let $\mathbb{R}_{0,m}$ be the universal Clifford algebra constructed over $\mathbb{R}^{0,m}$. The non-commutative multiplication in $\mathbb{R}_{0,m}$, the so-called geometric product, is governed by the rules

$$e_i^2 = -1, \quad i = 1, 2, \cdots, m \quad \text{and} \quad e_i e_j + e_j e_i = 0, \quad 1 \leq i \neq j \leq m.$$

A basis for the Clifford algebra $\mathbb{R}_{0,m}$ is obtained by considering for any set $A = \{i_1, \cdots, i_h\} \subset \{1, \cdots, m\}$ with $1 \leq i_1 < i_2 < \cdots < i_h \leq m$ the element $e_A = e_{i_1} e_{i_2} \cdots e_{i_h}$. For the empty set ϕ, we put $e_\phi = 1$, the latter being the identity element; then any $a \in \mathbb{R}_{0,m}$ may thus be written as

$$a = \sum_A a_A \, e_A, \qquad a_A \in \mathbb{R}, \tag{2.1}$$

or still as $a = \sum_{k=0}^{m} [a]_k$, where $[a]_k = \sum_{|A|=k} a_A \, e_A$ is a k-vector $(k = 0, 1, \cdots, m)$. If we denote the space of k-vectors by $\mathbb{R}^k_{0,m}$, then $\mathbb{R}_{0,m} = \sum_{k=0}^{m} \oplus \mathbb{R}^k_{0,m}$, leading to the identification of $\mathbb{R}$ and $\mathbb{R}^{0,m}$ with $\mathbb{R}^0_{0,m}$ and $\mathbb{R}^1_{0,m}$ respectively. We will also identify an element $\underline{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$ with the Clifford one-vector (or vector for short)

$$\underline{x} = \sum_{j=1}^{m} x_j \, e_j.$$

For the geometric product of two vectors, one gets $\underline{x}\,\underline{y} = \underline{x} \bullet \underline{y} + \underline{x} \wedge \underline{y}$, where

$$\underline{x} \bullet \underline{y} = -\langle \underline{x}, \underline{y} \rangle = -\sum_{j=1}^{m} x_j y_j = \frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\underline{x})$$

is a scalar and

$$\underline{x} \wedge \underline{y} = \sum_{i<j} e_i e_j (x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\underline{x})$$

is a 2-vector, also called a bivector. In particular $\underline{x}^2 = \underline{x} \bullet \underline{x} = -|\underline{x}|^2$.

Conjugation in $\mathbb{R}_{0,m}$ is defined as the anti-involution for which $\overline{e}_j = -e_j$, $j = 1, \cdots, m$. In particular for a vector $\underline{x}$, we have $\overline{\underline{x}} = -\underline{x}$.

The Dirac operator in $\mathbb{R}^m$ is the first order vector valued differential operator

$$\underline{\partial} = \sum_{j=1}^{m} e_j \, \partial_{x_j},$$

its fundamental solution being given by $E_m(\underline{x}) = \frac{1}{a_m} \frac{\overline{\underline{x}}}{|\underline{x}|^m}$, with a_m the area of the unit sphere S^{m-1} in $\mathbb{R}^m$.

Considering functions f defined on $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$, i.e. $f : \mathbb{R}^m \longrightarrow \mathbb{R}_{0,m}$, we say that such a function is left monogenic, respectively right monogenic, in the open region Ω of $\mathbb{R}^m$ iff f is continuously differentiable in Ω and satisfies in Ω the equation

$$\underline{\partial}\, f := \sum_A \sum_{i=1}^{m} e_i e_A \partial_{x_i} f_A(\underline{x}) = 0,$$

respectively

$$f\, \underline{\partial} := \sum_A \sum_{i=1}^{m} e_A e_i \partial_{x_i} f_A(\underline{x}) = 0.$$

Here, $f_A(\underline{x})$ are the real valued components of f with respect to the chosen Clifford basis, see (**??**).

As $\overline{\underline{\partial}\, f} = \overline{f}\, \overline{\underline{\partial}} = -\overline{f}\underline{\partial}$, a function f is left monogenic in Ω iff $\overline{f}$ is right monogenic in Ω. As moreover the Dirac operator factorizes the Laplace operator, i.e. $-\underline{\partial}^2 = \underline{\partial}\,\overline{\underline{\partial}} = \overline{\underline{\partial}}\,\underline{\partial} = \Delta$, where $\Delta = \sum\limits_{j=1}^{m} \partial_{x_j}^2$, a (left or right) monogenic function in Ω is harmonic and hence C_∞ in Ω. Introducing spherical co-ordinates $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} \in S^{m-1}$, gives rise to the Clifford vector valued locally integrable function $\underline{\omega}$, which is to be seen as the higher dimensional analogue of the signum distribution on the real line.

In a similar way we consider the Cauchy-Riemann operator D in $\mathbb{R}^{m+1}$, defined as

$$D = \partial_{x_0} + \underline{\partial} = \partial_{x_0} + \sum_{j=1}^{m} e_j \, \partial_{x_j},$$

where $(x_0, \cdots, x_m) \in \mathbb{R}^{m+1}$ and $(e_0, \cdots, e_m)$ is the corresponding basis of $\mathbb{R}^{0,m+1}$. We then introduce a second notion of monogenicity, viz monogenicity with respect to the Cauchy-Riemann operator D, for functions defined on $\mathbb{R}^{m+1}$. Such a function is said to be left monogenic, respectively right monogenic, w.r.t D in the open region Ω of $\mathbb{R}^{m+1}$ iff f is continuously differentiable in Ω and satisfies in Ω the equation $Df = 0$, respectively $fD = 0$.

§ 3. The Distributions T^*_λ and U^*_λ

In [3] and [4], the distributions $T_\lambda = Fp\ r^\lambda$ and $U_\lambda = Fp\ r^\lambda \underline{\omega}$, $(r = |\underline{x}|,\ \lambda \in \mathbb{C})$ and their generalized versions are thoroughly studied. The distributions T_λ are of course very classical. The distributions U_λ are Clifford vector valued; they have vectorial analogues which were introduced in the 1950's. For a short historical comment we refer the reader to Section 6.

We normalize these two families of distributions as follows:

$$\begin{cases} T^*_\lambda = \pi^{\frac{\lambda+m}{2}} \dfrac{T_\lambda}{\Gamma\left(\frac{\lambda+m}{2}\right)}, & \lambda \neq -m-2l,\ l = 0,1,2,\cdots, \\[2ex] T^*_{-m-2l} = \dfrac{\pi^{\frac{m}{2}-l}}{2^{2l}\Gamma\left(\frac{m}{2}+l\right)} (-\Delta)^l \delta(\underline{x}), & l = 0,1,2,\cdots \end{cases}$$

and

$$\begin{cases} U^*_\lambda = \pi^{\frac{\lambda+m+1}{2}} \dfrac{U_\lambda}{\Gamma\left(\frac{\lambda+m+1}{2}\right)}, & \lambda \neq -m-2l-1,\ l=0,1,2,\cdots, \\ U^*_{-m-2l-1} = -\dfrac{\pi^{\frac{m}{2}-l}}{2^{2l+1}\,\Gamma\left(\frac{m}{2}+l+1\right)}\, \underline{\partial}^{2l+1}\delta(\underline{x}), & l=0,1,2,\cdots. \end{cases}$$

Thus we obtain two entire mappings $\lambda \mapsto T^*_\lambda$ and $\lambda \mapsto U^*_\lambda$ from $\mathbb{C}$ to $\mathcal{S}'(\mathbb{R}^m)$.

The fundamental properties of the tempered distributions T^*_λ and U^*_λ are listed in the following lemma; their proofs are rather straightforward. For the properties in frequency space we adopt as definition of the Fourier transform:

$$\mathcal{F}[f(\underline{x})](\underline{y}) = \int_{\mathbb{R}^m} f(\underline{x}) \exp(-2\pi i \langle \underline{x}, \underline{y}\rangle) dV(\underline{x}).$$

Lemma 3.1. *For all $\lambda \in \mathbb{C}$, one has*

(i) $\underline{x}\, T^*_\lambda = \frac{\lambda+m}{2\pi}\, U^*_{\lambda+1}$; $\quad \underline{x}\, U^*_\lambda = U^*_\lambda\, \underline{x} = -T^*_{\lambda+1}$;

(ii) $\underline{\partial}\, T^*_\lambda = \lambda\, U^*_{\lambda-1}$; $\quad \underline{\partial}\, U^*_\lambda = U^*_\lambda\, \underline{\partial} = -2\pi\, T^*_{\lambda-1}$;

(iii) $\mathcal{F}[T^*_\lambda] = T^*_{-\lambda-m}$; $\quad \mathcal{F}[U^*_\lambda] = -i\, U^*_{-\lambda-m}$.

As convolution kernels they give rise to what we call the Riesz potentials of the first and of the second kind:

$$\mathcal{P}^\gamma_T[f] = T^*_{\gamma-m} * f, \quad \mathcal{P}^\gamma_U[f] = U^*_{\gamma-m} * f, \qquad f \in \mathcal{S}(\mathbb{R}^m).$$

For the connection with the traditional Riesz potentials, see Section 6.

As particular cases the definitions of complex powers of the Laplace operator (see e.g. [**?**]) and of the Dirac operator (see [9]) arise; for $f \in \mathcal{S}(\mathbb{R}^m)$ one has, for $\beta \in \mathbb{C}\backslash\{-\frac{m}{2}-l,\ l \in \mathbb{N}_0\}$,

$$(-\Delta)^\beta[f] = \frac{2^{2\beta}\,\Gamma\left(\frac{m}{2}+\beta\right)}{\pi^{\frac{m}{2}-\beta}} T^*_{-m-2\beta} * f,$$

and for $\alpha \in \mathbb{C}\backslash\{-m-l,\ l \in \mathbb{N}_0\}$,

$$\underline{\partial}^\alpha[f] = \frac{1+\exp(i\pi\alpha)}{2}\, \frac{2^\alpha\,\Gamma\left(\frac{m+\alpha}{2}\right)}{\pi^{\frac{m-\alpha}{2}}}\, T^*_{-m-\alpha} * f - \frac{1-\exp(i\pi\alpha)}{2}\, \frac{2^\alpha\,\Gamma\left(\frac{m+\alpha+1}{2}\right)}{\pi^{\frac{m-\alpha+1}{2}}}\, U^*_{-m-\alpha} * f.$$

Note that in particular the "square root of the negative Laplacian" is given by

$$(-\Delta)^{\frac{1}{2}}[f](\underline{x}) = \frac{4\pi}{a_{m+1}} T^*_{-m-1} * f(\underline{x}) = -\frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}}\, Fp \int_{\mathbb{R}^m} \frac{f(\underline{u})}{|\underline{x}-\underline{u}|^{m+1}}\, dV(\underline{u}),$$

which is a scalar valued convolution operator, as opposed to the Clifford vector valued Dirac operator $\underline{\partial}$ for which also holds $\underline{\partial}^2 = -\Delta$.

§ 4. Hilbert-Dirac Operators

4.1. Introduction

It was shown in [3] that the "finite part" distribution $U_{-m} = U^*_{-m}$ is nothing else but the "principal value" distribution $Pv\frac{\omega}{r^m}$, enabling a comprehensive and elegant definition of the

Hilbert transform in m-dimensional Euclidean space, viz the convolution with the Hilbert kernel

$$\mathcal{H} = \frac{2}{a_{m+1}} \; Pv \; \frac{\underline{\omega}}{r^m} = -\frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} \; U^*_{-m}$$

(see [7, 10] and Section 6 for a historical comment). For a suitable function f, its Hilbert transform is defined as a Riesz potential of the first kind, by

$$H[f] = \mathcal{H} * f = \frac{2}{a_{m+1}} \; Pv \; \frac{\underline{\omega}}{r^m} * f = -\frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} \; \mathcal{P}^0_U[f]$$

or

$$\begin{aligned} H[f](\underline{x}) &= \frac{2}{a_{m+1}} \lim_{\varepsilon \to 0+} \int_{|\underline{x}-\underline{y}|>\varepsilon} \frac{\overline{\underline{x}} - \overline{\underline{y}}}{|\underline{x} - \underline{y}|^{m+1}} f(\underline{y}) dV(\underline{y}) \\ &= \frac{2}{a_{m+1}} Pv \int_{\mathbb{R}^m} \frac{\overline{\underline{u}}}{|\underline{u}|^{m+1}} f(\underline{x} - \underline{u}) dV(\underline{u}). \end{aligned}$$

In [1] it has been investigated on which maximal space of functions this Hilbert transform can be defined; in particular the Hilbert operator H is a bounded linear operator on $L_2(\mathbb{R}^m; \mathbb{R}_{0,m})$.

The Hilbert transform arises in a natural way by considering boundary values (in L_2 or in distributional sense) of the Cauchy transform in $\mathbb{R}^{m+1}$ of an appropriate function or distribution in $\mathbb{R}^m$. This Cauchy transform may be defined for a suitable function or distribution f as the convolution

$$\mathcal{C}[f](x_0, \underline{x}) = E(x_0, \cdot\,) * f(\,\cdot\,)(\underline{x})$$

with the Cauchy kernel

$$E(x_0, \underline{x}) = \frac{1}{a_{m+1}} \frac{x_0 - \underline{x}}{|x_0 + \underline{x}|^{m+1}},$$

which is the fundamental solution in $\mathbb{R}^{m+1}$ of the Cauchy-Riemann operator $D = \partial_{x_0} + \underline{\partial}$. Taking limits for $x_0 \to 0$ and identifying $\mathbb{R}^m$ with the hyperplane $\{x_0 = 0\}$ in $\mathbb{R}^{m+1}$, the following distributions in $\mathbb{R}^m$ are obtained:

$$E(0+, \underline{x}) = \lim_{x_0 \to 0+} E(x_0, \underline{x}), \qquad E(0-, \underline{x}) = \lim_{x_0 \to 0-} E(x_0, \underline{x}).$$

They are the counterparts in $\mathbb{R}^m$ of the distributions $\frac{1}{x \pm i0}$ on the real line and satisfy the well-known relations

$$\begin{aligned} E(0+, \underline{x}) &= \frac{1}{2}\delta(\underline{x}) + \frac{1}{2}\mathcal{H}(\underline{x}), \\ E(0-, \underline{x}) &= -\frac{1}{2}\delta(\underline{x}) + \frac{1}{2}\mathcal{H}(\underline{x}), \end{aligned}$$

which are equivalent to the well-known distributional limits

$$\begin{aligned} \lim_{x_0 \to 0\pm} \mathcal{P}_{x_0}(\underline{x}) &= \lim_{x_0 \to 0\pm} \frac{1}{a_{m+1}} \frac{2x_0}{|x_0 + \underline{x}|^{m+1}} = \pm\delta(\underline{x}), \\ \lim_{x_0 \to 0\pm} \mathcal{Q}_{x_0}(\underline{x}) &= \lim_{x_0 \to 0\pm} \frac{1}{a_{m+1}} \frac{2\overline{\underline{x}}}{|x_0 + \underline{x}|^{m+1}} = \frac{2}{a_{m+1}} \; Pv \; \frac{\underline{\omega}}{r^m} = \mathcal{H}(\underline{x}), \end{aligned}$$

where $\mathcal{P}_{x_0}(\underline{x})$ is the Poisson kernel and $\mathcal{Q}_{x_0}(\underline{x})$ its conjugate harmonic in $\mathbb{R}^{m+1}\backslash\mathbb{R}^m$.

If in particular $f \in L_2(\mathbb{R}^m;\mathbb{R}_{0,m})$, then $\mathcal{C}[f](x_0,\underline{x})$ is in the Hardy spaces $H^2(\mathbb{R}^{m+1}_{\pm};\mathbb{R}_{0,m})$ and its non-tangential limits $\mathcal{C}^{\pm}[f]$ for $x_0 \to 0\pm$ satisfy the so-called Plemelj-Sokhotzki formula (see also [8, 10, 14]):

$$\mathcal{C}^{\pm}[f](\underline{x}) = \lim_{x_0\to 0\pm} \mathcal{C}[f](x_0,\underline{x}) = \pm\frac{1}{2}f(\underline{x}) + \frac{1}{2}H[f](\underline{x}) \qquad \text{for a.e. } \underline{x}\in\mathbb{R}^m.$$

4.2. Kernels involving natural powers of the Dirac operator

Let p be a natural number. We define in $\mathbb{R}^{m+1}$ the (p)-Cauchy kernels by

$$E^{(p)}(x_0,\underline{x}) = E(x_0,\underline{x})\,\underline{\partial}^p.$$

It is clear that in $\mathbb{R}^{m+1}\backslash\{0\}$, each $E^{(p)}(x_0,\underline{x})$ is left-monogenic w.r.t. the Cauchy-Riemann operator $\partial_{x_0}+\underline{\partial}$, while in distributional sense

$$(\partial_{x_0}+\underline{\partial})E^{(p)}(x_0,\underline{x}) = \underline{\partial}^p\delta(x_0,\underline{x}).$$

Now we investigate the distributional limits of $E^{(p)}(x_0,\underline{x})$ for $x_0\to 0\pm$.

Proposition 4.1. *For all $p\in\mathbb{N}$, one has*

$$E^{(p)}(0\pm,\underline{x}) = \lim_{x_0\to 0\pm} E^{(p)}(x_0,\underline{x}) = \pm\frac{1}{2}\underline{\partial}^p\delta(\underline{x}) + \frac{1}{2}\mathcal{H}\underline{\partial}^p(\underline{x}).$$

Proof. For any testing function ϕ, one has

$$\lim_{x_0\to 0\pm}\langle E^{(p)}(x_0,\underline{x}),\phi\rangle = \lim_{x_0\to 0\pm}\langle E(x_0,\underline{x})\underline{\partial}^p,\phi\rangle = \lim_{x_0\to 0\pm}(-1)^p\langle E(x_0,\underline{x}),\underline{\partial}^p\phi\rangle$$
$$= \Big\langle \pm\frac{1}{2}\delta(\underline{x}) + \frac{1}{2}\mathcal{H}(\underline{x}),\underline{\partial}^p\phi\Big\rangle = \Big\langle \pm\frac{1}{2}\underline{\partial}^p\delta(\underline{x}) + \frac{1}{2}\mathcal{H}\underline{\partial}^p(\underline{x}),\phi\Big\rangle.$$

Corollary 4.1. *For all $p\in\mathbb{N}$ the distributions $\underline{\partial}^p\delta(\underline{x})$ in $\mathbb{R}^m$ may be obtained by the distributional limits*

$$\underline{\partial}^p\delta(\underline{x}) = \lim_{\varepsilon\to 0+}\Big(\frac{1}{a_{m+1}}\frac{2\varepsilon}{|\varepsilon+\underline{x}|^{m+1}}\Big)\underline{\partial}^p.$$

In particular, for $p=1$, one gets

$$\underline{\partial}\delta = \lim_{\varepsilon\to 0+}\frac{m+1}{a_{m+1}}\,\frac{2\varepsilon\underline{x}}{|\varepsilon+\underline{x}|^{m+3}}.$$

Corollary 4.2. *For all $p\in\mathbb{N}$ the distributions $\mathcal{H}\,\underline{\partial}^p$ in $\mathbb{R}^m$ may be obtained by the distributional limits*

$$\mathcal{H}\underline{\partial}^p(\underline{x}) = \lim_{\varepsilon\to 0+}\frac{1}{a_{m+1}}\Big(\frac{-2\underline{x}}{|\varepsilon+\underline{x}|^{m+1}}\Big)\underline{\partial}^p.$$

In particular, for $p=1$, one gets

$$\lim_{\varepsilon\to 0+}\frac{2}{a_{m+1}}\frac{m\varepsilon^2-|\underline{x}|^2}{|\varepsilon+\underline{x}|^{m+3}} = \mathcal{H}\underline{\partial}.$$

The convolution kernel $\mathcal{H}\underline{\partial} = \underline{\partial}\mathcal{H}$ appearing in the particular case of the above Proposition **??** is the real convolution kernel of the "square root of the negative Laplace operator", since

$$\begin{aligned}\mathcal{H}\underline{\partial} = \underline{\partial}\mathcal{H} &= \underline{\partial}\Big(- \frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} U^*_{-m}\Big) \\ &= 2\pi \frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} T^*_{-m-1} = \frac{4\pi}{a_{m+1}} T^*_{-m-1} = -\frac{2}{a_{m+1}} Fp\, \frac{1}{r^{m+1}},\end{aligned}$$

or

$$\underline{\partial}\mathcal{H} * f = \mathcal{H}\underline{\partial} * f = (-\Delta)^{\frac{1}{2}}[f].$$

This convolution kernel gives rise to the composition of the Hilbert and the Dirac operator, which are known to be commutative operators (see e.g. [**?**])

$$H[\underline{\partial} f] = \mathcal{H} * \underline{\partial} f = \mathcal{H}\underline{\partial} * f = \underline{\partial}\mathcal{H} * f = \underline{\partial}\, H[f].$$

We call $\mathcal{H}\underline{\partial} = \underline{\partial}\mathcal{H}$ the Hilbert-Dirac convolution kernel and $H\underline{\partial} = \underline{\partial}H$ the Hilbert-Dirac operator:

$$H\underline{\partial}[f] = \underline{\partial}H[f] = \mathcal{H}\underline{\partial} * f = \underline{\partial}\mathcal{H} * f.$$

In the same order of ideas, we readily obtain the following results concerning the Hilbert operator and the natural powers of the Dirac operator.

Proposition 4.2. *For all $p \in \mathbb{N}$, the operators H and $\underline{\partial}^p$ are commuting operators; their composition $H\underline{\partial}^p$ is the convolution operator with kernel $\mathcal{H}\underline{\partial}^p = \underline{\partial}^p\mathcal{H}$:*

$$H\underline{\partial}^p[f] = H[\underline{\partial}^p f] = \underline{\partial}^p H[f] = \mathcal{H}\underline{\partial}^p * f = \underline{\partial}^p\mathcal{H} * f.$$

For $p = 2k+1$ odd, $\mathcal{H}\underline{\partial}^{2k+1}$ is the scalar convolution kernel

$$\begin{aligned}\mathcal{H}\underline{\partial}^{2k+1} &= \underline{\partial}^{2k+1}\mathcal{H} = \frac{4\pi}{a_{m+1}} T^*_{-m-1}\underline{\partial}^{2k} \\ &= \frac{2(2\pi)^{k+1}}{a_{m+1}} (m+1)(m+3)\cdots(m+2k-1) T^*_{-m-2k-1}.\end{aligned}$$

For $p = 2k$ even, $\mathcal{H}\,\underline{\partial}^{2k}$ is the Clifford-vector convolution kernel

$$\begin{aligned}\mathcal{H}\underline{\partial}^{2k} &= \underline{\partial}^{2k}\mathcal{H} = \mathcal{H}(-\Delta)^k = -\frac{2}{a_{m+1}} U^*_{-m}\underline{\partial}^{2k} \\ &= -\frac{2(2\pi)^k}{a_{m+1}} (m+1)(m+3)\cdots(m+2k-1) U^*_{-m-2k}.\end{aligned}$$

Combining Propositions **??** and **??**, we obtain that the distributions $E^p(0\pm, \underline{x})$ may be expressed in terms of the T^*_λ and the U^*_λ distributions as follows:

$$\begin{aligned}E^{(2k)}(0\pm, \underline{x}) &= \pm \frac{2^{2k-1}\Gamma\left(\frac{m}{2}+k\right)}{\pi^{\frac{m}{2}-k}} T^*_{-m-2k} - \frac{2^{2k-1}\Gamma\left(\frac{m+1}{2}+k\right)}{\pi^{\frac{m+1}{2}-k}} U^*_{-m-2k}, \\ E^{(2k+1)}(0\pm, \underline{x}) &= \mp \frac{2^{2k}\Gamma\left(\frac{m}{2}+k+1\right)}{\pi^{\frac{m}{2}-k}} U^*_{-m-2k-1} + \frac{2^{2k}\Gamma\left(\frac{m+1}{2}+k\right)}{\pi^{\frac{m-1}{2}-k}} T^*_{-m-2k-1},\end{aligned}$$

which in frequency space read

$$\mathcal{F}[E^{(2k)}(0\pm,\underline{x})](\underline{y}) = \pm\ \frac{1\pm i\underline{\xi}}{2}(2\pi)^{2k}\rho^{2k},$$
$$\mathcal{F}[E^{(2k+1)}(0\pm,\underline{x})](\underline{y}) = \frac{1\pm i\,\underline{\xi}}{2}(2\pi)^{2k+1}\rho^{2k+1},$$

where $\psi^+ = \frac{1+i\underline{\xi}}{2}$ and $\psi^- = \frac{1-i\underline{\xi}}{2}$ are the so-called Clifford-Heaviside functions satisfying

$$\psi^+ + \psi^- = 1; \quad \psi^+\psi^- = \psi^-\psi^+ = 0; \quad (\psi^+)^2 = \psi^+; \quad (\psi^-)^2 = \psi^-.$$

The distributions ψ^+ and ψ^- were introduced independently by Sommen in [**?**] and McIntosh in [**?**] and are to be seen as the higher dimensional counterparts of the Heaviside distributions $Y(x)$ and $Y(-x)$ on the real line.

In their turn the distributions $E^{(p)}(0\pm,\underline{x})$ are the higher dimensional analogues of the one-dimensional distributions

$$\frac{1}{(x\pm i0)^p} = \frac{(-1)^p}{(p-1)!}\ \frac{d^{p-1}}{dx^{p-1}}\frac{1}{x\pm i0} = \frac{1}{x^p}\ \mp\ i\pi\ \frac{(-1)^p}{(p-1)!}\ \delta^{(p-1)}$$

with Fourier transform

$$\mathcal{F}\Big[\frac{1}{(x\pm i0)^p}\Big](y) = \frac{(-1)^p}{(p-1)!}\ (2\pi i)^{p+1}\ y^p\ Y(\mp y).$$

If in particular f is in $L_2(\mathbb{R}^m;\mathbb{R}_{0,m})$, then the above distributional considerations lead to the following result.

Proposition 4.3. *If the function f is such that for some $n\in\mathbb{N}$, $\underline{\partial}^p f \in L_2(\mathbb{R}^m;\mathbb{R}_{0,m})$ for all $p = 0,1,2,\cdots,n$, then its (p)-Cauchy transforms*

$$\mathcal{C}^{(p)}[f](x_0,\underline{x}) = E^{(p)}(x_0,\cdot\,) * f(\,\cdot\,)(\underline{x})$$

all belong to the Hardy spaces $H^2(\mathbb{R}^{m+1}_\pm;\mathbb{R}_{0,m})$, their non-tangential L_2 boundary values are given by

$$\mathcal{C}^{(p)\,\pm}[f](\underline{x}) := \lim_{x_0\to 0\pm}\mathcal{C}^{(p)}[f](x_0,\underline{x}) = \mathcal{C}^{\pm}[\underline{\partial}^p f],$$

and moreover

$$H[\underline{\partial}^p f] = \mathcal{H}\underline{\partial}^p * f.$$

Proof. Let $p\in\{0,1,2,\cdots,n\}$. As $\underline{\partial}^p f\in L_2(\mathbb{R}^m;\mathbb{R}_{0,m})$ its Cauchy transform

$$\mathcal{C}[\underline{\partial}^p f](x_0,\underline{x})$$

is in the Hardy spaces $H^2(\mathbb{R}^{m+1}_\pm;\mathbb{R}_{0,m})$ and its nontangential limits $\mathcal{C}^\pm[\underline{\partial}^p f]$ satisfy

$$\mathcal{C}^\pm[\underline{\partial}^p f](\underline{x}) = \lim_{x_0\to 0\pm}\mathcal{C}[\underline{\partial}^p f](x_0,\underline{x}) = \pm\frac{1}{2}\underline{\partial}^p f + \frac{1}{2}H[\underline{\partial}^p f](\underline{x}). \tag{4.1}$$

But

$$\mathcal{C}[\underline{\partial}^p f](x_0,\underline{x}) = E(x_0,\cdot\,) * \underline{\partial}^p f(\,\cdot\,)(\underline{x}) = E(x_0,\cdot\,)\underline{\partial}^p * f(\,\cdot\,)(\underline{x})$$
$$= E^{(p)}(x_0,\cdot\,) * f(\,\cdot\,)(\underline{x}) = \mathcal{C}^{(p)}[f](x_0,\underline{x}),$$

and hence

$$\lim_{x_0\to 0\pm} \mathcal{C}^{(p)}[f](x_0,\underline{x}) = \lim_{x_0\to 0\pm} \mathcal{C}[\underline{\partial}^p f](x_0,\underline{x}),$$

and also, by (**??**) and Proposition **??**,

$$\pm\frac{1}{2}\underline{\partial}^p\delta * f + \frac{1}{2}\mathcal{H}\underline{\partial}^p * f = \pm\frac{1}{2}\underline{\partial}^p f + \frac{1}{2}H[\underline{\partial}^p f].$$

Corollary 4.3. *The operators $H\underline{\partial}^p = \underline{\partial}^p H$, $p = 1,\cdots,n$ are bounded operators from the Sobolev space $W_2^n(\mathbb{R}^m;\mathbb{R}_{0,m})$ into $W_2^{n-p}(\mathbb{R}^m;\mathbb{R}_{0,m})$.*

4.3. Kernels involving complex powers of the Dirac operator

Using a result on the convolution of the T^*_λ and U^*_λ distributions (see [5]), viz

$$T^*_{-m-\mu} * U^*_{-m} = U^*_{-m} * T^*_{-m-\mu} = \pi^{\frac{m}{2}} \frac{\Gamma\left(\frac{m+\mu+1}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+\mu}{2}\right)} U^*_{-m-\mu},$$

$$U^*_{-m} * U^*_{-m-\mu} = U^*_{-m-\mu} * U^*_{-m} = \pi^{\frac{m}{2}+1} \frac{\Gamma\left(\frac{m+\mu}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+\mu+1}{2}\right)} T^*_{-m-\mu},$$

it is easily shown that the Hilbert operator H and complex powers of the Dirac operator $\underline{\partial}^\mu$, $\mu\in\mathbb{C}$, are commuting operators; their composition $H[\underline{\partial}^\mu\cdot\,] = \underline{\partial}^\mu H[\,\cdot\,]$ is a convolution operator with kernel

$$\underline{\partial}^\mu\mathcal{H} = \mathcal{H}\underline{\partial}^\mu = \frac{1-\exp(i\pi\mu)}{2} 2^\mu \frac{\Gamma\left(\frac{m+\mu}{2}\right)}{\pi^{\frac{m-\mu}{2}}} T^*_{-m-\mu} - \frac{1+\exp(i\pi\mu)}{2} 2^\mu \frac{\Gamma\left(\frac{m+\mu+1}{2}\right)}{\pi^{\frac{m-\mu+1}{2}}} U^*_{-m-\mu}.$$

The convolution kernel $\underline{\partial}^\mu\mathcal{H} = \mathcal{H}\underline{\partial}^\mu$ also arises as part of the boundary value of the (μ)-Cauchy kernel

$$E^{(\mu)}(x_0,\underline{x}) = E(x_0,\underline{x})\underline{\partial}^\mu, \qquad \mu\in\mathbb{C},$$

yielding the distributions in $\mathbb{R}^m$:

$$E^{(\mu)}(0\pm,\underline{x}) = \lim_{x_0\to 0\pm} E^{(\mu)}(x_0,\underline{x}) = \pm\,\frac{1}{2}\,\underline{\partial}^\mu\delta + \frac{1}{2}\,\mathcal{H}\underline{\partial}^\mu,$$

or

$$E^{(\mu)}(0+,\underline{x}) = 2^{\mu-1}\frac{\Gamma(\frac{m+\mu}{2})}{\pi^{\frac{m-\mu}{2}}} T^*_{-m-\mu} - 2^{\mu-1}\frac{\Gamma(\frac{m+\mu+1}{2})}{\pi^{\frac{m-\mu+1}{2}}} U^*_{-m-\mu}$$

and

$$E^{(\mu)}(0-,\underline{x}) = -\exp(i\pi\mu)\Big(2^{\mu-1}\frac{\Gamma(\frac{m+\mu}{2})}{\pi^{\frac{m-\mu}{2}}} T^*_{-m-\mu} + 2^{\mu-1}\frac{\Gamma(\frac{m+\mu+1}{2})}{\pi^{\frac{m-\mu+1}{2}}} U^*_{-m-\mu}\Big),$$

which in frequency space take the form

$$\mathcal{F}[E^{(\mu)}(0+,\underline{x})](\underline{y}) = (2\pi)^\mu T_\mu \frac{1+i\underline{\xi}}{2}$$

and

$$\mathcal{F}[E^{(\mu)}(0-,\underline{x})](\underline{y}) = -\exp(i\pi\mu)(2\pi)^\mu T_\mu \frac{1-i\underline{\xi}}{2}.$$

For natural values of μ all these formulae reduce to the similar expressions in Subsection 4.2.

§ 5. A Reflection on the Nature of Some Operators

All operators discussed above are translation-invariant linear operators which thus can be represented by a multiplication operator on the Fourier transform side, i.e. for such an operator L there is a function $\alpha(\underline{y})$, called multiplier or symbol of the operator, such that

$$\mathcal{F}[L[f]](\underline{y}) = \alpha(\underline{y})\mathcal{F}[f](\underline{y}),$$

or alternatively, at least on test functions $f \in \mathcal{S}$:

$$L[f] = K * f,$$

where the distribution K is given by $\mathcal{F}[K] = \alpha$. One could say that the Fourier transform acts as a mirror between "convolution space" and "multiplication space". The symbols of the operators considered so far may also be found, up to constants, in a formal way, as we will explain now.

Take e.g. the scalar operator $r^2 = \sum_{j=1}^{m} x_j^2 = (i\underline{x})^2$ acting as a multiplication operator by juxtaposition. Replacing formally x_j by ∂_{x_j}, for all $j = 1, \cdots, m$, the Laplace operator $\Delta = \sum_{j=1}^{m} \partial_{x_j}^2$ is obtained, which we interpret as a convolution operator for which

$$\mathcal{F}[(-\Delta)\delta * f] = 4\pi r^2 \mathcal{F}[f].$$

The same holds for the Clifford vector multiplication operator $i\underline{x} = i\sum_{j=1}^{m} e_j x_j$ and its convolution counterpart, the Dirac operator $\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j}$, for which

$$\mathcal{F}[\underline{\partial}\delta * f] = 2\pi i\underline{x}\mathcal{F}[f].$$

The Hilbert-Dirac operator $H\underline{\partial} = \underline{\partial}H = (-\Delta)^{1/2}$ is a scalar convolution operator with convolution kernel $\mathcal{H}\underline{\partial} = \underline{\partial}\mathcal{H}$; clearly its counterpart in "multiplication space" is $|i\underline{x}| = r$ and indeed:

$$\mathcal{F}[\mathcal{H}\underline{\partial} * f] = 2\sqrt{\pi} r \mathcal{F}[f].$$

Now observing that in "multiplication space" $\frac{i\underline{x}}{r} = i\underline{\omega}$, its counterpart in "convolution space" should be obtained in a formal way by dividing $\underline{\partial}$ by $H\underline{\partial} = (-\Delta)^{1/2}$:

$$\frac{\underline{\partial}}{(-\Delta)^{1/2}} = \frac{\underline{\partial}}{H\underline{\partial}} = \frac{H\underline{\partial}}{H^2\underline{\partial}} = \frac{H\underline{\partial}}{\underline{\partial}} = H,$$

since, as is well known,

$$H^2[f] = H[H[f]] = \mathcal{H} * (\mathcal{H} * f) = (\mathcal{H} * \mathcal{H}) * f$$

and

$$\mathcal{H} * \mathcal{H} = \left(\frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} \right)^2 U^*_{-m} * U^*_{-m} = \delta.$$

And indeed, this formal reasoning is given sense by

$$\mathcal{F}[H[f]] = \mathcal{F}[\mathcal{H} * f] = i\underline{\omega}\mathcal{F}[f].$$

So we can draw the "Fourier-conversion" table (see Table 1), where we have used the notation $z^{\pm}_{\mu} = \frac{1 \pm \exp(i\pi\mu)}{2}$.

Table 1. Fourier Conversion Table

convolution operators	multiplication operators
$\underline{\partial} = -2\pi \frac{\Gamma\left(\frac{m}{2}+1\right)}{\pi^{\frac{m}{2}+1}} U^*_{-m-1} *$	$i\underline{x} = iU_1$
$-\Delta = \underline{\partial}^2 = (2\pi)^2 \frac{\Gamma\left(\frac{m}{2}+1\right)}{\pi^{\frac{m}{2}+1}} T^*_{-m-2} *$	$r^2 = (i\underline{x})^2 = T_2$
$\underline{\partial}^{2k+1} = -(2\pi)^{2k+1} \frac{\Gamma\left(\frac{m}{2}+k+1\right)}{\pi^{\frac{m}{2}+k+1}} U^*_{-m-2k-1} *$	$(i\underline{x})^{2k+1} = iU_{2k+1}$
$\underline{\partial}^{2k} = (2\pi)^{2k} \frac{\Gamma\left(\frac{m}{2}+k\right)}{\pi^{\frac{m}{2}+k}} T^*_{-m-2k} *$	$r^{2k} = (i\underline{x})^{2k} = T_{2k}$
$H\underline{\partial} = (-\Delta)^{1/2} = \lvert\underline{\partial}\rvert = 2\pi \frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} T^*_{-m-1} *$	$r = \lvert i\underline{x}\rvert = T_1$
$\frac{\underline{\partial}}{\lvert\underline{\partial}\rvert} = \frac{\underline{\partial}}{(-\Delta)^{1/2}} = H = -\frac{2}{a_{m+1}} U_{-m} *$	$\frac{i\underline{x}}{\lvert\underline{x}\rvert} = \frac{i\underline{x}}{r} = i\underline{\omega} = iU_0$
$H\underline{\partial}^{2k+1} = \frac{2(2\pi)^{k+1}}{a_{m+1}} (m+1)(m+3)\cdots(m+2k-1) T^*_{-m-2k-1} *$	$\lvert i\underline{x}\rvert^{2k+1} = r^{2k+1} = T_{2k+1}$
$H\underline{\partial}^{2k} = -\frac{2(2\pi)^{k}}{a_{m+1}} (m+1)(m+3)\cdots(m+2k-1) U^*_{-m-2k} *$	$i\underline{\omega}\lvert i\underline{x}\rvert^{2k} = iU_{2k}$
$\underline{\partial}^{\mu} = z^+_{\mu}\ a_{\mu}\ T_{-m-\mu} * - z^-_{\mu}\ b_{\mu}\ U_{-m-\mu} *$	$(i\underline{x})^{\mu} = z^+_{\mu}\ T_{\mu} + i\ z^-_{\mu}\ U_{\mu}$
$(-\Delta)^{\mu} = c_{\mu}\ T_{-m-2\mu} *$	$r^{2\mu} = T_{2\mu}$
$\mathcal{H}\underline{\partial}^{\mu} = z^-_{\mu}\ a_{\mu}\ T_{-m-\mu} * - z^+_{\mu}\ b_{\mu}\ U_{-m-\mu} *$	$i\underline{\omega}(i\underline{x})^{\mu} = z^-_{\mu}\ T_{\mu} + i z^+_{\mu} U_{\mu}$

Note that the symbol of the Hilbert operator H, viz $i\underline{\omega} = iU_0$, is a bounded function, which reflects the boundedness of the operator $H : L_2(\mathbb{R}^m; \mathbb{R}_{0,m}) \longrightarrow L_2(\mathbb{R}^m; \mathbb{R}_{0,m})$. Moreover $i\underline{\omega}$ is the only multiplication operator in the table which is a homogeneous function of order zero, which is in accordance with the dilation-invariance of H and the fact that $H[f] = \mathcal{H} * f$ is a singular integral operator.

Note also the polynomial character of the symbols of the operators $H\underline{\partial}^p = \underline{\partial}^p H$, $p \in \mathbb{N}$, reflecting their boundedness between the Sobolev spaces W_2^n and W_2^{n-p} $(n \geq p)$.

Finally, note the strongly unifying character of the T^*_λ and U^*_λ operators in all of these.

§ 6. Historical Comment

The radial distributions $T_\lambda = Fp\, r^\lambda$, $r = \lvert\underline{x}\rvert$, $\lambda \in \mathbb{C}$, are of course well known. In [**?**] Riesz introduced their normalizations

$$R_\alpha = \frac{1}{2^\alpha \pi^{\frac{m}{2}}} \frac{\Gamma\left(\frac{m-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} Fp r^{\alpha-m} = \frac{\Gamma\left(\frac{m-\alpha}{2}\right)}{2^\alpha \pi^{\frac{\alpha+m}{2}}} T^*_{\alpha-m}, \qquad \alpha \neq -2l,\ \alpha \neq m+2l,\ l \in \mathbb{N}_0,$$

$$R_{-2l} = (-\Delta)^l \delta = \frac{2^{2l}\Gamma\left(\frac{m}{2}+l\right)}{\pi^{\frac{m}{2}-l}} T^*_{-m-2l}, \qquad l \in \mathbb{N}_0,$$

$$R_{m+2l} = \frac{2(-1)^l}{\pi^{\frac{m}{2}} 2^{m+2l} \Gamma\left(\frac{m}{2}+l\right) l!} r^{2l} \Big(\log \frac{1}{\pi r} + A_{m,l} \Big), \qquad l \in \mathbb{N}_0,$$

where

$$A_{m,l} = \frac{1}{2}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{l} - C\Big) + \frac{1}{2}\frac{\Gamma'\left(\frac{m}{2}+l\right)}{\Gamma\left(\frac{m}{2}+l\right)}$$

and C is Euler's constant. The classical Riesz potential then reads

$$I^{\gamma}[f] = R_{\gamma} * f, \qquad f \in \mathcal{S}.$$

Defined in this way these Riesz normalisations satisfy convolution formulae of a very simple form

$$R_{\alpha} * R_{\beta} = R_{\alpha+\beta},$$

and their Fourier transforms are simply given by

$$\mathcal{F}[R_{\alpha}] = \frac{1}{(2\pi)^m}\frac{1}{\rho^{\alpha}}.$$

However note that, when ignoring the additional definition of R_{m+2l}, R_{α} shows simple poles at $\alpha = m+2l$, $l \in \mathbb{N}_0$. Moreover $\mathcal{F}[R_{-2l}]$ and $\mathcal{F}[R_{m+2l}]$, $l \in N_0$ are no Riesz kernels anymore, whereas T^*_{λ} is an entire function of $\lambda \in \mathbb{C}$ and the Fourier transform is a bijection in the family $\{T^*_{\lambda} : \lambda \in \mathbb{C}\}$.

In [**?**] Horváth introduced the vectorial kernels

$$\vec{N}_{\alpha} = -\vec{\nabla} R_{\alpha+1},$$

which, for $\alpha \neq -2l-1$ and $\alpha \neq m+2l+1$, $l \in \mathbb{N}_0$, are given by

$$\vec{N}_{\alpha} = \frac{1}{2^{\alpha}\pi^{\frac{m}{2}}}\,\frac{\Gamma\left(\frac{m-\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)}\,\frac{\vec{x}}{r^{m-\alpha+1}}.$$

These kernels satisfy the convolution formulae

$$\vec{N}_{\alpha} * \vec{N}_{\beta} = -R_{\alpha+\beta},$$

where the convolution of the two vector valued distributions has to be taken in the sense of a scalar product.

If the Euclidean vector $\vec{x}$ is identified with the Clifford vector $\underline{x}$, then the Horváth kernels $\vec{N}_{\alpha}$ correspond to the Clifford distributions

$$\vec{N}_{\alpha} \approx \frac{\Gamma\left(\frac{m-\alpha+1}{2}\right)}{2^{\alpha}\pi^{\frac{\alpha+m+1}{2}}} U^*_{\alpha-m}.$$

Again note that $\vec{N}_{\alpha}$ shows simple poles at $\alpha = m+2l+1$, $l \in \mathbb{N}_0$, whereas $U^*_{\alpha-m}$ is an entire function of $\lambda \in \mathbb{C}$. The convolution of $\vec{N}_{\alpha}$ with an appropriate function, say a rapidly decreasing one, then gives rise to the vectorial counterpart of our Clifford vector valued Riesz potentials of the second kind.

For $\alpha = 1$ the Horváth kernel turns into the vectorial kernel of Newtonian force

$$\vec{N}_1 = \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{\frac{m}{2}}}\frac{\vec{x}}{r^m},$$

in which one recognizes, after identification and up to a minus sign, the fundamental solution of the Dirac operator $\underline{\partial}$:

$$\vec{N}_1 \approx \frac{1}{a_m}\,\frac{\underline{\omega}}{r^{m-1}} = \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{\frac{m}{2}+1}}\,U^*_{1-m}.$$

The relation $\vec{\nabla} \circ \vec{N}_1 = \delta$ in [**?**], where the nabla operator $\vec{\nabla}$ is acting as a divergence, is then the vectorial equivalent of the formula

$$\underline{\partial}\Big(- \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{\frac{m}{2}+1}} U^*_{1-m} \Big) = \delta$$

(see Lemma **??**(ii)), where the Dirac operator $\underline{\partial}$ acts under geometric multiplication.

In the special case where $\alpha = 0$, the Horváth kernel

$$\vec{N}_0 = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} \, \frac{\vec{x}}{r^{m+1}} = \frac{2}{a_{m+1}} \, \frac{\vec{x}}{r^{m+1}}$$

is a vectorial generalization to $\mathbb{R}^m$ of the $Pv\frac{1}{x}$ kernel on the real line. It satisfies the reciprocity formula

$$\vec{N}_0 * \vec{N}_0 = -\delta,$$

which was proved first in [**?**]. This kernel leads to the vectorial Hilbert transform in $\mathbb{R}^m$, the components of which are the so-called Riesz-transforms R_j, $j = 1, \cdots, m$, given by

$$R_j f(\vec{x}) = \lim_{\varepsilon \to 0+} \frac{2}{a_{m+1}} \int_{|\vec{x}-\vec{y}|>\varepsilon} \frac{x_j - y_j}{|\vec{x}-\vec{y}|^{m+1}} f(\vec{y}) dV(\vec{y}).$$

For a nice historical background of the Hilbert transform we refer the reader to [7] and [8]. Note that $\vec{N}_0$ corresponds, up to a minus sign, to the Hilbert convolution kernel in the Clifford setting

$$-\vec{N}_0 \approx \frac{2}{a_{m+1}} \; Pv \; \frac{\underline{\omega}}{r^m} = -\frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} U^*_{-m},$$

which was introduced in the 1980's when exploring the deeper relationship between Clifford analysis and classical analysis in Euclidean space. However, to the authors' knowledge, the first one to have considered the Hilbert convolution kernel as a function taking values in the subspace $\mathbb{R}^1_{0,m}$ of Clifford vectors, is Horváth in [**?**]; there he establishes the reciprocity property $\mathcal{H} * \mathcal{H} = \delta$ by proving that the bivector part of $\mathcal{H} * \mathcal{H}$ is indeed zero.

Finally, the scalar convolution operator

$$\Lambda = \vec{\nabla} \circ \vec{N}_0 = -\vec{\nabla} \circ (\vec{\nabla} R_1) = (-\Delta) R_1 = R_{-1}$$

satisfies

$$\Lambda * \Lambda = -\Delta,$$

a result which already figures in [6]; one recognizes in Λ the Hilbert-Dirac kernel

$$\mathcal{H}\underline{\partial} = \underline{\partial}\mathcal{H} = \frac{4\pi}{a_{m+1}} T^*_{-m-1}$$

discussed above.

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