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ETA INVARIANTS, DIFFERENTIAL CHARACTERS AND FLAT VECTOR BUNDLES**

J. M. BISMUT*

Abstract

The purpose of this paper is to give a refinement of the Atiyah-Singer families index theorem at the level of differential characters. Also a Riemann-Roch-Grothendieck theorem for the direct image of flat vector bundles by proper submersions is proved, with Chern classes with coefficients in $\mathbf{C}/\mathbf{Q}$. These results are much related to prior work of Gillet-Soulé, Bismut-Lott and Lott.

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§ 0. Introduction and Main Theorems

The purpose of this paper is to establish a version of the Atiyah-Singer families index theorem (see [3]), where the Chern character takes its values in the $\mathbf{C}/\mathbf{Q}$ differential characters of Cheeger-Simons [16]. As a non trivial corollary, we obtain a version of a Riemann-Roch-Grothendieck theorem for the direct image of complex flat vector bundles by proper submersions, where the Chern classes are cohomology classes with coefficients in $\mathbf{C}/\mathbf{Q}$. This last result complements a result of Bismut-Lott [14, Theorem 0.1] for the corresponding imaginary parts.

Let $\pi : M \to B$ be a proper submersion, with a compact oriented spin even dimensional fibre Z. Let $T^H M$ be a vector subbundle of TM such that $TM = T^H M \oplus TZ$. Let g^{TZ} be a metric on TZ. Let ∇^{TZ} be the Euclidean correction on (TZ, g^{TZ}) constructed in [6, Section 1] which is canonically associated to $(\pi, T^H M, g^{TZ})$. Let S^{TZ} be the $\mathbf{Z}_2$-graded vector bundle of (TZ, g^{TZ}) spinors. Let $(\xi, g^{\xi}, \nabla^{\xi})$ be a complex hermitian vector bundle with connection. Let D^Z be the family of Dirac operators acting on smooth sections of $S^{TZ} \otimes \xi$ along the fibres Z.

Assume that the dimension of $\ker D^Z$ is locally constant, i.e. $\ker D^Z$ is a vector bundle on B. Then $\ker D^Z$ inherits a natural L_2 metric. Let $\nabla^{\ker D^Z}$ be the unitary connection on $\ker D^Z$ one obtains by projecting the unitary connection on the bundle of sections of $S^{TZ} \otimes \xi$ along the fibres Z constructed in [6].

Let $\tilde{\eta}$ be the odd real form on B of [7] such that

$$d\tilde{\eta} = \pi_*[\widehat{A}(TZ, \nabla^{TZ})\mathrm{ch}\,(\xi, \nabla^{\xi})] - \mathrm{ch}\,(\ker D^Z, \nabla^{\ker D^Z}). \tag{0.1}$$

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*Département de Mathématique, Université Paris-Sud, Bâtiment 425, F-91405 Orsay Cedex, France.
E-mail: Jean-Michel.Bismut@math.u-psud.fr
**with an Appendix by K. Corlette and H. Esnault

Let $\widehat{H}(M, \mathbf{R}/\mathbf{Q})$ be the ring of differential characters of Cheeger-Simons [16] with coefficients in $\mathbf{R}/\mathbf{Q}$. If P is a characteristic polynomial with coefficients in $\mathbf{Q}$, and if (E, ∇^E) is a vector bundle equipped with a metric preserving connection, we denote by $\widehat{P}(E, \nabla^E) \in \widehat{H}(M, \mathbf{R}/\mathbf{Q})$ the differential character constructed in [16, Section 2], which lifts the obvious Chern-Weil real closed form $P(E, \nabla^E)$. We use a similar notation on B.

Theorem 0.1. *The following identity holds,*

$$\widehat{\mathrm{ch}}(\ker D^Z, \nabla^{\ker D^Z}) + \tilde{\eta} = \pi_*[A(TZ, \nabla^{TZ})\widehat{\mathrm{ch}}(\xi, \nabla^\xi)] \qquad \textit{in} \;\; \widehat{H}(B, \mathbf{R}/\mathbf{Q}). \tag{0.2}$$

In fact Theorem 0.1 is a straightforward consequence of the original expression given by Cheeger-Simons for the eta invariant in terms of differential characters (see [16]), and of the adiabatic limits results of Bismut-Cheeger [7] and Dai [17], of which (0.2) is a considerable weakening.

Now we make the same assumptions as before, except that we no longer assume the fibres Z to be oriented, spin, or even dimensional.

Let (F, ∇^F) be a complex flat vector bundle on M. If P is taken as before, a simple modification of [16] allows us to define $\widehat{P}(F, \nabla^F) \in H^{\mathrm{odd}}(M, \mathbf{C}/\mathbf{Q})$. Let e be the Euler class, and c be the Chern class. Then $\hat{e}(TZ, \nabla^{TZ})\hat{c}(F, \nabla^F) \in \widehat{H}(M, \mathbf{C}/\mathbf{Q})$ does not depend on the choice of the Euclidean connection ∇^{TZ}, and will just be denoted by $\hat{e}(TZ)\hat{c}(F, \nabla^F)$.

Let $H(Z, F_{|Z})$ be the direct image of F by π. Then $H(Z, F_{|Z})$ is a complex $\mathbf{Z}$-graded vector bundle on M, equipped with a flat connection $\nabla^{H(Z, F_{|Z})}$.

Theorem 0.2. *If the orientation bundle $o(TZ)$ is the lift of a $\mathbf{Z}_2$-line bundle on B, then*

$$\hat{c}(H(Z, F_{|Z}), \nabla^{H(Z, F_{|Z})}) = \pi_*[\hat{e}(TZ)\hat{c}(F, \nabla^F)] \qquad \textit{in} \;\; \widehat{H}(B, \mathbf{C}/\mathbf{Q}). \tag{0.3}$$

To establish Theorem 0.2, we use a result of Bismut-Lott [14, Theorem 0.1], which says that in full generality, the imaginary parts of (0.3) coincide. As to the real parts, we use essentially a version of Theorem 0.1, for another family of Dirac operator $D^{Z,dR}$. Also, by using a result of [14], we show that $\tilde{\eta} = 0$, from which (0.3) follows.

We refer to the appendix by K. Corlette and H. Esnault for more precise statements when (F, ∇) is the trivial bundle with connection $(\mathbf{C}, d)$, so that $H^i(Z, \mathbf{C})$ is equipped with the classical Gauss-Manin connection. In fact when π is a proper submersion with orientable fibres, then the total class $\hat{c}(H^i(Z, \mathbf{C}) \oplus H^{\dim Z - i}(Z, \mathbf{C}))$ vanishes, and if π is a proper smooth morphism of complex manifolds with Kähler fibres, then the total class $\hat{c}(H^i(Z, \mathbf{C}))$ of the individual $H^i(Z, \mathbf{C})$ vanishes.

Theorem 0.1 has already appeared in the literature. In fact, in degree 1, (0.2) is a mod $(\mathbf{Q})$ version of the holonomy theorem of [9] (this mod $(\mathbf{Q})$ version being quite easy, the whole purpose of [9] being to lift the mod $(\mathbf{Q})$ ambiguity). Also in [22], when M and B are Kähler and ξ is holomorphic, Gillet and Soulé have given a proof of Theorem 0.1, based on the classical Riemann-Roch-Grothendieck theorem, on the existence of Bott-Chern classes (see [10]), of analytic torsion forms (see [11, 21], and the Hodge decomposition on B. Using the existence of a map from their arithmetic Chow groups (see [18, 19]) to the differential characters, they could give a direct verification of the weaker differential character version of their conjectural formula for arithmetic Riemann-Roch in Arakelov geometry (see [20]). This conjecture has later been proved in [21] for the first Chern class, using the analytic results of [13].

It thus seems natural to write a simple proof of Theorem 0.1 in full generality in the C^∞ category, in which the objects appearing in (0.2) live naturally. Incidently note that the adiabatic limit argument which allows us to pass from [16] to Theorem 0.1 cannot be extended to the Arakelov theoretic setting, even if one uses the results of [5] on the adiabatic limit of the Ray Singer holomorphic torsion of a fibered manifold, the obstruction being the Hodge conjecture.

In [23], Lott has developed a $\mathbf{R}/\mathbf{Z}$ index theory, for the direct image of a virtual bundle whose Chern character vanishes. The identity established in [23, Corollary 2] is an identity of odd classes in $H(B, \mathbf{R}/\mathbf{Q})$, which is essentially equivalent to Theorem 0.1 when $\mathrm{ch}(\xi)$ vanishes in $H(M, \mathbf{Q})$. The formula of Lott [23] also contains the form $\tilde{\eta}$, and its proof uses [7] and [17]. Finally, let us observe that in [23], by modifying the form $\tilde{\eta}$ "at infinity", Lott has extended his formula to the case where $\ker D^Z$ is not necessarily a vector bundle on B. A similar deformation argument can also be worked out in the context of Theorem 0.1.

When taking the direct image of a flat vector bundle F, the main point of Theorem 0.2 is that the form $\tilde{\eta}$ vanishes identically. Rather surprisingly, the work of Bismut-Lott [14], which is essentially devoted to a refinement of the equality of the imaginary parts of (0.3) at the level of differential forms, also provides us with the argument which shows the vanishing of the form $\tilde{\eta}$.

This paper is organized as follows. In Section 1, we prove Theorem 0.1. In Section 2, we construct differential characters with values in $\mathbf{C}/\mathbf{Q}$, and we recall a few properties of the Euler class and of the corresponding Euler character. Finally in Section 3, we prove Theorem 0.2.

§ 1. Differential Characters and the $\tilde{\eta}$ Form

The purpose of this section is to establish Theorem 0.1. This result is a straightforward application of a formula of Cheeger and Simon [16] for the eta invariants in terms of differential characters, and of the results on adiabatic limits of eta invariants of Bismut-Cheeger [7] and Dai [17]. Also we discuss briefly the relation of Theorem 0.1 to corresponding results of Gillet-Soulé [22].

This section is organized as follows. In (a), we briefly recall the construction by Cheeger and Simons [16] of the differential characters. In (b), we state the result of [16] expressing the eta invariant in terms of differential characters. In (c), we give the construction in [7] of the form $\tilde{\eta}$. Finally in (d), we prove Theorem 0.1.

(a) The differential characters of Cheeger and Simons

Let M be a compact oriented manifold. Let $\mathbf{A} = \mathbf{R}$ or $\mathbf{C}$. Let

$$\widehat{H}^{\bullet}(M, \mathbf{A}/\mathbf{Q}) = \bigoplus_{0}^{\dim M} \widehat{H}^{k}(M, \mathbf{A}/\mathbf{Q})$$

be the ring of differential characters of Cheeger and Simons [16] with values in $\mathbf{A}/\mathbf{Q}$. To make the reading of this paper easier, we briefly recall the definition of the differential characters and some of their properties.

Let $C_{\bullet}(M)$ (resp. $C^{\bullet}(M)$) be the group of singular chains (resp. cochains) in M. Let $Z_{\bullet}(M) \subset C_{\bullet}(M)$ be the subgroup of cycles in M. Let $\Lambda^{\mathbf{A}}(M)$ be the set of smooth $\mathbf{A}$-valued differential form on M, let $\Lambda_0^{\mathbf{A}}(M)$ be the subgroup of smooth $\mathbf{A}$-valued closed differential forms on M with periods lying in $\mathbf{Q}$. In particular, if $\omega \in \Lambda_0^{\mathbf{C}}(M)$, $\mathrm{Im}(\omega)$ is exact.

Put

$$\widehat{H}^{\bullet}(M, \mathbf{A}/\mathbf{Q}) = \{f \in \mathrm{Hom}(Z_{\bullet}(M), \mathbf{A}/\mathbf{Q}), df \in \Lambda^{\mathbf{A}}(M)\}. \tag{1.1}$$

Let $\int_M : \widehat{H}^\bullet(M, \mathbf{A}/\mathbf{Q}) \to \mathbf{A}/\mathbf{Q}$ be the obvious evaluation map. Then by [16, Theorem 1.1], there are natural exact sequences

$$\begin{aligned} &0 \to H^\bullet(M, \mathbf{A}/\mathbf{Q}) \to \widehat{H}^\bullet(M, \mathbf{A}/\mathbf{Q}) \xrightarrow{\delta_1} \Lambda_0^{\mathbf{A}}(M)[1] \to 0, \\ &0 \to \frac{\Lambda^{\mathbf{A}}(M)}{\Lambda_0^{\mathbf{A}}(M)} \to \widehat{H}^\bullet(M, \mathbf{A}/\mathbf{Q}) \xrightarrow{\delta_2} H^\bullet(M, \mathbf{Q})[1] \to 0. \end{aligned} \tag{1.2}$$

The construction of (δ_1, δ_2) is as follows. If $f \in \widehat{H}(M, \mathbf{A}/\mathbf{Q})$, let T be a $\mathbf{A}$-valued cochain whose $\mathbf{A}/\mathbf{Q}$ reduction $\widetilde{T}$ is such that $\widetilde{T}_{|Z} = f$. Then there is $(\omega, c) \in \Lambda_0^{\mathbf{A}}(M) \times Z^\bullet(M, \mathbf{Q})$ such that

$$dT = \omega - c. \tag{1.3}$$

Let $[c] \in H^\bullet(M, \mathbf{Q})$ be the class of c. Then by [16, Proof of Theorem 1.1],

$$\delta_1 f = \omega, \qquad \delta_2 f = [c]. \tag{1.4}$$

Also in [16, Section 1], Cheeger and Simons have constructed a ring structure on $\widehat{H}(M, \mathbf{A}/\mathbf{Q})$, so that δ_1 and δ_2 are ring homomorphisms.

Assume now that $\mathbf{A} = \mathbf{R}$. Let (E, g^E, ∇^E) be a complex Hermitian vector bundle on M with unitary connection. Let P be an invariant polynomial with rational coefficients. Let $P(E, \nabla^E) \in \Lambda_0^{\mathbf{R}}(M)$ be the corresponding Chern-Weil closed differential form, which represents the characteristic class $P(E) \in H^{\mathrm{even}}(M, \mathbf{Q})$. In [16, Section 2], Cheeger and Simons have constructed a differential character $\widehat{P}(E, \nabla^E) \in \widehat{H}^{\mathrm{odd}}(M, \mathbf{R})$, which is "natural", and such that

$$\delta_1 \widehat{P}(E, \nabla^E) = P(E, \nabla^E). \tag{1.5}$$

(b) Differential characters and the eta invariant

Let M be a compact oriented spin manifold of odd dimension. Let g^{TM} be a Riemannian metric on TM. Let ∇^{TM} be the Levi-Civita connection on (TM, g^{TM}).

Let S^{TM} be the Hermitian bundle of (TM, g^{TM}) spinors. The connection ∇^{TM} lifts to a unitary connection $\nabla^{S^{TM}}$ on S^{TM}.

Let ξ be a complex vector bundle on M, let g^ξ be a Hermitian metric on ξ and let ∇^ξ be a unitary connection on (ξ, g^ξ).

Let $c(TM)$ be the Clifford algebra of (TM, g^{TM}), i.e., the algebra spanned by 1 and $X \in TM$, with the commutation relations

$$XY + YX = -2\langle X, Y\rangle, \qquad X, Y \in TM. \tag{1.6}$$

Then $S^{TM} \otimes \xi$ is a $c(TM)$ Clifford module.

Let D^M be the Dirac operator acting on $\Gamma(S^{TM} \otimes \xi)$. Namely, if $e_1, \cdots, e_n$ is an orthonormal frame in TM, then

$$D^M = \sum_1^n c(e_i) \nabla_{e_i}^{S^{TM} \otimes \xi}. \tag{1.7}$$

Let $\mathrm{Sp}(D^M)$ be the spectrum of D^M. Let $\eta(s)$ be the eta function of D^M defined by Atiyah-Patodi-Singer [2]. Namely, for $s \in \mathbf{C}, \mathrm{Re}(s) \gg 0$, set

$$\eta(s) = \sum_{\substack{\lambda \in \mathrm{Sp}(D^M) \\ \lambda \neq 0}} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}. \tag{1.8}$$

By [2, Theorem 4.2], $\eta(s)$ extends to a holomorphic function near $s = 0$. Set

$$\bar{\eta}(s) = \frac{1}{2}(\eta(s) + \dim \ker D^M). \tag{1.9}$$

Then $\bar{\eta}(0)$ is called the (reduced) eta invariant of D^M.

Let $\widehat{A}$ and ch be the Hirzebruch genus and the Chern character respectively. Let $\widehat{\widehat{A}}(TM, \nabla^{TM})$, $\widehat{\mathrm{ch}}(\xi, \nabla^\xi)$ denote the lifts in $\widehat{H}^{\mathrm{odd}}(M, \mathbf{R}/\mathbf{Q})$ of the Chern-Weil representatives $\widehat{A}(TM, \nabla^{TM})$, $\mathrm{ch}(\xi, \nabla^\xi)$. Now, we recall a result of Cheeger and Simons [16].

Theorem 1.1. (Cheeger and Simons) *The following identity holds,*

$$\bar{\eta}(0) = \int_M \widehat{\widehat{A}}(TM, \nabla^{TM}) \widehat{\mathrm{ch}}(\xi, \nabla^\xi) \qquad \textit{in} \ \ \mathbf{R}/\mathbf{Q}. \tag{1.10}$$

Proof. The proof of [16, Theorem 9.2] is an easy consequence of the index theorem of Atiyah- Patodi-Singer for manifolds with boundary [2], and of the definition of the differential characters.

(c) The local families index theorem and the form $\tilde{\eta}$

Let now $\pi : M \to B$ be a submersion of smooth manifolds with compact even dimensional fibre Z. We assume that TZ is oriented and spin. Let g^{TZ} be a metric on TZ. Let $S^{TZ} = S^{TZ}_+ \oplus S^{TZ}_-$ be the corresponding $\mathbf{Z}_2$-graded Hermitian vector bundle of (TZ, g^{TZ}) spinors. Let $\nabla^{TZ}_{|Z}$ be the Levi-Civita connection on (TZ, g^{TZ}) along the fibres Z. The connection $\nabla^{TZ}_{|Z}$ induces a fibrewise connection $\nabla^{S^{TZ}}_{|Z}$ on S^{TZ}. Let ξ be a complex vector bundle on M, let g^ξ be a Hermitian metric on ξ, and let ∇^ξ be a unitary connection on ξ. Let $c(TZ)$ be the Clifford algebra of TZ. Then $S \otimes \xi$ is a $c(TZ)$ Clifford module.

Definition 1.2. *For $b \in B$, let $H_b = H_b^+ \oplus H_b^-$ be the vector space of smooth sections of $(S^{TZ} \otimes \xi)_{|Z_b}$ on Z_b.*

Then $H = H^+ \oplus H^-$ is a $\mathbf{Z}_2$-graded vector bundle on B. Let dv_Z be the volume form along the fibre Z. We equip H with the L_2 Hermitian product

$$s, s' \in H \to \langle s, s' \rangle = \int_Z \langle s, s' \rangle_{S^{TZ} \otimes \xi} dv_Z. \tag{1.11}$$

For $b \in B$, let D_b^Z be the Dirac operator acting on H_b. If $e_1, \cdots, e_n$ is an orthonormal frame in TZ, then

$$D^Z = \sum c(e_i) \nabla^{S^{TZ} \otimes \xi}_{|Z, e_i}. \tag{1.12}$$

Moreover D^Z exchanges H_+ and H_-, so that

$$D^Z = \begin{bmatrix} 0 & D_-^Z \\ D_+^Z & 0 \end{bmatrix}. \tag{1.13}$$

Let $T^H M$ be a subbundle of TM such that

$$TM = T^H M \oplus TZ. \tag{1.14}$$

Let $P^{TZ} : TM \to TZ$ be the associated projection. Let ∇^{TZ} be the connection on (TZ, g^{TZ}) associated to $(T^H M, g^{TZ})$, which is constructed in [6, Theorem 1.9]. In other words, let g^{TB} be a metric on TB. Let

$$g^{TM} = \pi^* g^{TB} \oplus g^{TZ} \tag{1.15}$$

be the obvious metric on $TM = T^H M \oplus TZ$. Let $\nabla^{TM,L}$ be the Levi-Civita connection on (TM, g^{TM}). Set

$$\nabla^{TZ} = P^{TZ} \nabla^{TM,L}. \tag{1.16}$$

Then by [6, Theorem 1.9], the connection ∇^{TZ} does not depend on g^{TB}. Also ∇^{TZ} preserves g^{TZ} and restricts to the Levi-Civita connection $\nabla^{TZ}_{|Z}$ along the fibres Z. Let $\nabla^{S^{TZ}}$ be the connection induced by ∇^{TZ} on S^{TZ}.

If $U \in TB$, let U^H be the lift of U in $T^H M$, so that $\pi_* U^H = U$.

Definition 1.3. *Let ∇^H be the connection on H, such that if s is a smooth section of $S^{TZ} \otimes \xi$ over M, if $U \in TB$, then*

$$\nabla^H_U s = \nabla^{S^{TZ} \otimes \xi}_{U^H} s. \tag{1.17}$$

If $U \in TB$, the Lie derivative operator L_{U^H} acts on the tensor algebra of TZ. If $U \in TB$, let $k(U)$ be given by

$$\frac{1}{2} L_{U^H} dv_Z = k(U) dv_Z. \tag{1.18}$$

Definition 1.4. *Set*

$$\nabla^{H,u} = \nabla^H + k. \tag{1.19}$$

Then the connection $\nabla^{H,u}$ preserves the Hermitian product (1.11). If $U, V \in TB$, set

$$T(U,V) = -P^{TZ}[U^H, V^H]. \tag{1.20}$$

Now we use the superconnection formalism of Quillen [26]. First we recall the definition of the Levi-Civita superconnection of [6, Section 3].

Definition 1.5. *For $t > 0$, let A_t be the Levi-Civita superconnection on H,*

$$A_t = \nabla^{H,u} + \sqrt{t} D^Z - \frac{c(T)}{4\sqrt{t}}. \tag{1.21}$$

We fix once and for all a square root $\sqrt{i}$ of i. Our formulas will not depend on this choice. Let $\varphi : \Lambda(T^*M) \to \Lambda(T^*M)$ be given by $\alpha \to (2\pi i)^{-\deg \alpha / 2} \alpha$.

Definition 1.6. *For $t > 0$, set*

$$\alpha_t = \varphi \mathrm{Tr}_s[\exp(-A_t^2)]. \tag{1.22}$$

By [6, Theorem 3.4], [8, Theorem 1.5], the forms α_t are real, even, closed, and represent $\mathrm{ch}(\ker D^Z_+ - \ker D^Z_-)$ in cohomology. In the sequel, the convergence of forms on B is taken in the sense of uniform convergence over compact sets together with their derivatives. First we recall the local families index theorems of [6, Theorems 4.12 and 4.16].

Theorem 1.7. *As $t \to 0$,*

$$\alpha_t \to \alpha_0 = \pi_*[\widehat{A}(TZ, \nabla^{TZ}) \mathrm{ch}(\xi, \nabla^\xi)]. \tag{1.23}$$

Definition 1.8. *Set*

$$\beta_t = (\sqrt{2i\pi})^{-1}\varphi\mathrm{Tr}_s\Big[\Big(\frac{D^Z}{2\sqrt{t}} + \frac{c(T)}{8t^{3/2}}\Big)\exp(-A_t^2)\Big]. \tag{1.24}$$

Now we recall some results of [11, Theorems 2.11 and 2.20] and [7, Theorem 4.35].

Theorem 1.9. *The forms β_t are odd and real. Moreover for $t > 0$,*

$$\frac{\partial\alpha_t}{\partial t} = -d\beta_t. \tag{1.25}$$

Finally, as $t \to 0$,

$$\beta_t = \mathcal{O}(1). \tag{1.26}$$

Assume now that $\ker D_+^Z$, $\ker D_-^Z$ have locally constant dimension over B. Then they form smooth complex vector bundles on B. Let $P^{\ker D^Z} : H \to \ker D^Z$ be the orthogonal projection operator. Let $g^{\ker D^Z}$ be the metric on $\ker D^Z$ induced by the Hermitian product (1.6) on $\ker D^Z$.

Definition 1.10. *Set*

$$\nabla^{\ker D^Z} = P^{\ker D^Z}\nabla^{H,u}. \tag{1.27}$$

Then $\nabla^{\ker D^Z} = \nabla^{\ker D_+^Z} \oplus \nabla^{\ker D_-^Z}$ preserves the metric $g^{\ker D^Z} = g^{\ker D_+^Z} \oplus g^{\ker D_-^Z}$. Put

$$\mathrm{ch}(\ker D^Z, \nabla^{\ker D^Z}) = \mathrm{ch}(\ker D_+^Z, \nabla^{\ker D_+^Z}) - \mathrm{ch}(\ker D_-^Z, \nabla^{\ker D_-^Z}). \tag{1.28}$$

Definition 1.11. *Set*

$$\alpha_\infty = \mathrm{ch}(\ker D^Z, \nabla^{\ker D^Z}). \tag{1.29}$$

Now we recall a result of [4, Theorem 9.23].

Theorem 1.12. *As $t \to +\infty$,*

$$\begin{aligned} \alpha_t &= \alpha_\infty + \mathcal{O}\Big(\frac{1}{\sqrt{t}}\Big), \\ \beta_t &= \mathcal{O}\Big(\frac{1}{t^{3/2}}\Big). \end{aligned} \tag{1.30}$$

Definition 1.13. *Set*

$$\tilde{\eta} = \int_0^{+\infty} \beta_t dt. \tag{1.31}$$

In view of Theorems 1.9 and 1.12, the form $\tilde{\eta}$ is well defined.

Theorem 1.14. *The smooth odd real form $\tilde{\eta}$ is such that*

$$d\tilde{\eta} = \pi_*[\widehat{A}(TZ, \nabla^{TZ})\mathrm{ch}(\xi, \nabla^\xi)] - \mathrm{ch}(\ker D^Z, \nabla^{\ker D^Z}). \tag{1.32}$$

Proof. This follows immediately from Theorems 1.7, 1.9 and 1.12.

(d) Differential characters and the form $\tilde{\eta}$

Theorem 1.15. *The following identity holds,*

$$\widehat{\mathrm{ch}}(\ker D^Z, \nabla^{\ker D^Z}) + \tilde{\eta} = \pi_*[\widehat{\widehat{A}}(T^Z, \nabla^{TZ})\widehat{\mathrm{ch}}(\xi, \nabla^\xi)] \qquad \text{in } \widehat{H}(B, \mathbf{R}/\mathbf{Q}). \tag{1.33}$$

Proof. We only need to prove (1.33) when B is orientable, the general case being obtained by replacing B by an oriented double cover. Set

$$\delta = \widehat{\mathrm{ch}}(\ker D^Z, \nabla^{\ker D^Z}) + \tilde{\eta} - \pi_*[\widehat{\widehat{A}}(TZ, \nabla^{TZ})\widehat{\mathrm{ch}}(\xi, \nabla^\xi)]. \tag{1.34}$$

In view of Theorem 1.14 and of [16, Theorem 1.1], it is clear that $\delta \in H^{\mathrm{odd}}(B, \mathbf{R}/\mathbf{Q})$. We will prove that

$$\delta = 0. \tag{1.35}$$

Assume first that B is compact and of odd dimension, and also that B is oriented and spin. Let $(\xi', g^{\xi'}, \nabla^{\xi'})$ be a complex vector bundle on B equipped with a metric $g^{\xi'}$ and a unitary connection $\nabla^{\xi'}$. Let g^{TB} be a metric on TB. Let g_ε^{TM} be the metric on $TM = T^H M \oplus TZ$,

$$g_\varepsilon^{TM} = \pi^* \frac{g^{TB}}{\varepsilon} \oplus g^{TZ}. \tag{1.36}$$

Let $\nabla_\varepsilon^{TM,L}$ be the Levi-Civita connection on (TM, g_ε^{TM}).

Clearly TM is oriented and spin. Let $\bar{\eta}_\varepsilon^{\xi \otimes \pi^* \xi'}(0)$ be the eta invariant of the Dirac operator on M associated to the metric g_ε^{TM} and the vector bundle $(\xi \otimes \pi^* \xi', \nabla^{\xi \otimes \pi^* \xi'})$. By Theorem 1.1,

$$\bar{\eta}_\varepsilon^{\xi \otimes \pi^* \xi'}(0) = \int_M \widehat{\widehat{A}}(TM, \nabla_\varepsilon^{TM})\widehat{\mathrm{ch}}(\xi, \nabla^\xi)\pi^*\widehat{\mathrm{ch}}(\xi', \nabla^\xi) \qquad \text{in } \mathbf{R}/\mathbf{Q}. \tag{1.37}$$

Let $\bar{\eta}^{\ker D_\pm^Z \otimes \xi'}(0)$ be the eta invariant of the Dirac operator on B associated to the metric g^{TB} and the vector bundle $(\ker D_\pm^Z \otimes \xi', \nabla^{\ker D_\pm^Z \otimes \xi'})$. Put

$$\bar{\eta}^{\ker D^Z \otimes \xi'}(0) = \bar{\eta}^{\ker D_+^Z \otimes \xi'}(0) - \bar{\eta}^{\ker D_-^Z \otimes \xi'}(0). \tag{1.38}$$

By proceeding as in [9, Equation (3.196)], we find that as $\varepsilon \to 0$,

$$\begin{aligned} &\int_M \widehat{\widehat{A}}(TM, \nabla_\varepsilon^{TM})\widehat{\mathrm{ch}}(\xi, \nabla^\xi)\pi^*\widehat{\mathrm{ch}}(\xi', \nabla^{\xi'}) \to \\ &\int_B \widehat{\widehat{A}}(TB, \nabla^{TB})\widehat{\mathrm{ch}}(\xi', \nabla^{\xi'})\pi_*[\widehat{\widehat{A}}(TZ, \nabla^{TZ})\widehat{\mathrm{ch}}(\xi, \nabla^\xi)] \qquad \text{in } \mathbf{R}/\mathbf{Q}. \end{aligned} \tag{1.39}$$

Now by the variational formula of [2], we know that as $\varepsilon \to 0$, $\bar{\eta}_\varepsilon^{\xi \otimes \pi^* \xi'}(0)$ converges in $\mathbf{R}/\mathbf{Z}$. More precisely, by results of Bismut-Cheeger [7, Theorem 4.35] and Dai [17, Theorem 0.1], we find that as $\varepsilon \to 0$,

$$\bar{\eta}_\varepsilon^{\xi \otimes \pi^* \xi'}(0) \to \int_B \widehat{A}(TB, \nabla^{TB})\mathrm{ch}(\xi', \nabla^{\xi'})\tilde{\eta} + \bar{\eta}^{\xi' \otimes \ker D^Z}(0) \qquad \mathrm{mod}\,(\mathbf{Z}). \tag{1.40}$$

Strictly speaking, we cannot directly apply [7] or [17], since in [7], it is assumed that $\ker D^Z = 0$ and in [17], that $\ker D^Z$ is a vector bundle, and that as $\varepsilon \to 0$, the dimension of $\ker D_\varepsilon^M$ stabilizes. Here, in general, this assumption is not verified. However, as explained in [23, Section 4], by a simple modification of the argument of [17], one shows easily that if one disregards small eigenvalues of D^M_ε, one introduces an error in the evaluation of $\bar\eta^{\xi\otimes\pi^*\xi'}_\varepsilon(0)$, which lies in $\mathbf{Z}$, without modifying the rest of the argument. Also by Theorem 1.1, we get

$$\bar\eta^{\ker D^Z\otimes\xi'}(0)=\int_B \widehat{\widehat{A}}(TB,g^{TB})\widehat{\mathrm{ch}}(\xi',\nabla^\xi)\widehat{\mathrm{ch}}(\ker D^Z,\nabla^{\ker D^Z}) \qquad \text{in } \mathbf{R}/\mathbf{Q}. \tag{1.41}$$

From (1.37)–(1.41), we find that for any $(\xi',\nabla^{\xi'})$,

$$\int_B \widehat{\widehat{A}}(TB,\nabla^{TB})\widehat{\mathrm{ch}}(\xi',\nabla^{\xi'})\delta=0 \qquad \text{in } \mathbf{R}/\mathbf{Q}. \tag{1.42}$$

Let $c_{\xi'}$ be an even cocycle in $H^{\mathrm{even}}(B,\mathbf{Q})$ representing $\widehat{A}(TB)\mathrm{ch}(\xi)$. Then (1.36) is equivalent to

$$\int_B c_{\xi'}\sqcup\delta=0 \qquad \text{in } \mathbf{R}/\mathbf{Q}. \tag{1.43}$$

As $\xi'\in K_0(B)$, the $\mathrm{ch}(\xi')$ generate $H^{\mathrm{even}}(B,\mathbf{Q})$. Since $\widehat{A}$ is a stable class (i.e. $\widehat{A}(0)=1$), as $\xi'\in K_0(B)$, $\widehat{A}(TB)\mathrm{ch}(\xi')$ generates $H^{\mathrm{even}}(B,\mathbf{Q})$. By (1.43), we get (1.33).

In the case of a general manifold B, we proceed differently. In effect by a result of Thom [27, Theorem 2.29], if h is any homology class, there is $n\in\mathbf{Z}$ and an oriented compact submanifold S such that S represents nh.

So let h be an odd homology class and let S be an oriented compact submanifold taken as before. Put

$$M_S=\pi^{-1}(S). \tag{1.44}$$

Then M_S is a compact oriented manifold, which fibres over S with compact fibre Z. If S is spin, we proceed as before, and by following the arguments after (1.41), we get

$$\delta=0. \tag{1.45}$$

If S is not spin, we replace the spin complex of S by the signature complex of [2]. The only difference with respect to the previous arguments is that $\widehat{A}$ is replaced by the modified genus of Hirzebruch $\mathcal{L}$, which is still stable. Ultimately, we still get (1.45). The proof of our theorem is completed.

Remark 1.16. It should be pointed out that when M and B are compact and Kähler, and the vector bundle ξ is holomorphic, Gillet and Soulé [22] give an easy proof of (1.27). Their proof uses the existence of analytic torsion forms and of Bott-Chern classes (see [10]), and the Hodge decomposition.

The holonomy theorem of [9, Theorem 3.16] follows easily from (1.34). In degree 1, (1.27) is a mod $(\mathbf{Q})$ version of [9]. In [21], Gillet and Soulé get directly this mod $(\mathbf{Q})$ version from (1.27).

Also in [23], Lott has established a result closely related to Theorem 1.15, by considering the case where the Chern character of the virtual bundle ξ vanishes. Then (1.27) becomes an equation of cohomology classes with coefficients in $\mathbf{R}/\mathbf{Q}$. The formula of Lott [23] also involves the form $\tilde\eta$, and its proof uses the adiabatic limit results of [7] and [17]. Note that in [23], Lott also gives a formula when $\ker D^Z$ is not a vector bundle on B. A similar extension can be given of Theorem 1.15 along the lines of [23].

§ 2. Differential Characters, the Euler Class and Flat Vector Bundles

This section is mostly technical. In (a) we recall the construction given in [16, Section 3] of the Euler differential character. In (b) we modify the constructions of [16] in a straightforward way to obtain differential characters valued in

$$\frac{\mathbf{C}}{\mathbf{Q}} \simeq \frac{\mathbf{R}}{\mathbf{Q}} \oplus \mathbf{R}.$$

Finally in (c) we give more explicit formulas for the differential characters of flat vector bundles. In the whole section, we use the notation of Section 1(a).

(a) The Euler differential character

Let X be a smooth manifold. Let E be a real oriented vector bundle on X. Let $e(E) \in H^{\dim E}(X, \mathbf{Z})$ be the Euler class of E. If $\dim E$ is odd, $e(E)$ is torsion. More precisely,

$$2e(E) = 0. \tag{2.1}$$

Let ∇^E be a metric preserving connection on E. Let $e(E, \nabla^E) \in \Lambda^{\mathbf{R}, \dim E}(X)$ be the Chern-Weil representative of $e(E)$, and let $\hat{e}(E, \nabla^E) \in \widehat{H}^{\dim E-1}(X, \mathbf{R}/\mathbf{Z})$ be the corresponding lift of $e(E, \nabla^E)$. If $\dim E$ is odd, $e(E, \nabla^E) = 0$. By [16, Theorem 1.1] (which is Equation (1.2) with $\mathbf{A} = \mathbf{R}$, and $\mathbf{Q}$ replaced by $\mathbf{Z}$), if $\dim E$ is odd,

$$\hat{e}(E, \nabla^E) \in H^{\dim E-1}(X, \mathbf{R}/\mathbf{Z}). \tag{2.2}$$

In particular $\hat{e}(E, \nabla^E)$ does not depend on ∇^E.

Let $S(E)$ be the sphere bundle on E. Let $TS(E)$ be the relative tangent bundle to the spheres $S(E)$. Then $TS(E)$ is of rank $\dim E - 1$. Let $e(TS(E))$ be the corresponding Euler class.

Now we establish a simple result in the manner of Cheeger-Simons [16, Section 3].

Let $z \in Z_{\dim E-1}(X)$. Then by [16, Equation (3.1)], we can find $y \in Z_{\dim E-1}(SE), w \in C_{\dim E}(X)$, with

$$z = \pi_*(y) + \partial w. \tag{2.3}$$

Theorem 2.1. *If $\dim E$ is odd, then*

$$\langle \hat{e}(E, \nabla^E), z \rangle = -\frac{1}{2} \int_y e(TS(E)). \tag{2.4}$$

In particular

$$2\hat{e}(E, \nabla^E) = 0 \qquad \textit{in} \quad H(X, \mathbf{R}/\mathbf{Z}). \tag{2.5}$$

Proof. In [24, Section 7], using the Berezin integral formalism, Mathai and Quillen have associated to any real orientable vector bundle E with a metric g^E and an Euclidean connection ∇^E, a form $\psi(E, g^E, \nabla^E)$ on the sphere bundle $S(E)$, of degree $\dim E - 1$ such that

$$d\psi(E, g^E, \nabla^E) = \pi^* e(E, \nabla^E), \qquad \int_{S(E)} \psi(E, \nabla^E) = -1. \tag{2.6}$$

The construction of $\psi(E, g^E, \nabla^E)$ is functorial. It follows from (2.6) that modulo coboundaries, $\psi(E, g^E, \nabla^E)$ only depends on ∇^E, and not on g^E. We identify X to the zero section of E. Let δ_X be the current of integration on X. In [28, Section 3], $\psi(E, g^E)$ has been extended to a current on E, such that

$$d\psi(E, g^E, \nabla^E) = \pi^* e(E, \nabla^E) - \delta_X, \tag{2.7}$$

from which the second identity (2.6) follows.

Now we proceed as in [16, Section 3]. Take z, y, w as in (2.3). Assume that $\dim E$ is odd. Set

$$\widehat{\chi}(E, \nabla^E)(z) = \int_y \psi(E, g^E, \nabla^E) \in \mathbf{R}/\mathbf{Z}. \tag{2.8}$$

We claim that (2.8) is well defined. In effect if $z = 0$, then

$$\pi_*(y) = -\partial w. \tag{2.9}$$

and so by [16, Equation (3.1)] , there is $y' \in Z_{\dim E}(SE)$, and $k \in \mathbf{Z}$ such that if $S^{\dim E - 1}$ is one fibre SE,

$$y = \partial y' + k S^{\dim E - 1}. \tag{2.10}$$

From (2.6), (2.8) and (2.10), we deduce that

$$\widehat{\chi}(E, \nabla^E)(z) \in \mathbf{Z}. \tag{2.11}$$

So (2.8) is well defined.

From (1.4) and the equation of currents (2.7), we find easily that, with the notation of [16],

$$\begin{aligned} &\delta_1 \widehat{\chi}(E, \nabla^E) = 0, \\ &\delta_2 \widehat{\chi}(E, \nabla^E) = e(E) \qquad \text{in} \quad H(X, \mathbf{Z}). \end{aligned} \tag{2.12}$$

Using (2.12) and universality [16, Theorem 2.2], we deduce as in [16, Section 3] that

$$\widehat{\chi}(E, \nabla^E) = \hat{e}(E, \nabla^E) \qquad \text{in} \quad H(X, \mathbf{R}/\mathbf{Z}). \tag{2.13}$$

Moreover by a simple calculation given in [15, Equation (6.20)] (where ∇f should be formally replaced by the tautological section of E over SE), we find that on SE,

$$\psi(E, g^E, \nabla^E) = -\frac{1}{2} e(TS(E)) \qquad \text{mod coboundaries}. \tag{2.14}$$

From (2.13) and (2.14), we get (2.4). Equation (2.5) follows (2.4).

Remark 2.2. Of course (2.1) and (2.5) are compatible. Note that (2.5) can be proved directly by the same argument as in [16, Proposition 8.12], where only flat vector bundles were considered.

Let μ be a $\mathbf{Z}_2$-line bundle. Let $C_\bullet(X, \mu)$, and $C^\bullet(X, \mu)$ be the groups of chains and cochains with coefficients in μ. Let $Z_\bullet(X, \mu)$ be group of cycles in $C_\bullet(X, \mu)$. Let Λ be the group of smooth differential forms on X with values in μ.

Set

$$\widehat{H}(X, \mathbf{R}/\mathbf{Z} \otimes \mu) = \{ f \in \operatorname{Hom}(Z_\bullet(X, \mu), \mathbf{R}/\mathbf{Z}), \partial f \in \Lambda \}. \tag{2.15}$$

If μ is non trivial, another description of $\widehat{H}(X, \mathbf{R}/\mathbf{Z} \otimes \mu)$ is as follows. Let $\rho : T \to X$ be a double covering of X on which $\rho^* \mu$ is trivial. Then T has a natural involution ε. Clearly

$$\widehat{H}(X, \mathbf{R}/\mathbf{Z} \otimes \mu) = [\widehat{H}(T, \mathbf{R}/\mathbf{Z})]^{\text{antiinvariant}}. \tag{2.16}$$

Let now E be a real vector bundle on X, and assume that E is non-orientable. Let $\rho : T \to X$ be the double covering of X associated to the orientation bundle $o(E)$. Then $\hat{e}(\rho^*(E, \nabla^E)) \in [\widehat{H}(T, \mathbf{R}/\mathbf{Z})]^{\text{antiinvariant}}$. Let $\hat{e}(E, \nabla^E)$ be the corresponding differential character in $\widehat{H}(X, \mathbf{R}/\mathbf{Z} \otimes o(E))$.

(b) Differential characters with values in $\mathbf{C}/\mathbf{Q}$

Let X be a smooth manifold. Since

$$\frac{\mathbf{C}}{\mathbf{Q}} = \frac{\mathbf{R}}{\mathbf{Q}} \oplus \mathbf{R},$$

if $f \in \widehat{H}(X, \mathbf{C}/\mathbf{Q})$, then f can be written uniquely as

$$f = g + ih, \qquad g \in \widehat{H}(X, \mathbf{R}/\mathbf{Q}),\ h \in \frac{\Lambda^{\mathbf{R}(\mathbf{X})}}{d\Lambda^{\mathbf{R}}(X)}. \tag{2.17}$$

Let (E, ∇^E) be a complex vector bundle with connection over X. If ∇^E preserves a metric g^E, as we saw in Section 1(a), Cheeger and Simons [16, Section 2] have associated to an invariant polynomial P with rational coefficients a differential character $\widehat{P}(E, \nabla^E) \in \widehat{H}(X, \mathbf{R}/\mathbf{Q})$. Strictly speaking, the construction given in [16] does not work for non metric connections, since the theorem of Narasimhan-Ramanan [25] used in [16] does not apply to non metric connections. However if $\nabla^{E,u}$ is a metric preserving connection on E, there is an unambiguously defined Chern-Simons form $\widetilde{P}(E, \nabla^E, \nabla^{E,u})$ in $\frac{\Lambda^{\mathbf{C}(\mathbf{X})}}{d\Lambda^{\mathbf{C}}(X)}$ such that

$$d\widetilde{P}(E, \nabla^E, \nabla^{E,u}) = P(E, \nabla^{E,u}) - P(E, \nabla^E). \tag{2.18}$$

Also $\widetilde{P}(E, \nabla^E, \nabla^{E,u})$ is functorial.

Let $\widehat{P}(E, \nabla^E) \in \widehat{H}(X, \mathbf{C}/\mathbf{Q})$ be given by

$$\widehat{P}(E, \nabla^E) = \widehat{P}(E, \nabla^{E,u}) - \widetilde{P}(E, \nabla^E, \nabla^{E,u}). \tag{2.19}$$

One verifies easily that $\widehat{P}(E, \nabla^E)$ does not depend on $\nabla^{E,u}$.

Let c_k be the k^{th} Chern class, associated to the k^{th} symmetric function σ_k ($1 \leq k \leq \dim E$). Then one has the trivial identities

$$\begin{aligned}
\hat{c}_k(\overline{E^*}, \nabla^{\overline{E^*}}) &= \overline{\hat{c}_k(E, \nabla^E)}, \\
\hat{c}_k(E^*, \nabla^{E^*}) &= (-1)^k \hat{c}_k(E, \nabla^E), \\
\hat{c}_k(\overline{E}, \nabla^{\overline{E}}) &= (-1)^k \overline{\hat{c}_k(E, \nabla^E)}.
\end{aligned} \tag{2.20}$$

(c) Differential characters and flat vector bundles

Assume now that (F, ∇^F) is a flat complex vector bundle. Then

$$\delta_1 \widehat{P}(F, \nabla^F) = 0, \tag{2.21}$$

and so by (1.2), $\widehat{P}(F, \nabla^F) \in H(X, \mathbf{C}/\mathbf{Q})$.

Let g^F be a Hermitian metric on F. Set

$$\omega(F, g^F) = (g^F)^{-1} \nabla^F g^F. \tag{2.22}$$

Then $\omega(F, g^F)$ is a 1-form with values in self-adjoint endomorphisms of F. By [14, Proposition 1.3], we know that for any $k \in \mathbf{N}^*$,

$$\mathrm{Tr}[\omega^{2k}(F, g^F)] = 0. \tag{2.23}$$

For $k \in \mathbf{N}$, set

$$d_k(F, g^F) = \frac{1}{(2\pi i)^{k-1}} \mathrm{Tr}[\omega^{2k-1}(F, g^F)]. \tag{2.24}$$

Then by [14, Theorems 1.8 and 1.11], the $d_k(F, g^F)$ are real closed forms and their cohomology class $d_k(F)$ do not depend on g^F.

Let $\nabla^{F,u}$ be the connection

$$\nabla^{F,u} = \nabla^F + \frac{1}{2}\omega(F, g^F). \tag{2.25}$$

Then $\nabla^{F,u}$ preserves the metric g^F. Also

$$(\nabla^{F,u})^2 = -\frac{1}{4}\omega^2(F, g^F). \tag{2.26}$$

From (2.23) and (2.26), we get

$$\mathrm{ch}(F, \nabla^{F,u}) = \mathrm{rk}(F). \tag{2.27}$$

Now we state a result of [14, Proposition 1.14].

Theorem 2.3. *The following identities hold*

$$\begin{aligned} \mathrm{Re}\,\widehat{\mathrm{ch}}(F, \nabla^F) &= \widehat{\mathrm{ch}}(F, \nabla^{F,u}) && \text{in } \ H^{odd}(X, \mathbf{R}/\mathbf{Q}), \\ \mathrm{Im}\,\widehat{\mathrm{ch}}(F, \nabla^F) &= -\frac{1}{4\pi} \sum_{k=1}^{+\infty} \frac{(k-1)!}{(2k-1)!} d_k(F) && \text{in } \ H^{odd}(X, \mathbf{R}). \end{aligned} \tag{2.28}$$

Proof. [14, Proposition 1.14] only gives the second equality (2.29). However the proof of [14, Proposition 1.14] and (2.19) show that the first identity in (2.29) also holds.

If $(F, \nabla^F), (F', \nabla^{F'})$ are flat, if P and P' are invariant polynomials with rational coefficients, and have no constant term, by [16, Proposition 8.7],

$$\widehat{P}(F, \nabla^F)\widehat{P}'(F', \nabla^{F'}) = 0 \qquad \text{in } \ H(X, \mathbf{C}/\mathbf{Q}). \tag{2.29}$$

§ 3. Differential Characters, Flat Vector Bundles and Direct Images

The purpose of this section is to establish Theorem 0.2. Namely, if $\pi : M \to B$ is a proper submersion, if F is a flat vector bundle on B, we calculate the Chern class of $R\pi_* F$ in $H(B, \mathbf{C}/\mathbf{Q})$ in terms of the corresponding classes of F in $H(M, \mathbf{C}/\mathbf{Q})$. The imaginary part of the corresponding classes is easily dealt with by a result of Bismut-Lott [14]. A modification of Theorem 0.1 allows us to treat the $\mathbf{R}/\mathbf{Q}$ part of the equality.

This section is organized as follows. In (a), we describe the geometric setting. In (b), in full generality, we show the equality of the imaginary part in (0.3). In (c), we establish the corresponding equality of the $\mathbf{R}/\mathbf{Q}$ parts, when the fibres of Z are locally orientable over B. Finally in (d), we give a weaker version of Theorem 0.2 when this assumption is not verified. In this section, we use the notation of Sections 1 and 2.

(a) A proper submersion

Let $\pi : M \to B$ be a submersion of smooth manifolds, with compact fibre Z. Let (F, ∇^F) be a complex flat vector bundle on M.

Let $H(Z, F_{|Z})$ be the complex $\mathbf{Z}$-graded vector bundle on B, such that for any $b \in B$,

$$H(Z, F_{|Z})_b = H(Z_b, F_{|Z_b}). \tag{3.1}$$

The $\mathbf{Z}$- graded vector bundle $H(Z, F_{|Z})$ is equipped with a flat connection $\nabla^{H(Z,F_{|Z})}$.

Let

$$c = 1 + c_1 + \cdots + c_k + \cdots \tag{3.2}$$

be the Chern polynomial. Set

$$\hat{c}(H(Z, F_{|Z})) = \sum_{i=0}^{\dim Z} (-1)^i \hat{c}(H^i(Z, F_{|Z})) \in \widehat{H}(M, \mathbf{C}/\mathbf{Q}). \tag{3.3}$$

By [16, Theorem 1.1] (which is quoted in (1.2), with $\mathbf{Q}$ now replaced by $\mathbf{Z}$), since $\hat{c}(F, \nabla^F) \in H(M, \mathbf{C}/\mathbf{Z})$, if ∇^{TZ} is any metric preserving connection on TZ,

$$\hat{e}(TZ, \nabla^{TZ})\hat{c}(F, \nabla^F) \in H(M, \mathbf{C}/\mathbf{Z}). \tag{3.4}$$

In particular, $\hat{e}(TZ, \nabla^{TZ})\hat{c}(F, \nabla^F)$ does not depend on ∇^{TZ}. From now on, we will write $\hat{e}(TZ)\hat{c}(F, \nabla^F)$ instead of $\hat{e}(TZ, \nabla^{TZ})\hat{c}(F, \nabla^F)$.

By Theorem 2.1, if $\dim Z$ is odd,

$$2\hat{e}(TZ)\hat{c}(F, \nabla^F) = 0 \qquad \text{in } \ H(M, \mathbf{C}/\mathbf{Z}). \tag{3.5}$$

(b) The imaginary part of the C/Q classes

We recall a result of Bismut-Lott [14, Theorem 0.1].

Theorem 3.1. *The following identity holds,*

$$\operatorname{Im} \hat{c}(H(Z, F_{|Z}), \nabla^{H(Z,F_{|Z})}) = \pi_*[\hat{e}(TZ) \operatorname{Im} \hat{c}(F, \nabla^F)] \qquad \textit{in } \ H(B, \mathbf{R}). \tag{3.6}$$

Proof. Let g^F be a Hermitian metric on F. Let $n_k(x_1, \cdots, x_q)$ be the Newton polynomial

$$n_k(x_1, \cdots, x_q) = \sum_1^q x_i^k. \tag{3.7}$$

Then by (2.28), we find that

$$\operatorname{Im} \frac{\hat{n}_k}{k!}(F, \nabla^F) = -\frac{1}{4\pi} \frac{(k-1)!}{(2k-1)!} d_k(F) \qquad \text{in } \ H^{2k-1}(M, \mathbf{R}). \tag{3.8}$$

Also using Newton's formulas and (2.29), we see easily that

$$\hat{c}_k(F,\nabla^F) = \frac{(-1)^{k-1}}{k}\hat{n}_k(F,\nabla^F) \qquad \text{in} \;\; H(M,\mathbf{C}/\mathbf{Q}). \tag{3.9}$$

By a result of [14, Theorem 0.1], we know that for any $k \in \mathbf{N}$,

$$d_k(H(Z,F_{|Z})) = \pi_*[c(TZ)d_k(F)] \qquad \text{in} \;\; H(B,\mathbf{R}). \tag{3.10}$$

From (3.9)–(3.11), we get (3.6).

(c) The real part of the $\mathbf{C}/\mathbf{Q}$ classes

Theorem 3.2. *If $o(TZ)$ is the lift to M a $\mathbf{Z}_2$-line bundle on B, then*

$$\hat{c}(H(Z,F_{|Z}),\nabla^{H(Z,F_{|Z})}) = \pi_*[\hat{e}(TZ)\hat{c}(F,\nabla^F)] \qquad \textit{in} \;\; H(B,\mathbf{C}/\mathbf{Q}). \tag{3.11}$$

In particular, if $\dim Z$ *is odd, under the same assumptions as before,*

$$\hat{c}(H(Z,F_{|Z}),\nabla^{H(Z,F_{|Z})}) = 0 \qquad \textit{in} \;\; H(B,\mathbf{C}/\mathbf{Q}). \tag{3.12}$$

Proof. By Theorem 3.1, we only need to show that the real parts of (3.11) coincide. Let g^F be a Hermitian metric on F. Recall that the metric preserving connection $\nabla^{F,u}$ was defined in (2.25). Let g^{TZ} be a metric on the relative tangent bundle TZ.

For $b \in B$, let E_b be the vector space of smooth sections on $\Lambda(T^*Z)\otimes F$ on Z_b. Then the E_b's are the fibres of an infinite dimensional vector bundle E on B. The metrics g^{TZ}, g^F induce an L_2 Hermitian product on the fibres of E.

Let d^Z be the fibrewise de Rham operator acting on E, let d^{Z*} be the formal adjoint of d^Z. Put

$$D^{Z,dR} = d^Z + d^{Z*}. \tag{3.13}$$

By Hodge theory, for every $b \in B$,

$$H(Z_b,F_{|Z_b}) \simeq \ker D^{Z_b,dR}. \tag{3.14}$$

As a subbundle of E, $\ker D^{Z,dR}$ inherits a Hermitian metric from the L_2 metric of E. Let $g^{H(Z,F_{|Z})}$ be the corresponding Hermitian metric on $H(Z,F_{|Z})$.

Then $(H(Z,F_{|Z}),g^{H(Z,F_{|Z})})$ is a flat $\mathbf{Z}$-graded Hermitian vector bundle on B. Therefore we can construct on B objects which are the obvious analogues of the ones we constructed for F in (2.22)–(2.25).

By (2.27),

$$\delta_1\widehat{\mathrm{ch}}(F,\nabla^{F,u}) = 0$$

and so by (1.2),

$$\widehat{\mathrm{ch}}(F,\nabla^{F,u}) \in H(M,\mathbf{R}/\mathbf{Q}). \tag{3.15}$$

By (3.15), we find that $\hat{e}(TZ)\widehat{\mathrm{ch}}(F,\nabla^{F,u})$ is a well-defined element of $H(M,\mathbf{R}/\mathbf{Q})$. Similarly, $\widehat{\mathrm{ch}}(H(Z,F_{|Z}),\nabla^{H(Z,F_{|Z}),u}) \in H(B,\mathbf{R}/\mathbf{Q})$.

Put

$$\delta = \widehat{\mathrm{ch}}(H(Z,F_{|Z}),\nabla^{H(Z,F_{|Z}),u}) - \pi_*[\hat{e}(TZ)\widehat{\mathrm{ch}}(F,\nabla^{F,u})] \in H(B,\mathbf{R}/\mathbf{Q}). \tag{3.16}$$

We will show that

$$\delta = 0 \qquad \text{in} \;\; H(B,\mathbf{R}/\mathbf{Q}). \tag{3.17}$$

By using Theorem 2.3 and (3.17), (3.11) will follow.

Let ∇^{TZ} be the Levi-Civita connection on (TZ, g^{TZ}). Of course ∇^{TZ} is only defined fibrewise. Let $\nabla^{\Lambda(T^*Z)\otimes F,u}$ be the connection on $\Lambda(T^*Z)\otimes F$ along the fibres Z, induced by ∇^Z and $\nabla^{F,u}$.

Recall that $\Lambda(T^*Z)$ is a left and right TZ Clifford module. Namely by identifying TZ and T^*Z, if $X \in TZ$, put

$$\begin{aligned} c(X) &= X \wedge -i_X, \\ \hat{c}(X) &= X \wedge +i_X. \end{aligned} \tag{3.18}$$

Then if $X, Y \in TZ$,

$$\begin{aligned} c(X)c(Y) + c(Y)c(X) &= -2\langle X, Y\rangle, \\ \hat{c}(X)\hat{c}(Y) + \hat{c}(Y)\hat{c}(X) &= 2\langle X, Y\rangle, \\ c(X)\hat{c}(Y) + \hat{c}(Y)c(X) &= 0. \end{aligned} \tag{3.19}$$

Let $e_1, \cdots, e_n$ be an orthonormal frame in TZ. Let D^Z be the operator

$$D^Z = \sum_1^n c(e_i)\nabla_{e_i}^{\Lambda(T^*Z)\otimes F,u}. \tag{3.20}$$

Then D^Z is a standard Dirac operator acting on E, which is self-adjoint with respect to the L_2 Hermitian product of E.

Let V be the operator

$$V = -\frac{1}{2}\sum_1^n \hat{c}(e_i)\omega(F, g^F)(e_i). \tag{3.21}$$

We recall a result from [15, Proposition 4.12].

Proposition 3.3. *The following identity holds,*

$$D^{Z,dR} = D^Z + V. \tag{3.22}$$

Let T^HM be a smooth subbundle of TM such that

$$TM = T^HM \oplus TZ. \tag{3.23}$$

1. The case where dim Z is even and TZ is oriented

Assume that $\dim Z$ is even and TZ is oriented. For simplicity, we first assume that B is odd dimensional, compact, oriented and spin.

Let g^{TB} be a Riemannian metric on TB. We equip $TM = T^HM \oplus TZ$ with the metric

$$g^{TM} = \pi^* g^{TB} \oplus g^{TZ}. \tag{3.24}$$

Let $(\xi', g^{\xi'}, \nabla^{\xi'})$ be a vector bundle on B equipped with a metric $g^{\xi'}$ and a unitary connection $\nabla^{\xi'}$.

Let S^{TB} be the vector bundle on B of the (TB, g^{TB}) spinors. Then S^{TB} is a TB-Clifford module.

If $X \in TZ$, $c(X), \hat{c}(X)$ act naturally on $\Lambda(T^*Z) \otimes \pi^* S^{TB}$. Let N be the number operator of $\Lambda(T^*Z)$. If $f \in TB$, we make $c(f)$ act on $\Lambda(T^*Z) \otimes \pi^* S^{TB}$ by the formula

$$c(f) = (-1)^N \otimes c(f). \tag{3.25}$$

It is then trivial to see that with these conventions, if $X, Y \in TM$, the analogue of (3.22) holds, so that $\Lambda(T^*Z)\widehat{\otimes}\pi^* S^{TB}$ is a left TM-Clifford module. In particular, $c(f)$ anticommutes with the $c(X), \hat{c}(X)$ $(X \in TZ)$. Assume temporarily that TZ is spin. Let $S^{TZ} = S_+^{TZ} \oplus S_-^{TZ}$ be the corresponding vector bundle of (TZ, g^{TZ}) spinors. Then

$$\Lambda(T^*Z) = S^{TZ}\widehat{\otimes}S^{TZ*}. \tag{3.26}$$

Also, $S^{TM} = \pi^* S^{TB}\widehat{\otimes}S^{TZ}$ is the vector bundle of (TM, g^{TM}) spinors.

Let $\nabla^{TM,L}$ be the Levi-Civita connection on (TM, g^{TM}). Then $\nabla^{TM,L}$ induces a unitary connection $\nabla^{S^{TM}}$ on S^{TM}. Let ∇^{TB} be the Levi-Civita connection on (TB, g^{TB}), and let $\nabla^{S^{TB}}$ be the induced connection on S^{TB}.

Let ∇^{TZ} be the connection on TZ constructed in [6, Theorem 1.9] (and briefly described in Section 1(c)) which is associated to $(T^H M, g^{TZ})$. As the notation indicates, ∇^{TZ} restricts to the previously considered Levi-Civita connection along the fibres Z. Then ∇^{TZ} induces a unitary connection $\nabla^{S^{TZ*}}$ on S^{TZ*}, and a connection $\nabla^{\Lambda(T^*Z)}$ on $\Lambda(T^*Z)$.

Let $\nabla^{\Lambda(T^*Z)\widehat{\otimes}\pi^* S^{TB},L}$ be the connection on $\Lambda(T^*Z)\widehat{\otimes}\pi^* S^{TB} = S^{TM}\widehat{\otimes}S^{TZ*}$ induced by $\nabla^{S^{TM}}$ and $\nabla^{S^{TZ*}}$. Let $\nabla^{\Lambda(T^*Z)\widehat{\otimes}\pi^* S^{TB}}$ be the connection on $\Lambda(T^*Z)\widehat{\otimes}\pi^* S^{TB}$ induced by $\nabla^{\Lambda(T^*Z)}$ and $\nabla^{S^{TB}}$.

Let $e'_1, \cdots, e'_{n+m}$ be an orthonormal frame in TM. Let $\langle S(\,\cdot\,)\,\cdot\,, \,\cdot\,\rangle$ be the $(0,3)$ tensor constructed in [6, Section 1] which is associated to $(T^H M, g^{TZ})$. Then

$$\nabla^{\Lambda(T^*Z)\widehat{\otimes}\pi^* S^{TB},L} = \nabla^{\Lambda(T^*Z)\widehat{\otimes}\pi^* S^{TB}} + \frac{1}{4}\langle S(\,\cdot\,)e'_i, e'_j\rangle c(e'_i)c(e'_j). \tag{3.27}$$

Let ∇^u (resp. $\nabla^{L,u}$) be the connection on $\Lambda(T^*Z)\widehat{\otimes}F \otimes \pi^*(S^{TB} \otimes \xi')$ induced by

$$\nabla^{\Lambda(T^*Z)},\ \nabla^{F,u},\ \nabla^{\pi^* S^{TB}},\ \nabla^{\xi'} \qquad (\text{resp. } \nabla^{\Lambda(T^*Z)\otimes\pi^* S^{TB},L},\ \nabla^{F,u},\ \nabla^{\xi'}).$$

Definition 3.4. *Let D^M be the Dirac operator acting on smooth sections of $\Lambda(T^*Z) \otimes F \otimes \pi^*(S^{TB} \otimes \xi')$ over M,*

$$D^M = \sum_1^{n+m} c(e'_i)\nabla^{L,u}_{e'_i}. \tag{3.28}$$

Recall that the operator D^Z acting fibrewise is defined in (3.20). Clearly, the operator D^Z also acts on smooth sections of $\Lambda(T^*Z) \otimes F \otimes \pi^*(S^{TB} \otimes \xi')$. Let $e_1, \cdots, e_n$ be an orthonormal frame in TZ. If $U \in TB$, set as in (1.18),

$$\frac{1}{2} L_{U^H} dv_Z = k(U) dv_Z. \tag{3.29}$$

Let $f_1, \cdots, f_n$ be an orthonormal frame in TB. Set

$$D^H = c(f_\alpha)(\nabla^u_{f_\alpha^H} + k(f_\alpha^H)). \tag{3.30}$$

Put

$$c(T) = \frac{1}{2}\langle T^H(f_\alpha^H, f_\beta^H), e_i\rangle c(f_\alpha)c(f_\beta)c(e_i). \tag{3.31}$$

Then by proceeding as in [7, Equation (4.26)], we get

$$D^M = D^H + D^Z - \frac{c(T)}{4}. \tag{3.32}$$

The operator $D^{Z,dR}$ constructed in (3.16) also acts on smooth sections of $\Lambda(T^*Z) \otimes F \otimes \pi^*(S^{TB} \otimes \xi')$. Let D'^M be the operator

$$D'^M = D^H + D^{Z,dR} - \frac{c(T)}{4}. \tag{3.33}$$

Let $\bar{\eta}^{D^M}(0), \bar{\eta}^{D'^M}(0)$ be the eta invariants associated to D^M, D'^M.

Proposition 3.5. *The following identity holds,*

$$\bar{\eta}^{D^M}(0) = \bar{\eta}^{D'^M}(0) \qquad \operatorname{mod}(\mathbf{Z}). \tag{3.34}$$

Proof. By (3.13), (3.22), (3.32) and (3.33),

$$D'^M = D^M + V. \tag{3.35}$$

Now V is a matrix valued operator anticommuting with the $c(X)$ $(X \in TM)$.

Using the variation formula for the eta invariant (see [2]), and local index theory techniques as in [7, Proof of Theorem 2.7], we get (3.34).

Put

$$B(x_1, \cdots, x_q) = \prod_1^q 2\sinh\Big(\frac{x_i}{2}\Big). \tag{3.36}$$

Theorem 3.6. *The following identity holds,*

$$\bar{\eta}^{D'^M}(0) = \int_M \widehat{\widehat{A}}(TM, \nabla^{TM})\widehat{B}(TZ, \nabla^{TZ})\widehat{\mathrm{ch}}(F, \nabla^{F,u})\pi^*\widehat{\mathrm{ch}}(\xi', \nabla^{\xi'}) \qquad \textit{in } \mathbf{R}/\mathbf{Q}. \tag{3.37}$$

Proof. Let $\widetilde{D}^M_{\pm}$ be the standard Dirac operator acting on smooth sections of $S^{TM}\widehat{\otimes}S^{TZ*}_{\pm} \otimes\pi^*\xi'$, which is associated to the metric g^{TM} and to the connections $\nabla^{S^{TZ*}_{\pm}}, \nabla^{\xi'}$. Then by [7, Equation (4.26)], $\widetilde{D}^M_{\pm}$ verifies the obvious analogue of (3.32). Let $\widetilde{D}^M$ be the operator acting or smooth sections of $S^{TM}\widehat{\otimes}S^{TZ*} \otimes \pi^*\xi'$,

$$\widetilde{D}^M = \begin{bmatrix} \widetilde{D}^M_+ & 0 \\ 0 & -\widetilde{D}^M_- \end{bmatrix}. \tag{3.38}$$

Let τ be the operator defining the grading of S^{TZ}. If $f \in TB$, the natural action of $c(f)$ on S^{TM} is given by

$$c(f) = c(f) \otimes \tau. \tag{3.39}$$

We claim that D^M is unitarily equivalent to $\widetilde{D}^M$. In fact, let τ' be the operator defining the $\mathbf{Z}_2$-grading of S^{TZ*}. Clearly

$$(-1)^N = \tau\widehat{\otimes}\tau'. \tag{3.40}$$

By (3.25), (3.32) and (3.40),

$$\tau^{-1}D^M\tau = D^H - D^Z + \frac{c(T)}{4}. \tag{3.41}$$

By (3.25), (3.32), (3.39)–(3.41), we get

$$\begin{aligned} D_+^M &= \widetilde{D}_+^M, \\ \tau^{-1} D_-^M \tau &= -\widetilde{D}_-^M. \end{aligned} \tag{3.42}$$

By (3.42), we see that D^M and $\widetilde{D}^M$ are unitarily equivalent. In particular,

$$\bar{\eta}^{D^M}(0) = \bar{\eta}^{\widetilde{D}_+^M}(0) - \bar{\eta}^{\widetilde{D}_-^M}(0) \qquad \mathrm{mod}\,(\mathbf{Z}). \tag{3.43}$$

By a result of [16, Theorem 9.2] (see Theorem 1.1),

$$\bar{\eta}^{\widetilde{D}_\pm^M}(0) = \int_M \widehat{\widehat{A}}(TM, \nabla^{TM}) \widehat{\mathrm{ch}}(S_\pm^{TZ*}, \nabla^{S_\pm^{TZ*}}) \widehat{\mathrm{ch}}(F, \nabla^{F,u}) \pi^* \widehat{\mathrm{ch}}(\xi', \nabla^{\xi'}). \tag{3.44}$$

So using (3.44), we get

$$\begin{aligned} \bar{\eta}^{D^M}(0) = \int_M \widehat{\widehat{A}}(TM, \nabla^{TM}) (\widehat{\mathrm{ch}}(S_+^{TZ*}, \nabla^{S_+^{TZ*}}) - \widehat{\mathrm{ch}}(S_-^{TZ*}, \nabla^{S_-^{TZ*}})) \\ \cdot \widehat{\mathrm{ch}}(F, \nabla^{F,u}) \pi^* \widehat{\mathrm{ch}}(\xi', \nabla^{\xi'}) \qquad \text{in } \mathbf{R}/\mathbf{Q}. \end{aligned} \tag{3.45}$$

Now by [1, p.484],

$$\mathrm{ch}(S_+^{TZ*}, \nabla^{S_+^{TZ*}}) - \mathrm{ch}(S_-^{TZ*}, \nabla^{S_-^{TZ*}}) = B(TZ, \nabla^{TZ}). \tag{3.46}$$

Using Proposition 3.5, (3.45) and (3.46), we get (3.37). The proof is completed.

Now we replace g^{TB} by $\frac{g^{TB}}{\varepsilon}$, for $\varepsilon > 0$. We introduce the subscript ε to indicate the dependence on the parameter $\varepsilon > 0$. From (3.37), we get

$$\bar{\eta}^{D'^M_\varepsilon}(0) = \int_{M'} \widehat{\widehat{A}}(TM, \nabla_\varepsilon^{TM}) \widehat{B}(TZ, \nabla^{TZ}) \widehat{\mathrm{ch}}(F, \nabla^{F,u}) \pi^* \widehat{\mathrm{ch}}(\xi', \nabla^{\xi'}) \qquad \text{in } \mathbf{R}/\mathbf{Q}. \tag{3.47}$$

By proceeding as in [9, Equation (3.196)] and in (1.39), we find that as $\varepsilon \to 0$,

$$\begin{aligned} &\int_{M'} \widehat{\widehat{A}}(TM, \nabla_\varepsilon^{TM}) \widehat{B}(TZ, \nabla^{TZ}) \widehat{\mathrm{ch}}(F, \nabla^{F,u}) \pi^* \widehat{\mathrm{ch}}(\xi', \nabla^{\xi'}) \\ &\to \int_B \widehat{\widehat{A}}(TB, \nabla^{TB}) \widehat{\mathrm{ch}}(\xi', \nabla^{\xi'}) \pi_* [\hat{e}(TZ, \nabla^{TZ}) \widehat{\mathrm{ch}}(F, \nabla^{F,u})] \qquad \text{in } \mathbf{R}/\mathbf{Q}. \end{aligned} \tag{3.48}$$

Let $\nabla^{E,u}$ be the unitary connection on E, which is the obvious analogue of the connection $\nabla^{H,u}$ of Definition 1.4. Let A_t^{dR} be the analogue of the superconnection A_t in (1.21), i.e.,

$$A_t^{dR} = \nabla^{E,u} + \sqrt{t} D^{Z,dR} - \frac{c(T)}{4\sqrt{t}} \cdot \tag{3.49}$$

Since $D^{Z,dR}$ is a perturbation of D^Z by the matrix valued operator V, which anticommutes with the $c(X)$'s ($X \in TZ$), by proceeding as in [7, Proof of Theorem 2.7], one verifies easily that the superconnection A_t^{dR} is as good as the superconnection A_t from the point of view of the local families index theorem of [6].

By [14, Proposition 3.14], the orthogonal projection of the connection $\nabla^{E,u}$ on $\ker D^{Z,dR} \simeq H(Z, F_{|Z})$ coincides with $\nabla^{H(Z,F_{|Z}),u}$.

Let D^B be the Dirac operator on B acting on smooth sections of $S^{TB} \otimes H(Z, F_{|Z}) \otimes \xi'$, where $S^{TB} \otimes H(Z, F_{|Z}) \otimes \xi'$ is equipped with the obvious unitary connection. Let $\tilde{\eta}^{dR}$ be the analogue of the form $\tilde{\eta}$ constructed in Definition 1.13, which is now associated to the superconnection A_t^{dR}. Then the analogue of (1.32) is

$$d\tilde{\eta}^{dR} = \pi_*[e(TZ, \nabla^{TZ})\mathrm{ch}(F, \nabla^{F,u})] - \mathrm{ch}(H(Z, F_{|Z}), \nabla^{H(Z,F_{|Z}),u}). \tag{3.50}$$

Using (2.27) and (3.50), we find that

$$d\tilde{\eta}^{dR} = 0. \tag{3.51}$$

We claim that the arguments of Bismut-Cheeger [7] and Dai [17] still apply to $\bar{\eta}^{D'^M_\varepsilon}(0)$ as $\varepsilon \to 0$. In effect, as explained before, the perturbation of D^Z by V is irrelevant from the point view of local index theory. Moreover $\ker D^{Z,dR}$ is a vector bundle on B. Using the analogue of (1.40), we find that as $\varepsilon \to 0$,

$$\bar{\eta}^{D'^M_\varepsilon}(0) \to \int_B \widehat{A}(TB, \nabla^{TB})\mathrm{ch}(\xi', \nabla^{\xi'})\tilde{\eta}^{dR} + \bar{\eta}^{D^B}(0) \qquad \mathrm{mod}\,(\mathbf{Z}). \tag{3.52}$$

Again, by Theorem 1.1,

$$\bar{\eta}^{D^B}(0) = \int_B \widehat{\widehat{A}}(TB, \nabla^{TB})\widehat{\mathrm{ch}}(H(Z, F_{|Z}), \nabla^{H(Z,F_{|Z}),u})\widehat{\mathrm{ch}}(\xi', \nabla^{\xi'}) \qquad \text{in} \quad \mathbf{R}/\mathbf{Q}. \tag{3.53}$$

Theorem 3.7. *The following identity holds,*

$$\tilde{\eta}^{dR} = 0. \tag{3.54}$$

Proof. By [14, Theorem 3.15], we know that

$$\varphi \mathrm{Tr}_s\{\exp - A_t^{dR,2}\} = \mathrm{rk}(F)\chi(Z). \tag{3.55}$$

By understanding (3.55) correctly, we can derive from (3.49) that for $t > 0$,

$$\mathrm{Tr}_s\Big[\Big(\frac{D^{dR}}{2\sqrt{t}} + \frac{c(T)}{8t^{3/2}}\Big)\exp(-A_t^{dR,2})\Big] = 0. \tag{3.56}$$

Let us give a direct proof of (3.56). By [14, Equations (3.49), (3.50)], there are superconnection A'_t, A''_t such that

$$A'^2_t = 0, \quad A''^2_t = 0, \quad A_t^{dR} = \frac{1}{2}(A''_t + A'_t). \tag{3.57}$$

By [14, Equation (3.57)],

$$\begin{aligned} \frac{\partial A'_t}{\partial t} &= \Big[\frac{N}{2t}, A'_t\Big], \\ \frac{\partial A''_t}{\partial t} &= -\Big[\frac{N}{2t}, A''_t\Big]. \end{aligned} \tag{3.58}$$

Set

$$B_t = \frac{1}{2}(A''_t - A'_t). \tag{3.59}$$

From (3.57)–(3.59), we get

$$B_t^2 = -A_t^{dR,2}, \quad \frac{\partial A_t^{dR}}{\partial t} = \Big[-\frac{N}{2t}, B_t\Big]. \tag{3.60}$$

Now B_t is a fibrewise differential operator. Using (3.60) and the fact that supertraces vanish on supercommutators [26], we obtain

$$\mathrm{Tr}_s\Big[\frac{\partial A_t^{dR}}{\partial t}\exp(-A_t^{dR,2})\Big] = \mathrm{Tr}_s\Big[\Big[B_t, \frac{N}{2t}\Big]\exp(B_t^2)\Big] = \mathrm{Tr}_s\Big[\Big[B_t, \frac{N}{2t}\exp(B_t^2)\Big]\Big] = 0, \tag{3.61}$$

which is just (3.56). The proof of our theorem is completed.

By (3.16), (3.47), (3.48), (3.52)–(3.54), we get

$$\int_B \widehat{\widehat{A}}(TB, \nabla^{TB})\widehat{\mathrm{ch}}(\xi', \nabla^{\xi'})\delta = 0. \tag{3.62}$$

The proof of (3.17) continues as the proof of Theorem 1.15. Thus we obtain Theorem 3.2. This argument still applies to the case of a general manifold B.

2. The case where $\dim Z$ is odd and TZ is oriented

Now we assume that $\dim Z$ is odd and TZ is oriented. Again, we first assume that B is odd dimensional, compact, orientable and spin. We still introduce the objects appearing in (3.28), (3.33).

Theorem 3.8. *The following identity holds,*

$$\bar{\eta}^{D^M}(0) = \bar{\eta}^{D'^M}(0) \mod (\mathbf{Z}), \quad \bar{\eta}^{D^M}(0) = 0 \mod (\mathbf{Z}/2). \tag{3.63}$$

Proof. For $0 \le s \le 1$, set

$$D_s^M = D^M + sV. \tag{3.64}$$

Let $\bar{\eta}^{D_s^M}(0)$ be the eta invariant of D_s^M. Then by [2], the derivative mod $\mathbf{Z}$ of $\bar{\eta}^{D_s^M}(0)$ is exactly the constant term in the asymptotic expension as $t \to 0$ of

$$\sqrt{t}\mathrm{Tr}[V\exp(-tD_s^{M,2})]. \tag{3.65}$$

However since $\dim M$ is even, this constant term vanishes.

Let $e_1, \cdots, e_n$ be an oriented orthonormal frame in TZ. Set

$$\rho = \hat{c}(e_1)\cdots\hat{c}(e_n). \tag{3.66}$$

Then clearly

$$\nabla^{\Lambda(T^*Z)\otimes F, u}\rho = 0. \tag{3.67}$$

Moreover since n is odd, using (3.19), it is clear that if $X \in TM, c(X)$ anticommutes with ρ, and if $Y \in TZ$, $\hat{c}(Y)$ commutes with ρ. It follows that D^M anticommutes with ρ. So if λ lies in the spectrum of $D^M, -\lambda$ also lies in the spectrum, and so

$$\bar{\eta}^{D^M}(0) = \frac{1}{2}\dim\ker D^M. \tag{3.68}$$

The proof is completed.

Remark 3.9. In general D'^M does not anticommute with ρ.

As before, we replace g^{TB} by $\frac{g^{TB}}{\varepsilon}$ with $\varepsilon > 0$. From Theorem 3.8, we get

$$\bar{\eta}^{D^M_\varepsilon}(0) = 0 \qquad \mathrm{mod}\,(\mathbf{Z}/2). \tag{3.69}$$

By Hodge theory,

$$\ker D^{Z,dR} = H(Z, F_{|Z}). \tag{3.70}$$

We claim that all the arguments in part 1 of the proof of Theorem 3.2 can be applied to the problem under consideration. This is not as direct as before, since here $\dim M$ is even, which is a case not considered in [7]. The difficulty is essentially to show that the local index calculations carry through in this case. However recall that by [BF, Section 1b)], if $f_1 \cdots f_m$ is an orthonormal base of TB, among the monomials in the $c(f_\alpha)$'s, only 1 and $c(f_1)\cdots c(f_m)$ have a nonzero trace, when acting on S^{TB}. Any monomial in the $c(f_\alpha), c(e_i), \hat{c}(e_i)$ whose trace on $\pi^* S^{TB} \otimes \Lambda(T^*Z)$ is nonzero is a monomial in the $c(e_i), \hat{c}(e_i)$ which is a factor of 1 or of $c(f_1)\cdots c(f_m)$. If this monomial is odd, then it is either an odd monomial in the $c(e_i), \hat{c}(e_i)$ or it is an even monomial in the $c(e_i)$, $\hat{c}(e_i)$ which factors $c(f_1)\cdots c(f_m)$. Now an odd monomial in the $c(e_i), \hat{c}(e_i)$ acts as an odd operator on $\Lambda(T^*Z)$, i.e. it exchanges $\Lambda^{\mathrm{even}}(T^*Z)$ and $\Lambda^{\mathrm{odd}}(T^*Z)$ and its trace is 0. In view of [BF, Equation (1.7)] and of (3.28), we find that if N is a monomial in the $c(e_i), \hat{c}(e_i)$'s,

$$\mathrm{Tr}^{\pi^* S^{TB}\otimes\Lambda(T^*Z)}[i^{\frac{\dim B-1}{2}} c(f_1)\cdots c(f_m) N] = 2^{(\dim B-1)/2}\mathrm{Tr}_s^{\Lambda(T^*Z)}[N]. \tag{3.71}$$

Using (3.71) and [15, Proposition 4.9], we see that the only odd monomial in the $c(f_\alpha), c(e_i)$, $\hat{c}(e_i)$ whose trace is nonzero is

$$c(f_1)\cdots c(f_m) c(e_1)\cdots c(e_n)\hat{c}(e_1)\cdots\hat{c}(e_n). \tag{3.72}$$

Practically, this means that as in usual local index theory, we must use all the Clifford variables $c(f_\alpha), c(e_i)$ to get a nontrivial trace. The proof then continues as in the case where $\dim Z$ is even.

3. The general case

Assume now that the fibres Z are fibrewise orientable, i.e. the orientation bundle $o(TZ)$ descends to the base B. Let $\widehat{B}$ be a double covering of B, such that the fibration $\pi : Z \to B$ lifts to a fibration $\pi : \widehat{Z} \to \widehat{B}$, where the fibre $\widehat{Z}$ are now orientable. Over $\widehat{B}$, the obvious analogue of (3.11) holds in $H(\widehat{B}, \mathbf{C}/\mathbf{Q})$. Both sides of (3.11) are invariant under the obvious involution ε, so they descend to B. The proof of Theorem 3.2 is completed.

Remark 3.10. It might be possible to prove Theorem 3.2 in full generality. However, in part 1 of the proof, we can no longer use the Atiyah-Padodi-Singer index theorem (see [2]) to establish the Cheeger-Simons identity of Theorem 3.6. Similarly in part 2 of the proof, ρ in (3.60) is no longer well defined.

(d) Extensions

Although we do not prove (3.11) in full generality, i.e., when $o(TZ)$ is arbitrary, we will prove a weaker, but general statement.

Theorem 3.11. *In full generality, if* $\dim Z$ *is even, for any* $k \in \mathbf{N}$,

$$\begin{aligned}&\hat{c}_k(H(Z, F_{|Z}), \nabla^{H(Z,F_{|Z})}) + (-1)^k \hat{c}_k(H(Z, F^*_{|Z}), \nabla^{H(Z,F^*_Z)})\\ &= 2\pi_*[\hat{e}(TZ)\hat{c}_k(F, \nabla^F)] \qquad \textit{in} \quad H(B, \mathbf{C}/\mathbf{Q}).\end{aligned} \tag{3.73}$$

If $\dim Z$ *is odd, then for* $k \in \mathbf{N}$,

$$\hat{c}_k(H(Z,F_{|Z}),\nabla^{H(Z,F_{|Z})}) - (-1)^k \hat{c}_k(H(Z,F^*_{|Z}),\nabla^{H(Z,F^*_{|Z})}) = 0 \qquad in \ \ H(B,\mathbf{C}/\mathbf{Q}). \tag{3.74}$$

Proof. Let $\rho: \widehat{M} \to M$ be a double covering of M on which TZ lifts to an orientable bundle. Then $\widehat{M}$ still fibres over B, with a fibre $\widehat{Z}$ which double covers Z. Clearly

$$H(\widehat{Z},\rho^* F_{|Z}) = H(Z,F_{|Z}) + H(Z,F_{|Z}\otimes o(TZ)). \tag{3.75}$$

Also, by Poincaré duality,

$$H^p(Z(F,\otimes o(TZ))_{|Z}) = (H^{\dim Z - p}(Z,F^*_{|Z}))^*. \tag{3.76}$$

By Theorem 3.2, if $\widehat{\pi}: M \to B$ is the obvious projection,

$$\hat{c}(H(\widehat{Z},\rho^* F_{|Z}),\nabla^{H(\widehat{Z},\rho^* F_{|\widehat{Z}})}) = 2\pi_*[\hat{e}(TZ)\widehat{\mathrm{ch}}(F,\nabla^F)] \qquad \text{in} \ \ H^*(B,\mathbf{C}/\mathbf{Q}). \tag{3.77}$$

Also by (2.20) and (3.76),

$$\begin{aligned} & c_k(H(Z,(F\otimes o(TZ))_{|Z}),\nabla^{H(Z,(F_{|Z}\otimes o(TZ))_{|Z}}) \\ &= (-1)^{k+\dim Z} c_k(H(Z,F^*_{|Z}),\nabla^{H(Z,F^*_{|Z}}). \end{aligned} \tag{3.78}$$

From (3.75)–(3.78), we get (3.73) and (3.74).

Theorem 3.12. *If* $F \simeq F^*$, *then*

$$\begin{aligned} \hat{c}_k(H(Z,F_{|Z}),\nabla^{H(Z,F_{|Z})}) &= \pi_*[\hat{e}(TZ)c_k(F,\nabla^F)] \\ & \qquad in \ \ H(B,\mathbf{C}/\mathbf{Q}) \quad if \ \ \dim Z \ \ is\ even,\ k\ is\ even, \\ &= 0 \quad in \ \ H(B,\mathbf{C}/\mathbf{Q}) \quad if \ \ \dim Z \ \ is\ odd,\ \ k\ is\ odd. \end{aligned} \tag{3.79}$$

If $F \simeq \overline{F}^*$, *then*

$$\hat{c}_k(H(Z,F_{|Z}),\nabla^{H(Z,F_{|Z})}) = \pi_*[\hat{e}(TZ)\hat{c}_k(F,\nabla^F)] \quad if \ \ \dim Z \ \ is\ even. \tag{3.80}$$

Proof. If $F \simeq F^*$, we use Theorem 3.11. If $F \simeq \overline{F}^*$,

$$H(Z,F^*_{|Z}) = \overline{H(Z,F_{|Z})}. \tag{3.81}$$

Then we use (2.20) and Theorem 3.11 to obtain the equality of the real parts in (3.80), the imaginary parts (which both vanish in this case) being equal by Theorem 3.1. The proof of Theorem 3.12 is completed.

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Appendix: Classes of Local Systems of Hermitian Vector Spaces

(by K. CORLETTE[1] and H. ESNAULT[2])

For a local system V on a topological manifold S associated to a representation

$$\rho:\ \pi_1(S,s) \to GL(n,\mathbb{C})$$

of the fundamental group, we denote by

$$\hat{c}_i(V) = \hat{c}_i(\rho) = \beta_i + \gamma_i \in H^{2i-1}(S, \mathbb{C}/\mathbb{Z})$$

the class defined in [4, 1]:

$$\beta_i \in H^{2i-1}(S,\mathbb{R}) \qquad \text{(see [1, (2.20)])},$$

$$\gamma_i \in H^{2i-1}(S,\mathbb{R}/\mathbb{Z}) \qquad \text{(see [4, §4])}.$$

If $f : X \to S$ is a smooth proper morphism of $\mathcal{C}^\infty$ manifolds with orientable fibers, the Riemann-Roch theorem (see [1, Theorem (0.2) and Theorem (3.11)]) says

$$\hat{c}_i\Big(\sum_{j=0}^{\dim(X/S)} (-1)^j R^j f_*\mathbb{C}\Big) = 0$$

in $H^{2i-1}(S,\mathbb{C}/\mathbb{Q})$, for all $i \geq 1$.

The purpose of this short note is to show how to apply Reznikov's ideas (see [12]) to obtaining vanishing of the single classes $\hat{c}_i(R^j f_*\mathbb{C})$ under some assumptions.

Definition A.1. *Let A be a ring with $\mathbb{Z} \subset A \subset \mathbb{C}$. A local system of A hermitian vector spaces is a local system associated to a representation ρ whose image $\rho(\pi_1(S,s))$ lies in $GL_n(A) \subset GL_n(\mathbb{C})$ and $U(p,q) \subset GL_n(\mathbb{C})$ for some pair (p,q) with $n = p+q$, where $U(p,q)$ is the unitary group with respect to a non-degenerate hermitian form with p positive, and q negative eigenvalues.*

Theorem A.1. *Let S be a topological manifold and let $\rho : \pi_1(S,s) \to GL(n,F)$ be a representation of the fundamental group with values in a number field F. Assume that for all real and complex embeddings $\sigma : F \to \mathbb{R}$ $(\subset \mathbb{C})$ and $\sigma : F \to \mathbb{C}$, $\sigma \circ \rho : \pi_1(S,s) \to GL(n,\mathbb{C})$ is a local system of $\sigma(F)$ hermitian vector spaces. Then $\hat{c}_i(\rho) = 0$ in $H^{2i-1}(S,\mathbb{C}/\mathbb{Q})$ for all $i \geq 1$.*

[1] Department of Mathematics, The University of Chicago, Illinois 60637, USA.
Email: kevin@math.uchicago.edu
[2] Universität Essen, FB6, Mathematik, 45117 Essen, Germany. **Email:** esnault@uni-essen.de

Examples of local systems of $\mathbb{Q}$ hermitian vector spaces are provided by $\mathbb{Q}$ variations of Hodge structures (see [9, I.2]), whose main instances are the Gauß-Manin local systems $R^j f_*\mathbb{C}$, where $f : X \to S$ is a smooth proper morphism of complex manifolds with Kähler fibres. So Theorem A.1 implies

Theorem A.2. *Let $f : X \to S$ be a smooth proper morphism of complex manifolds with Kähler fibres. Then*

$$\hat{c}_i(R^j f_*\mathbb{C}) = 0$$

in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$ for all $i \geq 1,\ j \geq 0$.

In the $\mathcal{C}^\infty$ category, other examples are provided by Poincaré duality.

Theorem A.3. *Let $f : X \to S$ be a smooth proper morphism of C^∞ manifolds with orientable fibres. Then*

$$\hat{c}_i(R^j f_*\mathbb{C} \oplus R^{(\dim(X/S)-j)} f_*\mathbb{C}) = 0$$

in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$ and

$$\hat{c}_i(R^{\dim(X/R)/2} f_*\mathbb{C}) = 0$$

in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$ if $\dim(X/S)$ is even.

Proof of Theorem A.1. The $U(p, q)$ flat bundle being isomorphic to the conjugate of its dual, the formula (see [1, (2.21)]) says that $\beta_i = 0$. Thus we just have to consider γ_i.

We may first assume that $\Lambda^n \rho : \pi_1(S, s) \to \mathbb{C}^*$ is trivial. In fact, it is torsion as a unitary and rational representation, say of order N, and $V \oplus \cdots \oplus V$ (N times) has trivial determinant. On the other hand,

$$\hat{c}_i(V \oplus \cdots \oplus V) = N\hat{c}_i(V)$$

in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$, as

$$\hat{c}_i(V) \cdot \hat{c}_j(V) = 0$$

for $i \geq 1,\ j \geq 1$, in $H^{2(i+j)-1}(S, \mathbb{C}/\mathbb{Q})$. (The multiplication is defined by image ($\hat{c}_i(V)$ in $H^{2i}(S, \mathbb{Z})) \cdot \hat{c}_j(V)$ (see [4, (1.11)]).)

Furthermore, by adding trivial factors to V, one may assume that n is as large as one wants.

There is an open cover $S = \underset{\alpha}{\cup} S_\alpha$ trivializing V with transition functions

$$\lambda_{\alpha\beta} \in \Lambda(S_{\alpha\beta}, SL_n(F))$$

such that

$$\sigma \circ \lambda_{\alpha\beta} \in \Lambda(S_{\alpha\beta}, SL_n(\sigma(F)) \cap U(p, q)).$$

One has the continuous maps

$$\varphi : S_\bullet \xrightarrow{\alpha} BSL_n(F) \xrightarrow{\sigma} BSL_n(\sigma(F)) \xrightarrow{\tau} BSL_n(\mathbb{C})_\delta \xrightarrow{\iota} BSL_n(\mathbb{C}),$$

and

$$\psi : S_\bullet \xrightarrow{\sigma\circ\lambda} BSU(p,q) \xrightarrow{\mu} BSL_n(\mathbb{C}),$$

where $S_\bullet$ is the simplicial classifying manifold associated to the open cover S_α, $BSL_n(F)$ and $BSL_n(\sigma(F))$ are the simplicial classifying sets, BG is the simplicial ($\mathcal{C}^\infty$) classifying manifold for

$$G = SL_n(\mathbb{C}),\ SU(p,q),$$

$BSL_n(\mathbb{C})_\delta$ is the discrete simplicial classifying set. So $\varphi = \psi$.

By [4, §8], there is a class $\gamma_i^{\text{univ}} \in H^{2i-1}(BSL_n(\mathbb{C})_\delta, \mathbb{R})$, whose image

$$\bar{\gamma}_i^{\text{univ}} \in H^{2i-1}(BSL_n(\mathbb{C})_\delta, \mathbb{R}/\mathbb{Q}) = H^{2i-1}(BSL_n(\mathbb{C})_\delta, \mathbb{R})/H^{2i-1}(BSL_n(\mathbb{C})_\delta, \mathbb{Q})$$

verifies

$$\gamma_i = \lambda^*\sigma^*\tau^*\bar{\gamma}_i^{\text{univ}}.$$

We now apply Reznikov's idea to use Borel's theorem. By [2, (7.5) (11.3)] and [3, (6.4)iii, (6.5)], for n sufficiently large compared to i, $H^{2i-1}(BSL_n(F), \mathbb{R})$ is generated by

$$\Big(\bigotimes_\sigma \sigma^*\tau^*\iota^* H^\bullet(BSL_n(\mathbb{C}), \mathbb{R})\Big)^{(2i-1)},$$

where $(2i-1)$ denotes the part of the tensor product of degree $(2i-1)$. Thus $\sigma^*\tau^*\bar{\gamma}_i^{\text{univ}} \in H^{2i-1}(BSL_n(F), \mathbb{R})$ is a sum of elements of the shape $\bigotimes_\sigma \sigma^*\tau^*\iota^* x_\sigma$, where at least one $x_\sigma \in H^{2i-1}_{\text{cont}}(SL_n(\mathbb{C}), \mathbb{R})$, for some $j \le i$. This implies that

$$\gamma_i = \sum \bigotimes_\sigma (\sigma\circ\lambda)^*\mu^* x_\sigma,$$

and for each summand, there is at least one

$$\mu^* x_\sigma \in H^{2j-1}_{\text{cont}}(SU(p,q), \mathbb{R}).$$

It remains to observe that

$$\text{for}\ \ p+q=n\ \ \text{large} \qquad H^{2i-1}_{\text{cont}}(SU(p,q), \mathbb{R}) = 0.$$

In fact, if $p = q$, this is part of [2, 10.6]. In general, the continuous cohomology of the $\mathbb{R}$ valued points of the $\mathbb{R}$ algebraic group $SU(p,q)$ is computed by

$$H^\bullet_{\text{cont}}(SU(p,q), \mathbb{R}) = H^\bullet(\text{Hom}_K(\Lambda^\bullet \mathfrak{G}_c/\mathfrak{K}), \mathbb{R}),$$

where K is the maximal compact subgroup $SU(p,q) \cap (U(p) \times U(q))$, $\mathfrak{K}$ is its Lie algebra, $\mathfrak{G}$ is the Lie algebra of $SU(p,q)$. The right hand side equals

$$H^\bullet(\text{Hom}_K(\Lambda^\bullet \mathfrak{G}_c/\mathfrak{K}), \mathbb{R}),$$

where $\mathfrak{G}_c$ is the Lie algebra of the compact form $SU(p+q)$ of $SU(p,q)$. This group is the de Rham cohomology of the manifold $SU(p+q)/SU(p+q)\cap(U(p)\times U(q))$, a Grassmann manifold without odd cohomology.

Remark A.1. To a representation ρ, one may also associate the classes

$$c_i(\rho)\in H^{2i-1}(S,\mathbb{C}/\mathbb{Z}(i))$$

defined by $\lambda^* c_i^{\text{univ}}$, where $\lambda : S_\bullet \to BGL_n(\mathbb{C})_\delta$ is defined by locally constant transition functions of the local system, and

$$c_i^{\text{univ}}\in H^{2i-1}(BGL_n(\mathbb{C})_\delta,\mathbb{C}/\mathbb{Z}(i)) = H^{2i}_{\mathcal{D}}(BGL_n(\mathbb{C})_\delta,\mathbb{Z}(i)),$$

where $H_{\mathcal{D}}$ is the Deligne-Beilinson cohomology, where c_i^{univ} are the restriction to $BGL_n(\mathbb{C})_\delta$ of the Chern classes in the Deligne-Beilinson cohomology of the universal bundle on the simplicial algebraic manifold BGL_n. One does not know in all generality that $\lambda^* c_i^{\text{univ}} = \hat{c}_i(V)$.

Again writing c_i^{univ} as $b_i^{\text{univ}} + z_i^{\text{univ}}$, with

$$b_i^{\text{univ}}\in H^{2i-1}(BGL_n(\mathbb{C})_\delta,\mathbb{R}(i-1)),$$

$$z_i^{\text{univ}}\in H^{2i-1}(BGL_n(\mathbb{C})_\delta,\mathbb{R}(i)/\mathbb{Z}(i)),$$

one knows that by definition b_i^{univ} lies in the image of the continuous cohomology of $GL_n(\mathbb{C})$:

$$\begin{aligned}
&H^{2i-1}_{\mathcal{D}}(BGL_n(\mathbb{C})_\bullet,\mathbb{Z}(i)) \longrightarrow H^{2i}_{\mathcal{D}}(BGL_n(\mathbb{C})_\bullet,\mathbb{R}(i))\\
\longrightarrow\ & H^{2i-1}(BGL_n(\mathbb{C})_\bullet,\mathcal{S}^\infty_{\mathbb{R}(i-1)}) \cong H^{2i-1}_{\text{cont}}(GL_n(\mathbb{C}),\mathbb{R}(i-1))\\
\longrightarrow\ & H^{2i-1}(BGL_n(\mathbb{C})_\delta,\mathbb{R}(i-1)),
\end{aligned}$$

where $\mathcal{S}^\infty_{\mathbb{R}(i-1)}$ is the sheaf of $\mathbb{R}(i-1)$ valued $\mathcal{C}^\infty$ functions. (In fact Beilinson gave a precise identification of this class in terms of the Borel regulator. See [11] for details.) Thus by the previous argument, $\lambda^* b_i^{\text{univ}} = 0$.

As before, we may assume that ρ has $SU(p,q)$ values, since the multiplication

$$c_i(\rho)\cdot c_j(\rho)$$

factorizes through the Betti class in $H^{2i}(S,\mathbb{Z}(i))$ of ρ (see [6, Proof of (3.4)]). Furthermore, by definition, z_i^{univ} is a discrete cohomology class. Thus one can apply the same argument as in Theorem A.1 to prove.

Theorem A.4. *Let S be a topological manifold and let $\rho : \pi_1(S,s) \to GL(n,F)$ be a representation of the fundamental group with values in a number field F. Assume that for all real and complex embeddings $\sigma : F \to \mathbb{R}\,(\subset \mathbb{C})$ and $\sigma : F \to \mathbb{C}$, $\sigma\circ\rho : \pi_1(S,s) \to GL(n,\mathbb{C})$*

is a local system of $\sigma(F)$ *hermitian vector spaces. Then* $c_i(\rho) = 0$ *in* $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$ *for all* $i \geq 1$.

On the other hand, if S is an algebraic manifold, then the image of $c_i(\rho)$ under the map

$$H^{2i-1}(S, \mathbb{C}/\mathbb{Z}(i)) \longrightarrow H^{2i}_{\mathcal{D}}(S, \mathbb{Z}(i))$$

is the Chern class $c_i^{\mathcal{D}}(E)$ of the underlying algebraic vector bundle E on $V \oplus \mathcal{O}_{S^{\mathrm{an}}}$ (see [6, (3.5)]). So one has

Corollary A.1. *Let* S *be an algebraic manifold and let* $\rho : \pi_1(S, s) \to GL(n, F)$ *be a representation of the fundamental group with values in a number field* F. *Assume that for all real and complex embeddings* $\sigma : F \to \mathbb{R}\,(\subset \mathbb{C})$ *and* $\sigma : F \to \mathbb{C}$, $\sigma \circ \rho : \pi_1(S, s) \to GL(n, \mathbb{C})$ *is a local system of* $\sigma(F)$ *hermitian vector spaces. Then the Chern classes of the underlying algebraic bundle in the Deligne cohomology are torsion.*

Remark A.2. Let $f : X \to S$ be a proper equidimensional morphism of algebraic smooth complex proper varieties X and S, such that f is smooth outside a normal crossing divisor Σ, with $D := f^{-1}(\Sigma)$ a normal crossing divisor without multiplicities (that is, f is "semi-stable" in codimension 1). Then the Gauß-Manin bundles

$$\mathcal{H}^j = R^j f_* \Omega^{\bullet}_{X/S}(\log D)$$

have an integrable holomorphic (in fact algebraic) connection with logarithmic poles along Σ whose residues are nilpotent (monodromy theorem, see e.g. [8, (3.1)]). This implies [7, Appendix B], that the de Rham classes of $\mathcal{H}^j$ are zero. Therefore

$$c_i^{\mathcal{D}}(\mathcal{H}^j) \in H^{2i-1}(S, \mathbb{C}/\mathbb{Z}(i))/F^i \subset H^{2i-1}_{\mathcal{D}}(S, \mathbb{Z}(i)),$$

that is, modulo torsion, $c_i^{\mathcal{D}}(\mathcal{H}^j)$ lies in the intermediate Jacobian, and $c_i^{\mathcal{D}}(\mathcal{H}^j|_{S-\Sigma})$ is torsion (see Corollary A.1, Theorem A.2). It would be interesting to understand those classes, in particular as one knows that there are only finitely many such classes for $\mathcal{H}^j$ of a given rank, as there are, according to Deligne [5], finitely many $\mathbb{Z}$ variations of Hodge structures of a given rank on $S - \Sigma$, and $\mathcal{H}^j$ is the canonical extension of $R^j f|_{S-\Sigma^*}\mathbb{C}$.

In fact, if f has relative (complex) dimension 1, even the Chern classes of $\mathcal{H}^j$ in the Chow groups of S are torsion, as a consequence of Grothendieck-Riemann-Roch theorem (see [10, (5.2)]).

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This note has been written just after Reznikov's proof (see [12]) was written. It is an illustration of his ideas. So the authors dedicate it to his memory.

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