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η-INVARIANT AND CHERN-SIMONS CURRENT**

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Abstract

The author presents an alternate proof of the Bismut-Zhang localization formula of η invariants, when the target manifold is a sphere, by using ideas of mod k index theory instead of the difficult analytic localization techniques of Bismut-Lebeau. As a consequence, it is shown that the $\mathbf{R}/\mathbf{Z}$ part of the analytically defined η invariant of Atiyah-Patodi-Singer for a Dirac operator on an odd dimensional closed spin manifold can be expressed purely geometrically through a stable Chern-Simons current on a higher dimensional sphere. As a preliminary application, the author discusses the relation with the Atiyah-Patodi-Singer $\mathbf{R}/\mathbf{Z}$ index theorem for unitary flat vector bundles, and proves an $\mathbf{R}$ refinement in the case where the Dirac operator is replaced by the Signature operator.

Keywords Direct image, η-Invariant, Chern-Simons current, mod k index theorem
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§ 1 . Introduction

The η invariant of Atiyah-Patodi-Singer was introduced in [3] as the correction term on the boundary of the index theorem for Dirac operators on manifolds with boundary. Since then it has appeared in many parts of geometry, topology as well as physics. We first recall the definition of this η invariant.

Let M be an odd dimensional closed oriented spin Riemannian manifold. Let $S(TM)$ be the associated bundle of spinors. Let E be a Hermitian vector bundle over M carrying with a Hermitian connection. Then one can define canonically a Dirac operator D^E : $\Gamma(S(TM)\otimes E)\to\Gamma(S(TM)\otimes E)$. It is a formally self-adjoint first order elliptic differential operator.

Let $s\in\mathbf{C}$ with $\mathrm{Re}(s)>\frac{\dim M}{2}$. Following [3], one defines the η function of D^E by

$$\eta(D^E,s)=\sum_{\lambda\in\mathrm{Spec}(D^E)\backslash\{0\}}\frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}. \tag{1.1}$$

It is shown in [3] that $\eta(D^E,s)$ is a holomorphic function for $\mathrm{Re}(s)>\frac{\dim M}{2}$, and can be extended to a meromorphic function on $\mathbf{C}$. Moreover, it is holomorphic at $s=0$. The value

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of $\eta(D^E, s)$ at $s = 0$ is called the η invariant of D^E and is denoted by $\eta(D^E)$. Let $\bar{\eta}(D^E)$ be the reduced η invariant of D^E which is also defined in [3]:

$$\bar{\eta}(D^E) = \frac{\dim(\ker D^E) + \eta(D^E)}{2}. \tag{1.2}$$

It turns out that this analytically defined invariant may jump by integers as the metrics and connections on TM and E change. These jumps can be detected by spectral flows introduced in [5]. On the other hand, the mod $\mathbf{Z}$ component of $\bar{\eta}(D^E)$ is smooth with respect to the involved metrics and connections, and its variation can be expressed through Chern-Simons forms. However, whether $\bar{\eta}(D^E)$ mod $\mathbf{Z}$ itself can be expressed geometrically, without passing to the spectral set of D^E, remains a question for long time. Here we only mention that such a formula for $\bar{\eta}(D^E)$ mod $\mathbf{Q}$ was proved in [15, Theorem 9.1], expressing the $\mathbf{R}/\mathbf{Q}$ component of $\bar{\eta}(D^E)$ through the Cheeger-Simons differential characters.

The purpose of this short article is to show that there is indeed a purely geometric formula for $\bar{\eta}(D^E)$ mod $\mathbf{Z}$. More precisely, if we embed M into a higher odd dimensional sphere, then $\bar{\eta}(D^E)$ mod $\mathbf{Z}$ can be expressed through a Chern-Simons current on the sphere. Comparing with Cheeger-Simons' mod $\mathbf{Q}$ one, such a formula is more of K-theoretic nature, and should be viewed as an index theorem in some geometric K-theory (compare with [6] and [18]).

In fact, this formula, which will be stated in its precise form in Theorem 2.3, can be obtained as an immediate application of a localization formula for η invariants proved by Bismut and Zhang in [14]. Our simple observation is that if one applies the Bismut-Zhang formula to an embedding into a higher dimensional sphere, then a simple application of the Bott periodicity will lead us to a geometric formula for the $\mathbf{R}/\mathbf{Z}$ component of $\bar{\eta}(D^E)$.

Thus we will first recall in the next section the Bismut-Zhang localization formula for η invariants and prove the geometric formula for $\bar{\eta}(D^E)$ mod $\mathbf{Z}$.

Also recall that the proof given by Bismut-Zhang in [14] for their localization formula relies heavily on the difficult paper of Bismut-Lebeau [12], and might not be easy to follow. So in Section 3 we will give an alternate proof of this localization formula by making use of the Freed-Melrose index theorem for $\mathbf{Z}/k$ manifolds [17] instead.

In Section 4, we present some preliminary applications of our formula to the case of flat vector bundles. In particular, we show that our formula leads to an alternate formulation of the Atiyah-Patodi-Singer $\mathbf{R}/\mathbf{Z}$ index theorem for unitary flat vector bundles [5, Theorem 5.3]. Moreover, we show that when considering the Signature operator instead of the Dirac operator, one can refine the above index theorem to an $\mathbf{R}$ valued one.

§ 2. A Geometric Formula for η Invariants

In this section, we recall the localization formula for η invariants of Bismut-Zhang [14] and use it to deduce a geometric formula for η invariants.

This section is organized as follows. In (a), we recall the direct image construction of Atiyah-Hirzebruch [2] under real embeddings in a geometrical form. In (b), we recall from [8] and [14] the construction of the Chern-Simons current associated to the geometric direct image constructed in (a). In (c), we state the localization formula from [14]. In (d), we apply this localization formula to getting a geometric formula expressing the η invariants through Chern-Simons currents on spheres.

(a) A geometric construction of direct images

Let $i : Y \hookrightarrow X$ be an embedding between two smooth oriented manifolds. We make the assumption that $\dim X - \dim Y$ is even and that if N denotes the normal bundle to Y in X, then N is orientable, spin and carries an induced orientation as well as a (fixed) spin structure.

Let μ be a complex vector bundle over Y.

Atiyah and Hirzebruch have constructed in [2] an element $i_!\mu \in \widetilde{K}(X)$, called the direct image of μ under i. We here recall this construction in a geometric form.

Let g^N be a Euclidean metric on N and ∇^N a Euclidean connection on N preserving g^N. Let $S(N)$ be the vector bundle of spinors associated to (N, g^N). Then $S(N) = S_+(N) \oplus S_-(N)$ (resp. its dual $S^*(N) = S_+^*(N) \oplus S_-^*(N)$) is a $\mathbf{Z}_2$-graded complex vector bundle over Y carrying an induced Hermitian metric $g^{S(N)} = g^{S_+(N)} \oplus g^{S_-(N)}$ (resp. $g^{S^*(N)} = g^{S_+^*(N)} \oplus g^{S_-^*(N)}$) from g^N, as well as a Hermitian connection $\nabla^{S(N)} = \nabla^{S_+(N)} \oplus \nabla^{S_-(N)}$ (resp. $\nabla^{S^*(N)} = \nabla^{S_+^*(N)} \oplus \nabla^{S_-^*(N)}$) induced from ∇^N.

Let g^μ be a Hermitian metric on μ and ∇^μ a Hermitian connection on μ preserving g^μ.

For any $r > 0$, set $N_r = \{Z \in N : |Z| < r\}$. We make the assumption that there is $\varepsilon_0 > 0$ such that $N_{2\varepsilon_0}$ is diffeomorphic to an open neighborhood of Y in X. Without confusion we now view directly $N_{2\varepsilon_0}$ as an open neighborhood of Y in X.

Let $\pi : N \to Y$ denote the projection of the normal bundle N over Y.

If $Z \in N$, let $\tilde{c}(Z) \in \mathrm{End}(S^*(N))$ be the transpose of $c(Z)$ acting on $S(N)$. Let $\tau^{N*} \in \mathrm{End}(S^*(N))$ be the transpose of τ^N defining the $\mathbf{Z}_2$-grading of $S(N) = S_+(N) \oplus S_-(N)$.

Let $\pi^*(S^*(N))$ be the pull back bundle of $S^*(N)$ over N. For any $Z \in N$ with $Z \neq 0$, let $\tau^{N*}\tilde{c}(Z) : \pi^*(S_\pm^*(N))|_Z \to \pi^*(S_\mp^*(N))|_Z$ denote the corresponding pull back isomorphisms at Z.

Let F be a complex vector bundle over Y such that $S_-^*(N) \otimes \mu \oplus F$ is a trivial complex vector bundle over Y (cf. [1]). Then

$$\tau^{N*}\tilde{c}(Z) \oplus \pi^*\mathrm{Id}_F : \pi^*(S_+^*(N) \otimes \mu \oplus F) \to \pi^*(S_-^*(N) \otimes \mu \oplus F) \tag{2.1}$$

induces an isomorphism between two trivial vector bundles over $N_{2\varepsilon_0} \setminus Y$.

Let F admit a Hermitian metric g^F and a Hermitian connection ∇^F.

Clearly, $\pi^*(S_\pm^*(N) \otimes \mu \oplus F)|_{\partial N_{2\varepsilon_0}}$ extend smoothly to two trivial complex vector bundles over $X \setminus N_{2\varepsilon_0}$. Moreover, the isomorphism $\tau^{N*}\tilde{c}(Z) \oplus \pi^*\mathrm{Id}_F$ over $\partial N_{2\varepsilon_0}$ extends smoothly to an isomorphism between these two trivial vector bundles over $X \setminus N_{2\varepsilon_0}$.

In summary, what we get is a $\mathbf{Z}_2$-graded Hermitian vector bundle

$$\xi = \xi_+ \oplus \xi_-, \qquad g^\xi = g^{\xi_+} \oplus g^{\xi_-} \tag{2.2}$$

over X such that

$$\xi_\pm|_{N_{\varepsilon_0}} = \pi^*(S_\pm^*(N) \otimes \mu \oplus F)|_{N_{\varepsilon_0}}, \quad g^{\xi_\pm|_{N_{\varepsilon_0}}} = \pi^*(g^{S_\pm^*(N)\otimes\mu} \oplus g^F)|_{N_{\varepsilon_0}}, \tag{2.3}$$

where $g^{S^*_\pm(N)\otimes\mu}$ is the tensor product Hermitian metric on $S^*_\pm(N)\otimes\mu$ induced from $g^{S^*_\pm(N)}$ and g^μ. It is easy to see that there exists an odd self-adjoint endomorphism V of ξ such that it is invertible on $X\setminus Y$, and that

$$V|_{N_{\varepsilon_0}} = \tau^{N*}\tilde{c}(Z)\oplus \pi^*\mathrm{Id}_F. \tag{2.4}$$

Moreover, there is a $\mathbf{Z}_2$-graded Hermitian connection $\nabla^\xi = \nabla^{\xi_+}\oplus\nabla^{\xi_-}$ on $\xi=\xi_+\oplus\xi_-$ over X such that

$$\nabla^{\xi_\pm}|_{N_{\varepsilon_0}} = \pi^*(\nabla^{S^*_\pm(N)\otimes\mu}\oplus\nabla^F), \tag{2.5}$$

where $\nabla^{S^*_\pm(N)\otimes\mu}$ is the Hermitian connection on $\nabla^{S^*_\pm(N)\otimes\mu}$ defined by $\nabla^{S^*_\pm(N)\otimes\mu} = \nabla^{S^*_\pm(N)}\otimes \mathrm{Id}_\mu + \mathrm{Id}_{S^*_\pm(N)}\otimes\nabla^\mu$.

Clearly, $\xi_+ - \xi_- \in \widetilde{K}(X)$ is exactly the Atiyah-Hirzebruch direct image $i_!\mu$ of μ constructed in [2]. We call (ξ,∇^ξ,V) constructed above a geometric direct image of $(\mu,\,\nabla^\mu)$.

Remark 2.1. Note that here we have used the dual of spinor bundle of N, instead of spinor bundle of N as in [2], to construct the direct image. This is done for the reason of fixing the sign convention in (2.10) below.

(b) A Chern-Simons current associated to a geometric direct image

We make the same assumptions and use the same notations as in (a).

If E is a real vector bundle carrying a connection ∇^E, we denote by $\widehat{A}(E,\nabla^E)$ the Hirzebruch characteristic form defined by

$$\widehat{A}(E,\nabla^E) = \det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^E}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^E\right)}\right), \tag{2.6}$$

where $R^E=\nabla^{E,2}$ is the curvature of ∇^E. While if E' is a complex vector bundle carrying a connection $\nabla^{E'}$, we denote by $\mathrm{ch}(E',\nabla^{E'})$ the Chern character form associated to $(E',\nabla^{E'})$ (cf. [22, Section 1]).

Let $i^{1/2}$ be a fixed square root of $i=\sqrt{-1}$. The objects which will be considered in the sequel do not depend on this square root. Let φ be the map $\alpha\in\Lambda^*(T^*X)\to(2\pi i)^{-\frac{\deg\alpha}{2}}\alpha\in\Lambda^*(T^*X)$.

We now use Quillen's superconnection formalism [20]. For $T\geq 0$, let C_T be the superconnection on the $\mathbf{Z}_2$-graded vector bundle ξ defined by

$$C_T = \nabla^\xi + \sqrt{T}V. \tag{2.7}$$

The curvature C_T^2 of C_T is a smooth section of $(\Lambda^*(T^*M)\widehat{\otimes}\mathrm{End}(\xi))^{\mathrm{even}}$. By [Q], we know that for any $T>0$,

$$\frac{\partial}{\partial T}\mathrm{Tr}_s[\exp(-C_T^2)] = -\frac{d}{2\sqrt{T}}\mathrm{Tr}_s[V\exp(-C_T^2)], \tag{2.8}$$

where "Tr_s" is the supertrace in the sense of Quillen [20] associated to the $\mathbf{Z}_2$-grading of ξ.

Clearly, the technical assumptions in [14, (1.10)–(1.12)] hold for our constructions in (a). Thus one can proceed as in [7], [8] and [14, Definition 1.3] to construct the Chern-Simons current $\gamma^{\xi,V}$ as

$$\gamma^{\xi,V} = \frac{1}{\sqrt{2\pi i}} \int_0^{+\infty} \varphi \mathrm{Tr}_s[V \exp(-C_T^2)] \frac{dT}{2\sqrt{T}}. \tag{2.9}$$

Let δ_Y denote the current of integration over the oriented submanifold Y of X. Then by [14, Theorem 1.4], we have

$$d\gamma^{\xi,V} = \mathrm{ch}(\xi_+, \nabla^{\xi_+}) - \mathrm{ch}(\xi_-, \nabla^{\xi_-}) - \widehat{A}^{-1}(N, \nabla^N)\mathrm{ch}(\mu, \nabla^\mu)\delta_Y. \tag{2.10}$$

Moreover, as indicated in [14, Remark 1.5], by proceeding as in [11, Theorem 3.3], one can prove that $\gamma^{\xi,V}$ is a locally integrable current.

(c) A localization formula for η invariants

We assume in this subsection that $i : Y \hookrightarrow X$ is an embedding between two odd dimensional closed oriented spin manifolds. Then the normal bundle N to Y in X is even dimensional and carries a canonically induced orientation and spin structure. Let g^{TX} be a Riemannian metric on TX. Let g^{TY} be the restricted Riemannian metric on TY. Let ∇^{TX} (resp. ∇^{TY}) denote the Levi-Civita connection associated to g^{TX} (resp. g^{TY}).

Without loss of generality we may and we will make the assumption that the embedding $(Y, g^{TY}) \hookrightarrow (X, g^{TX})$ is totally geodesic. Let N carry the canonically induced Euclidean metric as well as the Euclidean connection.

Thus, we may and we will make the same construction as in (a), (b).

Recall that the definition of the reduced η invariant for a (twisted) Dirac operator on an odd dimensional spin Riemannian manifold has been recalled in Section 1.

Under our assumptions, we see easily that the localization formula for η invariants proved in [14] holds in a slightly simplified form. We recall it as follows.

Theorem 2.1. (cf. [14, Theorem 2.2]) *The following identity holds,*

$$\bar{\eta}(D^{\xi_+}) - \bar{\eta}(D^{\xi_-}) \equiv \bar{\eta}(D^\mu) + \int_X \widehat{A}(TX, \nabla^{TX})\gamma^{\xi,V} \qquad \mathrm{mod}\ \mathbf{Z}. \tag{2.11}$$

Remark 2.2. The extra Chern-Simons form in [14, Theorem 2.2] disappears here simply because we have made the simplifying assumption that the isometric embedding $(Y, g^{TY}) \hookrightarrow (X, g^{TX})$ is totally geodesic. Indeed, in view of [14, Remark 2.3], Theorem 2.1 is equivalent to [14, Theorem 2.2].

(d) A geometric formula for $\bar{\eta}(D^\mu)$

We continue the discussion in (c) and assume that $X = S^{2n-1}$, a higher odd dimensional sphere (but we do not assume that it admits the standard metric, this makes the isometric embedding $i : (Y, g^{TY}) \hookrightarrow (S^{2n-1}, g^{TS^{2n-1}})$ to be totally geodesic possible).

Now recall that by the Bott periodicity (cf. [1]), one has $\widetilde{K}(S^{2n-1}) = \{0\}$. Thus, in our case, we have $i_!\mu = 0$. This means that there is a trivial complex vector bundle θ over S^{2n-1} such that $\theta \oplus \xi_+$ is isomorphic to $\theta \oplus \xi_-$.

We equip θ with a Hermitian metric as well as a Hermitian connection ∇^{θ}.

Let $\xi' = \xi'_+ \oplus \xi'_-$ be the $\mathbf{Z}_2$-graded Hermitian vector bundle over S^{2n-1} with $\xi'_{\pm} = \theta \oplus \xi_{\pm}$. Then $\xi'_{\pm}$ and ξ' carry canonically induced Hermitian connections $\nabla^{\xi'_{\pm}}$ and $\nabla^{\xi'}$ respectively through direct sums.

Let W be an odd self-adjoint automorphism of ξ', which clearly exists by the above discussion.

For any $T \geq 0$, set

$$C'_T = \nabla^{\xi'} + \sqrt{T} W. \tag{2.12}$$

Similarly as in (2.9), we define the Chern-Simons form $\gamma^{\xi',W}$ as

$$\gamma^{\xi',W} = \frac{1}{\sqrt{2\pi i}} \int_0^{+\infty} \varphi \mathrm{Tr}_s[W \exp(-(C'_T)^2)] \frac{dT}{2\sqrt{T}}. \tag{2.13}$$

Since W is invertible, one has the following formula due to Bismut-Cheeger [10, Theorem 2.28], which corresponds to the case with $Y = \emptyset$ in (2.11),

$$\bar{\eta}(D^{\xi'_+}) - \bar{\eta}(D^{\xi'_-}) \equiv \int_{S^{2n-1}} \widehat{A}(TS^{2n-1}, \nabla^{TS^{2n-1}}) \gamma^{\xi',W} \qquad \text{mod } \mathbf{Z}. \tag{2.14}$$

On the other hand, one clearly has

$$\bar{\eta}(D^{\xi'_{\pm}}) = \bar{\eta}(D^{\xi_{\pm}}) + \bar{\eta}(D^{\theta}). \tag{2.15}$$

From (2.11), (2.14) and (2.15), one gets

$$\bar{\eta}(D^{\mu}) \equiv \int_{S^{2n-1}} \widehat{A}(TS^{2n-1}, \nabla^{TS^{2n-1}}) \gamma^{\xi',W} - \int_{S^{2n-1}} \widehat{A}(TS^{2n-1}, \nabla^{TS^{2n-1}}) \gamma^{\xi,V} \qquad \text{mod } \mathbf{Z}. \tag{2.16}$$

We can re-formulate (2.16) as follows.

Let $\widetilde{\xi} = \widetilde{\xi}_+ \oplus \widetilde{\xi}_-$ be the $\mathbf{Z}_2$-graded Hermitian vector bundle over S^{2n-1} defined by

$$\widetilde{\xi}_+ = \xi_+ \oplus \xi'_-, \qquad \widetilde{\xi}_- = \xi_- \oplus \xi'_+, \tag{2.17}$$

carrying the canonically induced Hermitian connection $\nabla^{\tilde{\xi}}$ through direct sums. Let $\widetilde{V} = V \oplus W^T$, where W^T is the transpose of W, be the odd self-adjoint endomorphism of $\widetilde{\xi}$. Then $(\widetilde{\xi}, \nabla^{\tilde{\xi}}, \widetilde{V})$ forms a geometric direct image of μ in the sense of Section 2(a).

Let $\gamma^{\tilde{\xi},\widetilde{V}}$ be the associated Chern-Simons current defined by (2.9). By (2.10) and the construction of $(\widetilde{\xi}, \nabla^{\tilde{\xi}}, \widetilde{V})$, one verifies easily that

$$d\gamma^{\tilde{\xi},\widetilde{V}} = -\widehat{A}^{-1}(N, \nabla^N) \mathrm{ch}(\mu, \nabla^{\mu}) \delta_Y. \tag{2.18}$$

We can now state the main result of this section as follows, which is simply a reformulation of (2.16).

Theorem 2.2. *The following identity holds,*

$$\bar{\eta}(D^{\mu}) \equiv -\int_{S^{2n-1}} \widehat{A}(TS^{2n-1}, \nabla^{TS^{2n-1}}) \gamma^{\tilde{\xi},\widetilde{V}} \qquad \text{mod } \mathbf{Z}. \tag{2.19}$$

Remark 2.3. If the embedding $Y \hookrightarrow S^{2n-1}$ is not totally geodesic, then the right hand side of (2.19) will contain an extra Chern-Simons term involving the second fundamental form of this embedding. With this extra term involved, one can then assume that S^{2n-1} carries the standard metric.

Remark 2.4. It is interesting that the right hand side of (2.19) does not involve any spectral information of D^μ. It is purely geometric/topological and resembles well the K-theoretic proof of the Atiyah-Singer index theorem [6]. On the other hand, the different choice of W may cause the right hand side of (2.19) an integer jump. This partly explains that (2.19) is in general a mod $\mathbf{Z}$ formula. While conversely, one can always find a W to make (2.19) a purely equality in $\mathbf{R}$. This may sound few sense as when g^{TY}, g^μ and ∇^μ vary, dim(ker D^μ) may jump. However, if we take $\mu = S(TY)$, then $D^{S(TY)}$ is equivalent to the Signature operator of (Y, g^{TY}) and thus $\bar{\eta}(D^\mu)$ becomes a smooth invariant. Moreover, since $K^1(S^{2n-1}) = \mathbf{Z}$, one can choose W so that (2.19) is an equality without mod $\mathbf{Z}$. In such a form (2.19) becomes an equality in $\mathbf{R}$ not depending on the variation of g^{TY}.

Remark 2.5. If $\dim Y \equiv 3 \bmod 8\mathbf{Z}$ and μ is a complexification of a Euclidean vector bundle carrying a Euclidean connection, then one can embed Y into a higher $8l+3$ dimensional manifold and proceed as in [21, Section 3] to improve (2.19) to a mod $2\mathbf{Z}$ formula.

Remark 2.6. As have been mentioned in Section 1, the proof of Theorem 2.1 given in [14] relies heavily on the difficult paper of Bismut-Lebeau [12]. So in the next section, we will give an alternate proof of Theorem 2.1 for the case where X is a higher dimensional sphere.

§ 3. An Alternate Proof of Theorem 2.1 when $X = S^{2n-1}$

As in Section 2, let Y be an odd dimensional closed oriented spin manifold carrying a Riemannian metric g^{TY} and the associated Levi-Civita connection ∇^{TY}. Let μ be a complex vector bundle over Y carrying a Hermitian metric g^μ and a Hermitian connection ∇^μ.

In case when there will be no confusion, we will use the notations in Section 2 without further explanation.

Since $\dim Y$ is odd, by a well-known result in bordism/cobordism theory, there is a positive integer k such that the k disjoint copies of Y bound a compact oriented spin manifold $\widehat{Y}$ of dimension $\dim Y + 1$ such that the boundary $\partial\widehat{Y}$ does not contain other components. Moreover, there is a complex vector bundle $\hat{\mu}$ over $\widehat{Y}$ such that when restricted to boundary, it is just μ on each copy of Y (We thank Fuquan Fang for confirming this to us).

Clearly, $(\widehat{Y}, Y)$ is a $\mathbf{Z}/k$ manifold in the sense of Sullivan (cf. [17, 21]).

Let $g^{\widehat{Y}}$ be a Riemannian metric on $T\widehat{Y}$ which is of product nature near $\partial\widehat{Y}$ and which on $\partial\widehat{Y}$ is exactly g^{TY} on each copy of the boundary. Let $\nabla^{\widehat{Y}}$ be the associated Levi-Civita connection. Similarly, let $g^{\hat{\mu}}$ (resp. $\nabla^{\hat{\mu}}$) be a Hermitian metric on $\hat{\mu}$ such that it is of product nature near $\partial\widehat{Y}$ and that on $\partial\widehat{Y}$ it is exactly g^μ (resp. ∇^μ) on μ over each copy of Y.

Let $S^{2n,k}$ be the $\mathbf{Z}/k$ manifold obtained by removing k balls D^{2n} from the $2n$-sphere. Then the boundary $\partial S^{2n,k}$ consists of k disjoint copies of S^{2n-1}. Let $i : \widehat{Y} \hookrightarrow S^{2n,k}$ be a $\mathbf{Z}/k$

embedding (cf. [17, 21]). The existence of such an embedding is clear when n is sufficiently large.

Let $g^{TS^{2n,k}}$ be a $\mathbf{Z}/k$ metric on $TS^{2n,k}$ which is of product nature near $\partial S^{2n,k}$, such that $g^{TS^{2n,k}}|_{T\widehat{Y}} = g^{T\widehat{Y}}$ and that the isometric embedding $i : (\widehat{Y}, g^{T\widehat{Y}}) \hookrightarrow (S^{2n,k}, g^{TS^{2n,k}})$ is totally geodesic. Let $\nabla^{TS^{2n,k}}$ be the associated Levi-Civita connection.

Let $\widehat{N}$ denote the normal bundle to $\widehat{Y}$ in $S^{2n,k}$. Then $\widehat{N}$ carries an induced orientation and spin structure, as well as a $\mathbf{Z}/k$ Euclidean metric (resp. connection) $g^{\widehat{N}}$ (resp. $\nabla^{\widehat{N}}$).

We can then apply the constructions in Sections 2(a), (b) to the embedding $i : (\widehat{Y}, g^{T\widehat{Y}}) \hookrightarrow (S^{2n,k}, g^{TS^{2n,k}})$ in a $\mathbf{Z}/k$ manner (that is, preserving all the $\mathbf{Z}/k$ structures), such that all the metrics, connections and maps involved are of product nature near boundary.

We denote the resulting $\mathbf{Z}/k$ geometric direct image of $(\hat{\mu}, \nabla^{\hat{\mu}})$ by $(\widehat{\xi} = \widehat{\xi}_+ \oplus \widehat{\xi}_-, \nabla^{\hat{\xi}}, \widehat{V})$. When restrict to each copy of the boundary, we denote the restricted geometric direct image of (μ, ∇^{μ}) by $(\xi = \xi_+ \oplus \xi_-, \nabla^{\xi}, V)$.

Let $\mathrm{ind}_k(D^{\hat{\xi}_\pm})$, $\mathrm{ind}_k(D^{\hat{\mu}})$ be the mod k indices defined by (cf. [17, (5.2), and 21]),

$$\mathrm{ind}_k(D^{\hat{\xi}_\pm}) \equiv \int_{S^{2n,k}} \widehat{A}(TS^{2n,k}, \nabla^{TS^{2n,k}})\mathrm{ch}(\widehat{\xi}_\pm, \nabla^{\hat{\xi}_\pm}) - k\,\bar{\eta}(D^{\xi_\pm}) \qquad \mathrm{mod}\ k\mathbf{Z}, \tag{3.1}$$

$$\mathrm{ind}_k(D^{\hat{\mu}}) \equiv \int_{\widehat{Y}} \widehat{A}(T\widehat{Y}, \nabla^{\widehat{Y}})\mathrm{ch}(\widehat{\mu}, \nabla^{\hat{\mu}}) - k\,\bar{\eta}(D^{\mu}) \qquad \mathrm{mod}\ k\mathbf{Z}. \tag{3.2}$$

By the mod k index theorem of Freed and Melrose [17, (5.5)], one knows that

$$\mathrm{ind}_k(D^{\hat{\xi}_+}) - \mathrm{ind}_k(D^{\hat{\xi}_-}) = \mathrm{ind}_k(D^{\hat{\mu}}) \tag{3.3}$$

in $\mathbf{Z}/k\mathbf{Z}$.

Now if we denote $\gamma^{\hat{\xi},\widehat{V}}$ the Chern-Simons current constructed in Section 2(b), then its restriction to the boundary consists of k copies of the Chern-Simons current $\gamma^{\xi,V}$.

By applying the transgression formula (2.10) to $\gamma^{\hat{\xi},\widehat{V}}$ and integrate over $S^{2n,k}$, one gets

$$\begin{aligned}&\int_{S^{2n,k}} \widehat{A}(TS^{2n,k}, \nabla^{TS^{2n,k}})\mathrm{ch}(\widehat{\xi}_+, \nabla^{\hat{\xi}_+}) - \int_{S^{2n,k}} \widehat{A}(TS^{2n,k}, \nabla^{TS^{2n,k}})\mathrm{ch}(\widehat{\xi}_-, \nabla^{\hat{\xi}_-}) \\ &- \int_{\widehat{Y}} \widehat{A}(T\widehat{Y}, \nabla^{\widehat{Y}})\mathrm{ch}(\widehat{\mu}, \nabla^{\hat{\mu}}) = k\int_{S^{2n-1}} \widehat{A}(TY, \nabla^{TY})\gamma^{\xi,V}.\end{aligned} \tag{3.4}$$

From (3.1)–(3.4), one deduces that

$$\bar{\eta}(D^{\xi_+}) - \bar{\eta}(D^{\xi_-}) \equiv \bar{\eta}(D^{\mu}) + \int_{S^{2n-1}} \widehat{A}(TY, \nabla^{TY})\gamma^{\xi,V} \qquad \mathrm{mod}\ \mathbf{Z}, \tag{3.5}$$

which is exactly the Bismut-Zhang formula (2.11) in the case where $X = S^{2n-1}$.

Remark 3.1. The relation between the Bismut-Zhang formula (2.11) and the Freed-Melrose mod k index theorem [17] was exploited in [21] where (2.11) is used to give an alternate proof of a mod k equality between the right hand sides of (3.1) and (3.2). Now such an equality can be proved directly by applying the Riemann-Roch property for Dirac operators on manifolds with boundary proved by Dai and Zhang in [16], without using the results in [14]. In fact, this can be done by first applying [16, Theorem 1.2 and Lemma 4.6] to get (3.3). The mod k equality between the right hand sides of (3.1) and (3.2) is then an easy consequence of the Atiyah-Patodi-Singer index theorem [3] (This observation grew out of discussions with Xianzhe Dai).

Remark 3.2. One should also note that the proof of (3.5) given in this section holds only for those embeddings which can be obtained through an embedding between $\mathbf{Z}/k$ manifolds, while (2.11) is much more general. Still we hope the simplified proof in this section would be helpful for a good feeling of (2.11). In fact, for many applications, the existence for such an embedding is suffice, as our main result (2.19) holds for it. While on the other hand, one should be able to apply (3.5) to give a proof of (2.11) by embedding X there into a higher dimensional sphere. This will be discussed elsewhere.

§ 4. Some Applications

In this section, we discuss some immediate applications of formula (2.19).

This section is organized as follows. In (a), we discuss briefly the relationship between the Chern-Simons current and the Cheeger-Simons differential character [15]. In (b), we discuss the Atiyah-Patodi-Singer $\mathbf{R}/\mathbf{Z}$ index theorem for unitary flat vector bundles [5] from the point of view of (2.19). In (c), we show that by replacing the Dirac operator by the Signature operator, one may refine the above $\mathbf{R}/\mathbf{Z}$ formula to an $\mathbf{R}$ valued one.

We make the same assumptions and use the same notation as in Section 2.

(a) Chern-Simons current and the Cheeger-Simons differential character

As was indicated by Bismut in [8], the Chern-Simons currents constructed in Section 2 are closely related to the differential characters introduced by Cheeger and Simons in [15]. This becomes clearer if we compare the transgression formula (2.18) with the one in [15, (4.3)]. The difference is that in [15, (4.3)], the transgression formula holds on different Stiefel manifolds, while our formula holds universally on a single sphere. Moreover, the differential characters for Chern character forms in [15] are defined mod $\mathbf{Q}$, while our formula is clearly of an $\mathbf{R}/\mathbf{Z}$ nature (as the construction of the Chern-Simons current $\gamma^{\tilde{\xi},\widetilde{V}}$ in Section 2(d) depends on the choice of an automorphism W).

More precisely, if we denote by $\widehat{\widehat{A}}(TY,\nabla^{TY})$ (resp. $\widehat{\mathrm{ch}}(\mu,\nabla^{\mu})$) the Cheeger-Simons differential character associated to $\widehat{A}(TY,\nabla^{TY})$ (resp. $\mathrm{ch}(\mu,\nabla^{\mu})$) constructed in [15], then by [15, Theorem 9.1], one has, in using the product notation as in [15],

$$\bar{\eta}(D^{\mu}) \equiv \langle \widehat{\widehat{A}}(TY,\nabla^{TY}) * \widehat{\mathrm{ch}}(\mu,\nabla^{\mu}),[Y]\rangle \qquad \mathrm{mod}\ \mathbf{Q}. \tag{4.1}$$

From (2.19) and (4.1), one gets

$$\int_{S^{2n-1}} \widehat{A}(TS^{2n-1},\nabla^{TS^{2n-1}})\gamma^{\tilde{\xi},\widetilde{V}} + \langle \widehat{\widehat{A}}(TY,\nabla^{TY}) * \widehat{\mathrm{ch}}(\mu,\nabla^{\mu}),[Y]\rangle \equiv 0 \qquad \mathrm{mod}\ \mathbf{Q}. \tag{4.2}$$

(b) The Atiyah-Patodi-Singer index theorem for flat bundles revisited

In this section, we replace the Hermitian vector bundle μ in Section 2 by $\mu\otimes\rho$, where ρ is a unitary flat vector bundle with the flat connection denoted by ∇^{ρ}. We equip $\mu\otimes\rho$ with the induced tensor product Hermitian metric as well as the tensor product Hermitian connection $\nabla^{\mu\otimes\rho} = \nabla^{\mu}\otimes \mathrm{Id}_{\rho} + \mathrm{Id}_{\mu}\otimes\nabla^{\rho}$.

By [5],

$$\bar{\eta}_{\mu,\rho} := \bar{\eta}(D^{\mu\otimes\rho}) - \mathrm{rk}(\rho)\bar{\eta}(D^{\mu}) \qquad \bmod \mathbf{Z} \tag{4.3}$$

is a smooth invariant with respect to $(g^{TM}, g^{\mu}, \nabla^{\mu})$. Moreover, [5, Theorem 5.3] provides a topological interpretation for this invariant.

We now examine this invariant by using (2.19).

Let $(\widetilde{\xi}_{\rho}, \nabla^{\tilde{\xi}_{\rho}}, \widetilde{V}_{\rho})$ be a geometric direct image of $(\mu\otimes\rho, \nabla^{\mu\otimes\rho})$ constructed similarly as that for (μ, ∇^{μ}) in Section 2(d). Let $\gamma^{\tilde{\xi}_{\rho}, \widetilde{V}_{\rho}}$ be the associated Chern-Simons current defined in (2.9). By (2.18), $\gamma^{\tilde{\xi}_{\rho}, \widetilde{V}_{\rho}}$ verifies the transgression formula

$$d\gamma^{\tilde{\xi}_{\rho}, \widetilde{V}_{\rho}} = -\widehat{A}^{-1}(N, \nabla^{N})\mathrm{ch}(\mu, \nabla^{\mu})\mathrm{rk}(\rho)\delta_{Y}, \tag{4.4}$$

as ρ is a flat bundle.

By (2.19), one also has

$$\bar{\eta}(D^{\mu\otimes\rho}) \equiv -\int_{S^{2n-1}} \widehat{A}(TS^{2n-1}, \nabla^{TS^{2n-1}})\gamma^{\tilde{\xi}_{\rho}, \widetilde{V}_{\rho}} \qquad \bmod \mathbf{Z}. \tag{4.5}$$

From (2.19), (4.3) and (4.5), one gets

$$\bar{\eta}_{\mu,\rho} \equiv \int_{S^{2n-1}} \widehat{A}(TS^{2n-1}, \nabla^{TS^{2n-1}})(\mathrm{rk}(\rho)\gamma^{\tilde{\xi}, \widetilde{V}} - \gamma^{\tilde{\xi}_{\rho}, \widetilde{V}_{\rho}}) \qquad \bmod \mathbf{Z}. \tag{4.6}$$

On the other hand, from (2.18) and (4.4), one finds

$$d(\mathrm{rk}(\rho)\gamma^{\tilde{\xi}, \widetilde{V}} - \gamma^{\tilde{\xi}_{\rho}, \widetilde{V}_{\rho}}) = 0. \tag{4.7}$$

Thus, $\mathrm{rk}(\rho)\gamma^{\tilde{\xi}, \widetilde{V}} - \gamma^{\tilde{\xi}_{\rho}, \widetilde{V}_{\rho}}$ determines a cohomology class in $H^{\mathrm{odd}}_{\mathrm{dR}}(S^{2n-1}, \mathbf{R})$. One verifies easily that this cohomology class does not depend on the choice of ∇^{μ}. It only depends on the choices of the automorphisms W and W_{ρ} appearing in the construction of the geometric direct images of $(\widetilde{\xi}, \nabla^{\tilde{\xi}}, \widetilde{V})$ and $(\widetilde{\xi}_{\rho}, \nabla^{\tilde{\xi}_{\rho}}, \widetilde{V}_{\rho})$. The different choices of W and W_{ρ} cause a (possible) integer jump in integration term in the right hand side of (4.6).

Thus, (4.6) may be thought of in some sense as an alternate version of the Atiyah-Patodi-Singer index theorem for flat vector bundles stated in [5, Theorem 5.3]. Its conceptual novelty is that one need not divide the topological index into two parts (that is, a $\mathbf{Q}/\mathbf{Z}$ part plus an $\mathbf{R}$ part).

We believe that formulas (4.2) and (4.6) could be used to give a (possibly) alternate understanding of the Grothendieck-Riemann-Roch type formulas for flat vector bundles studied in [9, 13, 19, 23].

(c) Signature operator and an R valued index theorem for flat vector bundles

Now we set $\mu = S(TY)$ in the above subsection. In this case, D^{μ} is the Signature operator associated to (TY, g^{TY}), denoted by D_{Sign}, while $D^{\mu\otimes\rho}$ is now denoted by D^{ρ}_{Sign}.

Set

$$\bar{\eta}_{\mathrm{Sign},\rho} = \bar{\eta}(D^{\rho}_{\mathrm{Sign}}) - \mathrm{rk}(\rho)\bar{\eta}(D_{\mathrm{Sign}}). \tag{4.8}$$

Then $\bar{\eta}_{\mathrm{Sign},\rho}$ is a smooth invariant equivalent to what defined in [4, Theorem 2.4].

On the other hand, as indicated in Remark 2.4, one can choose the automorphisms W and W_ρ in the construction of the Chern-Simons current so that

$$\bar{\eta}(D^\rho_{\rm Sign}) = \int_{S^{2n-1}} \widehat{A}(TS^{2n-1}, \nabla^{TS^{2n-1}})\gamma^{\tilde{\xi}_\rho, \widetilde{V}_\rho}, \tag{4.9}$$

$$\bar{\eta}(D_{\rm Sign}) = \int_{S^{2n-1}} \widehat{A}(TS^{2n-1}, \nabla^{TS^{2n-1}})\gamma^{\tilde{\xi}, \widetilde{V}}. \tag{4.10}$$

It is clear that (4.9), (4.10) are actually equalities not depending on the choice of g^{TM}. Moreover, since $K^1(S^{2n-1}) = \mathbf{Z}$, one sees that the choice of W and W_ρ is canonical (up to stable homotopy).

By (4.7), one also sees that $\gamma^{\tilde{\xi}_\rho, \widetilde{V}_\rho} - {\rm rk}(\rho)\gamma^{\tilde{\xi}, \widetilde{V}}$ determines canonically an element in $H^{\rm odd}_{\rm dR}(Y, \mathbf{R})$.

We can state our $\mathbf{R}$ valued refinement of (4.6) as follows.

Theorem 4.1. *The following K-theoretic formula for $\bar{\eta}_{{\rm Sign},\rho}$ holds,*

$$\bar{\eta}_{{\rm Sign},\rho} = \int_{S^{2n-1}} \widehat{A}(TS^{2n-1}, \nabla^{TS^{2n-1}})({\rm rk}(\rho)\gamma^{\tilde{\xi}, \widetilde{V}} - \gamma^{\tilde{\xi}_\rho, \widetilde{V}_\rho}). \tag{4.11}$$

Remark 4.1. Theorem 4.1 was proved for unitary flat vector bundles over spin manifolds. It is natural to ask whether there is still such a kind of formulas without the spin condition.

Remark 4.2. If one could find a purely topological way to identify the automorphisms W and W_ρ (or the difference element of ${\rm rk}(\rho)W$ and W_ρ in $K^1(S^{2n-1})$), then one would provide a positive answer to a question of Atiyah-Patodi-Singer stated implicitly in [4, p.406]. On the other hand, for any choice of W and W_ρ, the right hand side of (4.11) provides a smooth invariant of Y. So in some sense, $\bar{\eta}_{{\rm Sign},\rho}$ becomes one example of a series of smooth invariants associated to the unitary flat vector bundle ρ over Y.

Remark 4.3. Clearly, all the results of this paper can be extended to the case of spin^c manifolds. We leave this to the interested reader.

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