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SOME RESULTS OF A NEHARI FAMILY***

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Abstract

The authors study the Nehari family N_b and obtain some important properties for this family.

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§ 1. Schwarzian Derivative

Let $w = f(z)$ be a locally univalent analytic function in the unit disk D, and its Schwarzian derivative is defined by

$$S_f = \Big(\frac{f''}{f'}\Big)' - \frac{1}{2}\Big(\frac{f''}{f'}\Big)^2 = \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2. \tag{1.1}$$

Let $u(z) = f'(z)^{-\frac{1}{2}}$. Then

$$u'' = -\frac{1}{2}(f')^{-\frac{1}{2}}\Big[-\frac{3}{2}\Big(\frac{f''}{f'}\Big)^2 + \frac{f'''}{f'}\Big] = -\frac{1}{2}(f')^{-\frac{1}{2}}S_f(z),$$

hence $u'' + \frac{1}{2}S_f \cdot u = 0$.

On the contrary, if $q(z)$ is an analytic function, and u is the solution of the following equation

$$u'' + qu = 0,$$

denote

$$f(z) = \int_0^z u^{-2}(\zeta)d\zeta,$$

then $S_f = 2q$.

An important property of Schwarzian derivative is the invariance under Möbius transformation. If T is a Möbius transformation, then we have

$$S_{T\circ f} = S_f. \tag{1.2}$$

According to the above property, if $f(z)$ is a locally univalent meromorphic function, the Schwarzian derivatives of $f(z)$ at its poles can be defined by

$$S_f(z) = S_{\frac{1}{f}}(z).$$

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For the Schwarzian derivative of the composition of two analytic functions f, h, we have

$$S_{f\circ h} = S_f(h(z))(h')^2 + S_h. \tag{1.3}$$

In 1949, Nehari [**?**] proved the following theorem which reveals the connection between the univalence of analytic functions and their Schwarzian derivatives.

Theorem A. *If f is a univalent analytic function in the unit disk D, then*

$$|S_f(z)| \le \frac{6}{(1-|z|^2)^2}. \tag{1.4}$$

On the contrary, if f is a locally univalent analytic function in the unit disk D, and satisfies

$$|S_f(z)| \le \frac{2}{(1-|z|^2)^2}, \tag{1.5}$$

then f is univalent in the unit disk D.

Because the Schwarzian derivative of the Köbe function

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots \tag{1.6}$$

is

$$S_k(z) = \frac{-6}{(1-z^2)^2},$$

the constant 6 in the above necessity condition can not be improved.

At the same time, Hille [**?**] found that the function

$$f(z) = \Big(\frac{1-z}{1+z}\Big)^{i\gamma}, \qquad \gamma > 0 \tag{1.7}$$

is not univalent in the unit disk D, and its Schwarzian derivative is

$$S_f(z) = \frac{2(1+\gamma^2)}{(1-z^2)^2}.$$

This example shows that the constant 2 in the above sufficiency condition is also sharp.

Nehari's work has attracted the notice of many mathematicians. In 1962, Ahlfors and Weill [2] found the connection between the quasiconformal extensibility of conformal mappings and their Schwarzian derivatives.

Theorem B. (cf. [2]) *If*

$$|S_f(z)| \le \frac{2t}{(1-|z|^2)^2}, \qquad 0 \le t < 1, \tag{1.8}$$

then f can be extended to a $\frac{1+t}{1-t}$-quasiconformal mapping of the whole complex plane.

Following Nehari and Ahlfors, many scholars have made a lot of researches in this area.

§ 2 . Nehari Families

In 1979, Nehari [**?**] obtained other two important univalence criteria for analytic functions.

Theorem C. *If f is a locally univalent analytic function in the unit disk D, and satisfies*

$$|S_f(z)| \le \frac{4}{1-|z|^2}, \tag{2.1}$$

then f is univalent in the unit disk D.

Theorem D. *If f is a locally univalent analytic function in the unit disk D, and satisfies*

$$|S_f(z)| \le \frac{\pi^2}{2}, \tag{2.2}$$

then f is univalent in the unit disk D.

We used to call the families of univalent analytic functions which satisfy the following three Nehari's univalence criteria

(a) $|S_f(z)| \le \frac{2}{(1-|z|^2)^2}$,

(b) $|S_f(z)| \le \frac{4}{1-|z|^2}$,

(c) $|S_f(z)| \le \frac{\pi^2}{2}$

as Nehari families, and denote them by N_a, N_b and N_c respectively.

It should be noticed that the functions which satisfy the normalizations $f(0)=0,\ f'(0)=1,\ f''(0)=0$ and the following three equalities

$$S_f(z) = \frac{2}{(1-z^2)^2}, \quad S_f(z) = \frac{4}{1-z^2}, \quad S_f(z) = \frac{\pi^2}{2}$$

are

$$f(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \quad f(z) = H(z) = \frac{1}{4}\Big(\log\frac{1+z}{1-z} - \frac{1}{1+z} + \frac{1}{1-z}\Big), \quad f(z) = \frac{2}{\pi}\tan\frac{\pi z}{2},$$

which are called the extremal functions for the corresponding Nehari families, and the images of the unit disk under the extremal functions are called the extremal domains.

In 1985, Gehring and Pommerenke [**?**] studied Nehari family N_a, and obtained some important results.

Theorem E. *Let f be an analytic function in the unit disk which satisfies the normalization $f''(0)=0$. If f satisfies*

$$|S_f(z)| \le \frac{2}{(1-|z|^2)^2},$$

then either

$$f(z) = a\log\frac{e^{i\theta}+z}{e^{i\theta}-z} + b, \qquad a, b \in C,\ 0 \le \theta < 2\pi, \tag{2.3}$$

or $f(z)$ has a homeomorphic extension to $\overline{D}$ with

$$|f(z) - f(z')| \le M_1\Big(\log\frac{3}{|z-z'|}\Big)^{-1}, \qquad z, z' \in \overline{D},$$

$$|f(re^{i\theta}) - f(e^{i\theta})| \le M_2[\mathrm{dist}(f(re^{i\theta}), \partial f(D))]^{\frac{1}{2}}, \qquad 0 \le r < 1,\ 0 \le \theta < 2\pi.$$

Theorem F. *Let f be an analytic function in the unit disk. If f satisfies*

$$|S_f(z)| \le \frac{2b}{(1-|z|^2)^2}, \qquad b < 1,$$

then $f(D)$ is a quasidisk with constant $8(1-b)^{-\frac{1}{2}}$.

For Nehari family N_c, the following results are also known (see [5]).

Theorem G. *Let f be an analytic function in the unit disk which satisfies the normalizations $f(0)=0,\ f'(0)=1$ and $f''(0)=0$. If f satisfies*

$$|S_f(z)| \le \frac{\pi^2 t}{2}, \qquad 0 \le t < 1, \tag{2.4}$$

then f has a homeomorphic extension to $\overline{D}$, and

$$\frac{2}{\pi\sqrt{t}}\tanh\Big(\frac{\pi\sqrt{t}}{2}|z|\Big) \le |f(z)| \le \frac{2}{\pi\sqrt{t}}\tan\Big(\frac{\pi\sqrt{t}}{2}|z|\Big),$$
$$\cosh^{-2}\Big(\frac{\pi\sqrt{t}}{2}|z|\Big) \le |f'(z)| \le \cos^{-2}\Big(\frac{\pi\sqrt{t}}{2}|z|\Big).$$

§ 3 . Nehari Family N_b

For the Nehari families N_a and N_c, many results have already been obtained, but for the Nehari family N_b, much less is known.

We know that the analytic functions in the Nehari families which satisfy the following equations

$$S_f(z) = \frac{2t}{(1-z^2)^2}, \qquad S_f(z) = \frac{4t}{1-z^2}, \qquad S_f(z) = \frac{\pi^2 t}{2}$$

are closely related to the following three differential equations

(a)′ $u'' + \frac{t}{(1-z^2)^2}u = 0,$

(b)′ $u'' + \frac{2t}{1-z^2}u = 0,$

(c)′ $u'' + \frac{\pi^2 t}{4}u = 0,$

where $u(z) = [f'(z)]^{-\frac{1}{2}}$.

The solutions of differential equations (a)′, (c)′ can be expressed explicitly, but the solution of differential equation (b)′ can not be expressed explicitly, which makes the research for Nehari family N_b more difficult.

In this paper, we apply the method in [**?**] to studying the Nehari family N_b, and obtain some important properties for N_b.

If $f(z) = z + a_2 z^2 + \cdots$ is an analytic function in the unit disk D, let $g(z) = f(z)/(1 + a_2 f(z))$, then $g(z)$ is a meromorphic function in the unit disk, which satisfies the normalizations $g(0) = 0,\ g'(0) = 1$ and $g''(0) = 0$, and has the same Schwarzian derivative with $f(z)$.

Theorem 3.1. *Let $f(z)$ be a meromorphic function in the unit disk D, which satisfies $f(0)=0, f'(0)=1, f''(0)=0$, and*

$$(1-|z|^2)|S_f(z)| \le 4. \tag{3.1}$$

Then either

$$f(z) = H(z) = \int_0^z \frac{ds}{(1-s^2)^2} = \frac{1}{4}\Big(\log\frac{1+z}{1-z} - \frac{1}{1+z} + \frac{1}{1-z}\Big), \tag{3.2}$$

or $f(z)$ has a homeomorphic extension to $\overline{D}$ with

$$|f(z) - f(z')| < M_1|z - z'|, \qquad z, z' \in D, \tag{3.3}$$

$$|f(re^{i\theta}) - f(e^{i\theta})| = o[\mathrm{dist}(f(re^{i\theta}), \partial f(D))]^{\frac{1}{2}}, \qquad r \to 1-0,\ 0 \le \theta < 2\pi. \tag{3.4}$$

Proof. Let

$$t = H(z) = \int_0^z \frac{ds}{(1-s^2)^2},$$

whose Schwarzian derivative is $S_H(z) = \frac{4}{1-z^2}$. Suppose that the meromorphic function $f(z)$ satisfies the conditions of the theorem, and let

$$g(t) = f \circ h(t), \tag{3.5}$$

where $h(t) = e^{i\theta}H^{-1}(t)$. Then $g(t)$ is the meromorphic function in the extremal domain, and

$$|g'(t)| = (1-r^2)^2|f'(re^{i\theta})| \tag{3.6}$$

for $t \geq 0$ and $r = H^{-1}(t)$. After some computation, we have

$$\begin{aligned} h'(t) &= e^{i\theta}(1-r^2)^2, \qquad h''(t) = -4e^{i\theta}(1-r^2)^3 r, \\ S_h(z) &= -4(1-r^2)^3, \\ S_g(z) &= S_f \cdot h'^2 + S_h = e^{2i\theta}(1-r^2)^4 S_f - 4(1-r^2)^3. \end{aligned}$$

Hence

$$\mathrm{Re}\{S_g\} = (1-r^2)^3[\mathrm{Re}\{e^{2i\theta}(1-r^2)S_f\} - 4] \leq 0. \tag{3.7}$$

Define

$$v(t) = |g'(t)|^{-\frac{1}{2}}, \qquad t \geq 0. \tag{3.8}$$

Here $v(t)$ is 0 at the possible poles of $g(t)$. Then we have

$$\frac{v'}{v} = -\frac{1}{2}\mathrm{Re}\Big\{\frac{g''}{g'}\Big\}, \qquad \frac{v''}{v} - \Big(\frac{v'}{v}\Big)^2 = -\frac{1}{2}\mathrm{Re}\Big\{\frac{d}{dt}\frac{g''}{g'}\Big\}. \tag{3.9}$$

It follows that

$$v''(t) = p(t)v(t), \qquad t \geq 0, \tag{3.10}$$

where

$$p(t) = -\frac{1}{2}\mathrm{Re}\{S_g(t)\} + \Big[\frac{1}{2}\mathrm{Im}\Big\{\frac{g''(t)}{g'(t)}\Big\}\Big]^2 \geq 0 \tag{3.11}$$

by (3.7).

It is obvious that $v(t)$ is non-negative and convex for $t \geq 0$. This is also true when g has a possible pole at $t = t_0 > 0$ in which case $v(t_0) = 0$. It follows from $f''(0) = 0$ and (3.5) that $g''(0) = 0$. Then (3.9) implies that $v'(0) = 0$ which shows that $v(t)$ has its minimum $v(0) = 1$ at $t = 0$, so we conclude that $g(t) \neq \infty$, which implies that f is analytic in the unit disk.

If $v'(t_0) = 0$ for some $t = t_0 \neq 0$, by the convexity of $v(t)$ we obtain that $v'(t) \equiv 0, t \in [0, t_0]$, hence $v''(t) \equiv 0,\ t \in [0, t_0]$. Then it follows from (3.9) that

$$\mathrm{Re}\Big\{\frac{g''}{g'}\Big\} \equiv 0, \quad \mathrm{Re}\Big\{\frac{d}{dt}\frac{g''}{g'}\Big\} \equiv 0, \qquad t \in [0, t_0].$$

Because

$$\mathrm{Re}\{S_g\} = \mathrm{Re}\Big\{\frac{d}{dt}\frac{g''}{g'}\Big\} - \frac{1}{2}\Big\{\mathrm{Re}\Big\{\frac{g''}{g'}\Big\}\Big\}^2 + \frac{1}{2}\Big\{\mathrm{Im}\Big\{\frac{g''}{g'}\Big\}\Big\}^2 \leq 0,$$

it also follows that

$$\mathrm{Im}\Big\{\frac{g''}{g'}\Big\} \equiv 0, \qquad t \in [0, t_0],$$

hence

$$g''(t) \equiv 0, \qquad t \in [0, t_0].$$

Thus by the isolatedness of zeros of analytic functions, we have

$$g''(t) \equiv 0, \qquad t \in G,$$

and hence

$$g'(t) \equiv 1, \qquad t \in G,$$

where G is the extremal domain. By (3.5) we obtain

$$f(z) = H(z) = \int_0^z \frac{ds}{(1-s^2)^2} = \frac{1}{4}\Big(\log\frac{1+z}{1-z} - \frac{1}{1+z} + \frac{1}{1-z}\Big).$$

On the contrary, if $f(z)$ is not of the form $H(z)$, then the above argument shows that for any choice of the constant θ, $v'(t) > 0$ for $t > 0$. So we have

$$v'(t) \geq \alpha > 0, \qquad t \in [1, +\infty),$$

hence

$$v(t) \geq v(1) + \alpha(t-1), \qquad t \in [1, +\infty),$$

or

$$|g'(t)| \leq \frac{1}{[v(1) + \alpha(t-1)]^2}. \tag{3.12}$$

It follows from (3.6) that

$$|f'(re^{i\theta})| \leq \frac{1}{(1-r^2)^2} \cdot \frac{1}{[v(1) + \alpha(t-1)]^2}.$$

When r is close to 1 sufficiently,

$$|f'(re^{i\theta})| \leq \frac{1}{(1-r^2)^2} \cdot \frac{16}{\alpha^2(\log\frac{1+r}{1-r} + \frac{2r}{1-r^2} - 4)^2}, \tag{3.13}$$

the right hand side of the above inequality is bounded when $r \to 1 - 0$. Thus there exists a constant $M > 0$ such that

$$|f'(re^{i\theta})| \leq M.$$

This is an unexpected fact! It tells us that in the Nehari family N_b, the derivatives of all the functions other than the extremal function $H(z)$ (up to a composition of a Möbius transformation) are bounded. On the contrary, according to the result of Gehring and Pommerenke [?], the derivative of the extremal function $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$ in the Nehari family N_a is $f'(z) = \frac{1}{1-z^2}$. For the derivatives of all the functions other than the extremal function in the Nehari family N_a, it holds that

$$|f'(z)| < \frac{a}{1-|z|}\Big(\log\frac{8}{1-|z|}\Big)^{-2} + b,$$

which shows that when $r \to 1 - 0$, the derivatives of the non-extremal functions in the Nehari family N_a are infinity of lower order compared with the extremal function.

Now we can derive the distortion formula of the theorem. For any two points $z, z' \in D$, consider the hyperbolic segment Γ joining z and z' in D. Then Γ has length $l \leq \frac{\pi}{2}|z - z'|$. Hence

$$|f(z) - f(z')| \leq \int_\Gamma |f'(\zeta)||d\zeta| \leq M_1|z - z'|.$$

Furthermore, we have

$$\int_{t_0}^{+\infty}|g'(t)|dt \le \int_{t_0}^{+\infty}\frac{dt}{[v(t_0)+\alpha(t-t_0)]^2} = \frac{1}{\alpha v(t_0)} = \frac{1}{\alpha}|g'(t_0)|^{\frac{1}{2}}$$

for $1 \le t_0 < \infty$. Then by (3.5), (3.6) and the Köbe Distortion Theorem

$$\frac{1}{4}(1-|z|^2)|f'(z)| \le \operatorname{dist}[f(z), \partial f(D)], \tag{3.14}$$

we obtain that when $r \to 1-0$,

$$|f(re^{i\theta}) - f(e^{i\theta})| = \Big|\int_r^1 f'(re^{i\theta})dr\Big| \le \int_{t_0}^{+\infty}|g'(t)|dt$$
$$\le \frac{2}{\alpha}(1-r^2)^{\frac{1}{2}}\Big[\frac{1}{4}(1-r^2)|f'(re^{i\theta})|\Big]^{\frac{1}{2}} = o[\operatorname{dist}(f(re^{i\theta}), \partial f(D))]^{\frac{1}{2}},$$

which completes the proof of the theorem.

For the discussion of the characterizations of quasidisks, we need the following definitions.

Definition 3.1. *A Jordan curve Γ is called a quasi-circle with constant M if it satisfies*

$$\min[\operatorname{diam}\Gamma_1, \operatorname{diam}\Gamma_2] \le M|w_1 - w_2|, \qquad w_1, w_2 \in \Gamma, \tag{3.15}$$

where Γ_1, Γ_2 are two components of $\Gamma \setminus \{w_1, w_2\}$. A domain bounded by a quasi-circle with constant M is called a quasidisk with constant M.

Definition 3.2. *If any two points w_1, w_2 on the boundary of domain $G(\subset C)$ can be joined by an open arc $L \subset G$ so that*

$$\min_{j=1,2}|w - w_j| \le c \cdot \operatorname{dist}(w, \partial G), \qquad w \in L, \tag{3.16}$$

then we say that the domain G has c-accessible boundary.

It is obvious that $c \ge 1$.

Theorem 3.2. *Let $f(z)$ be a meromorphic function in the unit disk D, and satisfy*

$$(1-|z|^2)|S_f(z)| \le 4b, \qquad b < 1. \tag{3.17}$$

Then $f(D)$ is a quasidisk with constant $\frac{2M}{(1-b)^2}$, where M is an absolute constant.

The proof of Theorem 3.2 needs the following lemmas.

Lemma 3.1. (cf. [**?**]) *Suppose that G is a Jordan domain in the complex plane C. If there is a constant c, such that for any Möbius transformation φ satisfying $\varphi(G) \subset C$, domain $\varphi(G)$ has c-accessible boundary, then ∂G is a quasi-circle with constant $2c$.*

Lemma 3.2. (cf. [5]) *Suppose that $f(z), F(z)$ are locally univalent analytic functions in the unit disk, satisfying the normalized conditions $f(0) = F(0) = 0$, $f'(0) = F'(0) = 1$, $f''(0) = F''(0) = 0$, and $|S_f(z)| \le S_F(|z|)$. Then $|f'(z)| \le F'(|z|)$.*

Lemma 3.3. (cf. [5]) *Suppose that $f(z), G(z)$ are locally univalent analytic functions in the unit disk, satisfying the normalized conditions $f(0) = G(0) = 0$, $f'(0) = G'(0) = 1$, $f''(0) = G''(0) = 0$, and $|S_f(z)| \le -S_G(|z|)$. Then $|f'(z)| \ge G'(|z|)$.*

Proof of Theorem 3.2. First we consider the differential equation

$$u''(r)+\frac{2b}{1-r^2}u(r)=0,\quad u(0)=1,\quad u'(0)=0,\qquad 0<b<1.$$

Suppose

$$u(r)=1+a_2r^2+a_3r^3+a_4r^4+\cdots+a_nr^n+\cdots.$$

Then we have

$$u''(r)=2a_2+6a_3r+12a_4r^2+\cdots+n(n-1)a_nr^{n-2}+\cdots,$$
$$\frac{2b}{1-r^2}u(r)=2b[1+(1+a_2)r^2+a_3r^3+(1+a_2+a_4)r^4+\cdots].$$

Comparing the coefficients, we obtain

$$u(r)=1-br^2-\frac{1}{6}b(1-b)r^4-\frac{1}{15}b(1-b)\Big(1-\frac{1}{6}b\Big)r^6 -\frac{1}{28}b(1-b)\Big(1-\frac{1}{6}b\Big)\Big(1-\frac{1}{15}b\Big)r^8-\cdots.$$

Thus we have the following estimation

$$\begin{aligned}u(r)&>1-b-\frac{1}{6}b(1-b)-\frac{1}{15}b(1-b)\Big(1-\frac{1}{6}b\Big)-\frac{1}{28}b(1-b)\Big(1-\frac{1}{6}b\Big)\Big(1-\frac{1}{15}b\Big)-\cdots\\&=(1-b)\Big(1-\frac{1}{6}b\Big)\Big(1-\frac{1}{15}b\Big)\Big(1-\frac{1}{28}b\Big)\cdots=\prod_{n=1}^{\infty}\Big(1-\frac{b}{n(2n-1)}\Big),\end{aligned}$$

hence $u(r)>(1-b)S$, where $S=\prod\limits_{n=2}^{\infty}\big(1-\frac{1}{n(2n-1)}\big)$.

Then we consider the differential equation

$$v''(r)-\frac{2b}{1-r^2}v(r)=0,\quad v(0)=1,\quad v'(0)=0,\qquad 0<b<1.$$

With the same method, we can obtain

$$v(r)=1+br^2+\frac{1}{6}b(1+b)r^4+\frac{1}{15}b(1+b)\Big(1+\frac{1}{6}b\Big)r^6 +\frac{1}{28}b(1+b)\Big(1+\frac{1}{6}b\Big)\Big(1+\frac{1}{15}b\Big)r^8+\cdots.$$

Thus we have the following estimation

$$\begin{aligned}v(r)&<1+b+\frac{1}{6}b(1+b)+\frac{1}{15}b(1+b)\Big(1+\frac{1}{6}b\Big)+\frac{1}{28}b(1+b)\Big(1+\frac{1}{6}b\Big)\Big(1+\frac{1}{15}b\Big)+\cdots\\&=(1+b)\Big(1+\frac{1}{6}b\Big)\Big(1+\frac{1}{15}b\Big)\Big(1+\frac{1}{28}b\Big)\cdots=\prod_{n=1}^{\infty}\Big(1+\frac{b}{n(2n-1)}\Big),\end{aligned}$$

hence $v(r)<T$, where $T=2\prod\limits_{n=2}^{\infty}\big(1+\frac{1}{n(2n-1)}\big)$.

Finally we consider the solutions of the following differential equations

$$u_1''(z)+\frac{1}{2}S_fu_1(z)=0,\quad u_2''(z)+\frac{-2b}{1-z^2}u_2(z)=0,\quad u_3''(z)+\frac{2b}{1-z^2}u_3(z)=0,$$

and denote the normalized functions whose Schwarzian derivatives are $\frac{-4b}{1-z^2}$ and $\frac{4b}{1-z^2}$ by $G(z)$ and $F(z)$ respectively. By Lemma 3.2 and Lemma 3.3, we have

$$G'(|z|) \leq |f'(z)| \leq F'(|z|),$$

hence

$$\frac{1}{T^2} < \frac{1}{v^2(r)} \leq |f'(z)| \leq \frac{1}{u^2(r)} < \frac{1}{S^2(1-b)^2}. \tag{3.18}$$

Now we can prove that $f(D)$ is a quasidisk, and obtain an estimation of its constant.

Because of the invariance of the Schwarzian derivative under Möbius transformation, we need only to consider such two points $z_1 = f(-1),\ z_2 = f(1)$. For any $z_0 \in (-1, +1)$, it follows from (3.18) and the Köbe Distortion Theorem (3.14) that

$$\begin{aligned}
&\min\{|z_j - f(z_0)|\} = \min\Big\{\Big|\int_{z_0}^{\pm 1} f'(z)dz\Big|\Big\} \\
&= \min\Big\{\frac{1}{4}(1-z_0^2)|f'(z_0)| \cdot \Big|\int_{z_0}^{\pm 1} \frac{f'(t)}{f'(z_0)}dt\Big| \cdot \frac{4}{1-z_0^2}\Big\} \\
&\leq \frac{4T^2}{S^2(1-b)^2}\operatorname{dist}[f(z_0), \partial f(D)] = \frac{M}{(1-b)^2}\operatorname{dist}[f(z_0), \partial f(D)],
\end{aligned}$$

where $M = \frac{4T^2}{S^2}$.

So we have proved that $f(D)$ has $\frac{M}{(1-b)^2}$-accessible boundary. By Lemma 3.1, we know that $f(D)$ is a quasidisk with constant $\frac{2M}{(1-b)^2}$, which completes the proof of the theorem.

In 1963, Ahlfors [1] proved that a necessary and sufficient condition for a Jordan curve Γ on the closed complex plane $\widehat{C}$ to be a quasicircle is there is a constant C such that

$$|(z_1, z_2, z_3, z_4)| = \Big|\frac{z_1 - z_2}{z_1 - z_3} \cdot \frac{z_3 - z_4}{z_2 - z_4}\Big| \leq C \tag{3.19}$$

holds for all ordered quadruples z_1, z_2, z_3, z_4 on Γ. Now let K be the constant C in (3.19) when Γ is the unit circle. Then we have

Theorem 3.3. *Suppose that $f(z)$ is a meromorphic function in the unit disk D, which satisfies $(1-|z|^2)|S_f(z)| \leq 4b < 4$. Then $f(D)$ is a quasidisk, and*

$$|(w_1, w_2, w_3, w_4)| = \Big|\frac{w_1 - w_2}{w_1 - w_3} \cdot \frac{w_3 - w_4}{w_2 - w_4}\Big| \leq \frac{1}{16}\exp\Big[(\pi + \log K)\frac{1+b}{1-b}\Big] \tag{3.20}$$

holds for all ordered quadruples w_1, w_2, w_3, w_4 on $\partial f(D)$.

For the proof of Theorem 3.3, we need the following lemma.

Lemma 3.4. (cf. [**?**]) *Let $A \subset \widehat{C}$. Suppose that the function $g = g(z, \lambda) : A \times D \to \widehat{C}$ is injective in z for fixed λ with $g(z, 0) = z$, and is meromorphic in λ for fixed z. Then $g(z, \lambda)$ can be extended continuously to $\overline{A} \times D$ so that $g(z, \lambda)$ is meromorphic in λ for $z \in \overline{A}$, and*

$$|(w_1, w_2, w_3, w_4)| \leq \frac{1}{16}\exp\Big[(\pi + \log^+ |(z_1, z_2, z_3, z_4)|)\frac{1+|\lambda|}{1-|\lambda|}\Big] \tag{3.21}$$

holds for all ordered quadruples z_1, z_2, z_3, z_4 on $\overline{A}$, where $w_i = g(z_i, \lambda)$.

Proof of Theorem 3.3. Without loss of gererality, we suppose that the function $f(z)$ satisfies the normalized conditions $f(0) = 0, f'(0) = 1$ and $f''(0) = 0$. For $\lambda \in D$, consider the differential equation

$$S_g(z) = \frac{\lambda}{b} S_f(z), \quad g(0) = 0, \quad g'(0) = 1, \quad g''(0) = 0.$$

According to the initial conditions, the differential equation has a unique solution $g = g(z, \lambda)$, which is meromorphic in λ for fixed z, is analytic in z for fixed λ, and when $\lambda = b$, $g(z, b) = f(z)$. It follows from Lemma 3.4 with A being the unit disk D that $f(D)$ is a quasidisk, and

$$|(w_1, w_2, w_3, w_4)| \leq \frac{1}{16} \exp\left[(\pi + \log K)\frac{1+b}{1-b}\right]$$

holds for all ordered quadruples w_1, w_2, w_3, w_4 on $\partial f(D)$.

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