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THE FUNCTIONAL DIMENSION OF SOME CLASSES OF SPACES***

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Abstract

The functional dimension of countable Hilbert spaces has been discussed by some authors. They showed that every countable Hilbert space with finite functional dimension is nuclear. In this paper the authors do further research on the functional dimension, and obtain the following results: (1) They construct a countable Hilbert space, which is nuclear, but its functional dimension is infinite. (2) The functional dimension of a Banach space is finite if and only if this space is finite dimensional. (3) Let B be a Banach space, B^* be its dual, and denote the weak $*$ topology of B^* by $\sigma(B^*, B)$. Then the functional dimension of $(B^*, \sigma(B^*, B))$ is 1. By the third result, a class of topological linear spaces with finite functional dimension is presented.

Keywords Functional dimension, Countable Hilbert space, Topological linear space
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§ 1. The Functional Dimension for a Class of Countable Hilbert Spaces

In [1] the calculation formula of the functional dimension of perfect countable Hilbert spaces was given. Let Φ be a countable Hilbert space. According to the definition (see [1]), there are countable inner products $\{\langle\,,\rangle, m \in \mathbb{N}\}$ in Φ, such that $\|\phi\|_1 \le \|\phi\|_2 \le \cdots$, and $\Phi = \bigcap\limits_{m=1}^{+\infty} \Phi_m$, where Φ_m is a Hilbert space as the completion of $\{\Phi, \langle\,,\rangle_m\}$, and $\Phi_1 \supset \Phi_2 \supset \cdots$. Suppose that Φ is perfect. Then the imbedding operator I_k^m from Φ_m to Φ_k is a compact operator for $m > k \ge 1$. Let the sequence of eigenvalues of $\sqrt{I_k^{m*} I_k^m}$ be $\{a_n^{(km)}, n \in \mathbb{N}\}$ with $a_n^{(km)} \downarrow 0$. Then the functional dimension $df\Phi$ can be calculated as follows (see [1]):

$$df\Phi = 1 + \sup_k \inf_{m>k} \tau_{km}, \tag{1.1}$$

where $\tau_{km} = \inf\Big\{\mu\Big| \sum\limits_{a_n^{(k,m)}<1} \Big(\ln \frac{1}{a_n^{(km)}}\Big)^{-\mu} < +\infty\Big\}$ is just the exponent of convergence of the sequence $\{\ln(1/a_n^{(km)})|_{n=1,2,\cdots}\}$.

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Now we give a concreate class of perfect countable Hilbert spaces, and obtain the calculation formula of their functional dimensions.

Let H be a Hilbert space, A be a self-adjoint operator on H, and $\mathcal{D}_{A^k}$ be the domain of A^k $(k=0,1,2,\cdots)$. Obviously, $\Phi_A=\bigcap\limits_{k=0}^{+\infty}\mathcal{D}_{A^k}$ is a linear subspace of H. On Φ_A we can define countable inner products as follows:

$$\langle x,y\rangle_m=\sum_{k=0}^{m}(A^kx,A^ky),\qquad m\in\mathbb{N},\ x,y\in\Phi_A,$$

where $(\cdot,\cdot)$ is the inner product on H. Since H is complete and A^k $(k=0,1,2,\cdots)$ is closed, it is clear that the completion of $\{\Phi_A,\langle\,,\rangle_m\}$ is just $\mathcal{D}_{A^m}$. It is denoted by Φ_m. From $\Phi_A=\bigcap\limits_{m=1}^{+\infty}\Phi_m$ it follows that Φ_A is a countable Hilbert space. Furthermore, suppose that A has spectral property C. By the definition (see [3]), we have $\sigma(A)=P_\sigma(A)=\{\lambda_n\,|\,n\in\mathbb{N}\}$, $|\lambda_n|\uparrow+\infty$, and also the multiplicity of each eigenvalue in $P_\sigma(A)$ is finite and is exactly the number of times the eigenvalue is repeated in the sequence $\{\lambda_n\}$. In this case Φ_A is a perfect countable Hilbert space by [3, Theorem 2.2].

Now we start to set up the formula about calculating $df\,\Phi_A$. By the spectrum analysis there exists an orthonormal basis $\{\phi_n;n\in\mathbb{N}\}$ in H, such that $A\phi_n=\lambda_n\phi_n, n\in\mathbb{N}$. Noticing that $\{\phi_n\}\subset\Phi_A$, we have

$$\langle\phi_i,\phi_j\rangle_m=\sum_{p=0}^{m}\lambda_j^{2p}\delta_{ij},\qquad i,j\in\mathbb{N}.$$

Let $\psi_n^{(m)}=\phi_n/\|\phi_n\|_m=\phi_n\Big/\Big(\sum\limits_{p=0}^{m}\lambda_n^{2p}\Big)^{1/2}$, $n\in\mathbb{N}$. Then

$$\langle\psi_i^{(m)},\psi_j^{(m)}\rangle_m=\delta_{ij},\qquad i,j\in\mathbb{N}.$$

This means that $\{\psi_n^{(m)},n\in\mathbb{N}\}$ is an orthonormal basis in Φ_m. Suppose that $k<m$, $x\in\Phi_m$. We have $x=\sum\limits_{n=1}^{+\infty}\langle x,\psi_n^{(m)}\rangle_m\psi_n^{(m)}=\sum\limits_{n=1}^{+\infty}\langle x,\psi_n^{(m)}\rangle\|\phi_n\|_k/\|\phi_n\|_m\psi_n^{(k)}$. Therefore

$$I_k^mx=\sum_{n=1}^{+\infty}\|\phi_n\|_k/\|\phi_n\|_m\langle x,\psi_n^{(m)}\rangle_m\psi_n^{(k)}.$$

From [1, I.2] it follows that I_k^m is a self-adjoint operator, and $\sigma(I_k^m)=P_\sigma(I_k^m)=\{a_n^{(km)}=\|\phi_n\|_k/\|\phi_n\|_m,n\in\mathbb{N}\}$. Furthermore

$$a_n^{(km)}=\Big(\sum_{p=0}^{k}\lambda_n^{2p}\Big)^{1/2}\Big/\Big(\sum_{p=0}^{m}\lambda_n^{2p}\Big)^{1/2}=\Big(\frac{\lambda_n^{2(k+1)}-1}{\lambda_n^{2(m+1)}-1}\Big)^{1/2}\sim\frac{1}{|\lambda_n|^{m-k}}\downarrow 0,\qquad n\to+\infty.$$

By this we have

$$\tau_{km}=\inf\Big\{\mu\Big|\sum_{a_n^{(km)}<1}\Big(\ln\frac{1}{a_n^{(km)}}\Big)^{-\mu}<+\infty\Big\}=\inf\Big\{\mu\Big|\sum_n(1/\ln|\lambda_n|)^{\mu}<+\infty\Big\}=\tau_A,$$

where τ_A is independent of k, m. From the formula (1.1) it follows that

$$df\,\Phi_A = 1 + \tau_A. \tag{1.2}$$

In [1, I.3.8] there was an assertion: Every countable Hilbert space with finite functional dimension is nuclear. Now we want to show that its converse is not true. For example, let A_0 be a self-adjoint operator on H with $\sigma(A_0) = P_\sigma(A_0) = \{\lambda_n = 2n+1, n \in \mathbb{N}\}$, the multiplicity of each λ_n being 1. It is easy to see that

$$a_n^{(k,k+2)} \sim 1/|\lambda_n|^2 = 1/(2n+1)^2, \qquad n, k \in \mathbb{N}.$$

Since $\sum\limits_{n=1}^{+\infty} a_n^{(k,k+2)} < +\infty$, it follows that I_k^{k+2} is a nuclear operator for $k \in \mathbb{N}$. Then Φ_{A_0} is a nuclear space (see [1]). On the other hand,

$$\tau_{A_0} = \inf\Big\{\mu \,\Big|\, \sum_{n=1}^{+\infty} [1/\ln(2n+1)]^\mu < +\infty\Big\} = +\infty.$$

So $df\,\Phi_{A_0} = 1 + \tau_{A_0} = +\infty$.

Remark 1.1. From the above example we can see that for countable Hilbert spaces the property finite functional dimension is definitely a more restrictive topological character than the property nuclearity.

§ 2. The Functional Dimension of Banach Spaces

In [1] the functional dimension of topological linear spaces was defined as follows.

Let Φ be a topological linear space, U be a neighborhood of zero in Φ, and M be a set in Φ. A set G in Φ is called an ε-set for M relative to U, if $M \subset \bigcup\limits_{x \in G} (x + \varepsilon U)$. Denote

$$N(\varepsilon, M, U) = \min\Big\{ r_G \,\Big|\, \begin{array}{l} G \text{ runs through the collection of} \\ \text{all } \varepsilon\text{-sets for } M \text{ relative to } U \end{array} \Big\}, \tag{2.1}$$

where r_G denotes the number of elements in G.

Let $\mathcal{U}(0)$ be the neighborhood system of zero in Φ. This means that $\mathcal{U}(0)$ is the family of all neighborhoods of zero in Φ. Suppose that $U, V \in \mathcal{U}(0)$. Denote $\sigma_{UV} = \varlimsup\limits_{\varepsilon \to 0_+} \ln\ln N(\varepsilon, V, U)/\ln\ln \varepsilon^{-1}$, where $N(\varepsilon, V, U)$ is as in (2.1). Let

$$df\,\Phi = \sup_U \inf_V \{\sigma_{UV} \mid U, V \in \mathcal{U}(0)\}. \tag{2.2}$$

The number $df\,\Phi$ is called the functional dimension of the space Φ.

We have the following propositions on the function N. It is easy to prove them by a simple analysis.

Proposition 2.1.
If $M_1 \supset M_2$, *then* $N(\varepsilon, M_1, U) \geq N(\varepsilon, M_2, U)$;
If $U_1 \supset U_2$, *then* $N(\varepsilon, M, U_1) \leq N(\varepsilon, M, U_2)$;
If $\varepsilon_1 \geq \varepsilon_2$, *then* $N(\varepsilon_1, M, U) \leq N(\varepsilon_2, M, U)$.

Proposition 2.2. *Let $\alpha > 0$. We have*

$$N(\varepsilon,\ M,\ \alpha U) = N(\varepsilon\alpha, M, U),$$
$$N(\alpha\varepsilon, \alpha M, U) = N(\varepsilon,\ M,\ U).$$

Lemma 2.1. *Let Φ be a topological linear space, $\mathcal{B}(0)$ be a base for the neighborhood system of zero in Φ, and σ_{UV} be as above. We have*

$$df\,\Phi = \sup_U \inf_{V\subset U} \{\sigma_{UV} \mid U, V \in \mathcal{B}(0)\},$$

and also $df\,\Phi$ is independent of the choice of $\mathcal{B}(0)$.

Proof. By Proposition 2.1, σ_{UV} is monotonously decreasing when V is descending. So it is easy to see that

$$\inf_{V\in\mathcal{U}(0)} \sigma_{UV} = \inf_{V\in\mathcal{B}(0)} \sigma_{UV} = \inf_{\substack{V\subset U\\ V\in\mathcal{B}(0)}} \sigma_{UV}.$$

Here the second equality holds for each fixed $U \in \mathcal{B}(0)$.

On the other hand, by Proposition 2.1, σ_{UV} is monotonously increasing when U is descending, and so does $\inf\limits_{V\in\mathcal{U}(0)} \sigma_{UV}$. Therefore we have

$$\sup_{U\in\mathcal{U}(0)} \Big\{ \inf_{V\in\mathcal{U}(0)} \sigma_{UV} \Big\} = \sup_{U\in\mathcal{B}(0)} \Big\{ \inf_{V\in\mathcal{U}(0)} \sigma_{UV} \Big\} = \sup_{U\in\mathcal{B}(0)} \Big\{ \inf_{\substack{V\subset U\\ V\in\mathcal{B}(0)}} \sigma_{UV} \Big\}.$$

Then $df\,\Phi = \sup\limits_U \inf\limits_{V\subset U} \{\sigma_{UV} \mid U, V \in \mathcal{B}(0)\}$.

From the above argument we can see that $df\,\Phi$ is independent of the choice of $\mathcal{B}(0)$.

Lemma 2.2. *Let Φ be a topological linear space, which satisfies the first axiom of countability. Suppose that $\mathcal{B}(0) = \{U_k \mid k \in \mathbb{N}\}$, where $U_1 \supset U_2 \supset \cdots$. Then*

$$df\,\Phi = \sup_k \inf_{m>k} \sigma_{U_k U_m},$$

and also $df\,\Phi$ is independent of the choice of $\{U_k \mid k \in \mathbb{N}\}$.

Proof. By Lemma 2.1, it is obvious.

Lemma 2.3. *Let B be a Banach space, and S be the unit ball of B. Then*

$$df\,B = \overline{\lim_{\varepsilon\to 0_+}} \ln\ln N(\varepsilon, S, S)/\ln\ln\varepsilon^{-1},$$

and also $df\,B$ is independent of the choice of S.

Proof. Take $U_k = \frac{1}{k}S,\ k = 1, 2, \cdots$. By Proposition 2.2, we have $N(\varepsilon, U_k, U_m) = N(\frac{k}{m}\varepsilon, S, S)$. Furthermore

$$\begin{aligned}\sigma_{U_k U_m} &= \overline{\lim_{\varepsilon\to 0_+}} \ln\ln N\Big(\frac{k}{m}\varepsilon, S, S\Big)\Big/\ln\ln\varepsilon^{-1}\\ &= \overline{\lim_{\varepsilon\to 0_+}} \ln\ln N(\varepsilon, S, S)/\ln\ln\varepsilon^{-1} \qquad \text{for}\quad m, k, \in \mathbb{N}.\end{aligned}$$

From Lemma 2.2 it follows that $df\,B = \overline{\lim\limits_{\varepsilon\to 0_+}} \ln\ln N(\varepsilon, S, S)/\ln\ln\varepsilon^{-1}$.

Now we study the functional dimension of Banach spaces.

Theorem 2.1. *The functional dimension of finite dimensional Banach spaces is* 1*. And the functional dimension of infinite dimensional Banach spaces is* $+\infty$*.*

Proof. Suppose B is an n-dimensional real linear space. Take

$$S = \Big\{ \max_{1\le j\le n} |x_j| < 1 \,\Big|\, (x_1, x_2, \cdots, x_n) \in B,\ x_j\ (1 \le j \le n) \in \mathbb{R} \Big\},$$

where $\mathbb{R}$ denotes the real field. Then

$$N(\varepsilon, S, S) \sim K_n(1/\varepsilon)^n,$$

and K_n is independent of ε (see [2, Formula (48)]). By Lemma 2.3, we have $df B = 1$.

Suppose B is an n-dimensional complex linear space. We can define an inner product as its topology. And also B with this inner product is a $2n$-dimensional real linear space. Denote it by B_r, and the natural map from B to B_r by J. Let S_B be the unit ball of B. Then $JS_B = S_{B_r}$ is exactly the unit ball of B_r. Obviously G is an ε-set for S_B relative to S_B if and only if JG is an ε-set for S_{B_r} relative to S_{B_r}. So $N(\varepsilon, S_B, S_B) = N(\varepsilon, S_{B_r}, S_{B_r}) \sim K_{2n}(1/\varepsilon)^{2n}$, where K_{2n} is independent of ε. By Lemma 2.3, we have $df B = 1$.

Suppose B is an infinite dimensional Banach space. It is known that the unit ball of an infinite dimensional Banach space must not be countably compact. Moreover, a set in a complete metric space is countably compact if and only if it is totally bounded (see [4, II, 4]). Then $S = \{x \,|\, x \in B, \|x\| < 1\}$ must not be totally bounded. So there exists some $\varepsilon_0 > 0$, such that any ε_0-set for S relative to S must not be finite set. That is, $N(\varepsilon_0, S, S) = +\infty$. By Proposition 2.1, we have $N(\varepsilon, S, S) = +\infty$ for any $\varepsilon < \varepsilon_0$. Then $df B = +\infty$.

Remark 2.1. From Section 1 and Section 2 we can see that the property finite functional dimension implies a very deep topological character. In the next section we will give a class of topological linear spaces, whose functional dimensions are finite.

§ 3. The Functional Dimension of $(B^*, \sigma(B^*, B))$

In the title of this section, B^* denotes the dual space of a Banach space B and $\sigma(B^*, B)$ is the weak-star topology for B^*. It is known that a base for the neigbborhood system of zero in $(B^*, \sigma(B^*, B))$ is as follows:

$$\mathcal{B}_1(0) = \{U(0; y_1, y_2, \cdots, y_m; \varepsilon) \,|\, m \in \mathbb{N},\ y_1, y_2, \cdots y_m \in B,\ \varepsilon > 0\},$$

where $U(0; y_1, y_2, \cdots, y_m; \varepsilon) = \{f \,|\, f \in B^*, |f(y_i)| < \varepsilon, 1 \le i \le m\}$.

Lemma 3.1. *Another base for the neighborhood system of zero in* $(B^*, \sigma(B^*, B))$ *is as follows:*

$$\mathcal{B}_2(0) = \Big\{U(0; x_1, x_2, \cdots, x_n; 1) \,\Big|\, \begin{matrix} n \in \mathbb{N},\ x_1, x_2, \cdots x_n \ \textit{are linearly} \\ \textit{independent elements in } B \end{matrix} \Big\},$$

where $U(0; x_1, x_2, \cdots, x_n; 1) = \{f \,|\, f \in B^*, |f(x_j)| < 1, 1 \le j \le n\}$.

Proof. Suppose that W is a neighborhood of zero in $(B^*, \sigma(B^*, B))$. Then we can find a neighborhood $V = U(0; y_1, y_2, \cdots, y_m; \varepsilon) \in \mathcal{B}_1(0)$, such that $W \supset V$. If $y_1, y_2, \cdots, y_m$ are linearly independent, letting $x_j = y_j/\varepsilon,\ 1 \le j \le m$, we have $V = U(0; x_1, x_2, \cdots, x_m; 1) \in \mathcal{B}_2(0)$. Otherwise, there exists $n < m$, such that $y_1, y_2, \cdots, y_n$ are linearly independent and $y_k = \sum\limits_{j=1}^{n} \alpha_{kj} y_j,\ n+1 \le k \le m$. Let $\eta_k = \sum\limits_{j=1}^{n} |\alpha_{kj}|,\ n+1 \le k \le m$, and $\delta = \min(\varepsilon, \varepsilon/\eta_{n+1}, \cdots \varepsilon/\eta_m)$. It is clear that $V \supset V_0 = \{f \,|\, f \in B^*, |f(y_j)| < \delta, 1 \le j \le n\}$. By the above analysis we have $V_0 \in \mathcal{B}_2(0)$. This comes to the conclusion.

Lemma 3.2. *Let $U, V \in \mathcal{B}_2(0)$. Suppose that*

$$V = \{f \mid f \in B^*,\ |f(x_j)| < 1,\ 1 \le j \le n\},$$

where $x_1, x_2, \cdots x_n$ are linearly independent elements in B;

$$U = \{f \mid f \in B^*,\ |f(y_j)| < 1,\ 1 \le i \le m\},$$

where $y_1, y_2, \cdots y_m$ are linearly independent elements in B. If $V \subset U$, then $y_1, y_2, \cdots y_m$ belong to the linear span of $x_1, x_2, \cdots, x_n$.

Proof. Denote the linear span of $x_1, x_2, \cdots, x_n$ by E. Without loss of generality, we assume that $y_1 \notin E$. Denote the $(n+1)$-dimensional linear span of $x_1, \cdots, x_n, y_1$ by E_1. Define a linear functional f_0 on E_1 as follows:

$$\begin{aligned} f_0(x_j) &= 0, \qquad 1 \le j \le n, \\ f_0(y_1) &= 1. \end{aligned}$$

Obviously f_0 is a bounded linear functional on E_1. By the Hahn-Banach Theorem we can find a bounded linear functional f on B, such that $f|E_1 = f_0,\ \|f\| = \|f_0\|$. The facts $f \in V$ and $f \notin U$ contradict the given condition $V \subset U$. Hence it must be that $y_1, y_2, \cdots y_m \in E$.

Theorem 3.1. *The functional dimension of $(B^*, \sigma(B^*, B))$ is 1.*

Proof. Take U, V as in Lemma 3.2. Suppose that $V \subset U$. By Lemma 3.2, we can write

$$y_i = \sum_{j=1}^{n} \beta_{ij} x_j, \qquad 1 \le i \le m,$$

where β_{ij} is a constant for each $(i, j),\ 1 \le i \le m, 1 \le j \le n$.

Denote the linear span of $x_1, x_2, \cdots x_n$ by E. Let $E^{\perp} = \{f \mid f \in B^*, f|_E = 0\}$, and $\tau : B^* \to B^*/E^{\perp}$ be the quotient map. That is, $\tau f = f + E^{\perp} = \hat{f}$ for every $f \in B^*$, where $\hat{f}$ denotes the equivalence class containing f. It is known that $B^*/E^{\perp} = E^* \cong \mathbb{F}^n$, where either $\mathbb{F} = \mathbb{R}$ if Banach space B is over the real field, or $\mathbb{F} = \mathbb{C}$ if B is over the complex field. Thus we can regard τf as an element $(f(x_1), f(x_2), \cdots f(x_n))$ in $\mathbb{F}^n$. Moreover, it is easy to see that $\ker \tau = \{f \mid f \in B^*, \tau f = 0\} = E^{\perp}$, and $\operatorname{ran} \tau = \{f \mid f \in B^*\} = \mathbb{F}^n$.

Write $\widehat{V} = \tau V,\ \widehat{U} = \tau U$. We have

$$\begin{aligned} \widehat{V} &= \{\tau f \mid f \in V\} = \{\zeta \mid \zeta \in \mathbb{F}^n,\ |\zeta_j| < 1,\ 1 \le j \le n\}, \\ \widehat{U} &= \{\tau f \mid f \in U\} = \Big\{\zeta \,\Big|\, \zeta \in \mathbb{F}^n,\ \Big|\sum_{j=1}^{n} \beta_{ij} \zeta_j\Big| < 1,\ 1 \le i \le m\Big\}, \end{aligned}$$

and also $\widehat{V} \subset \widehat{U}$.

Note the following claim: For any $\zeta \in \widehat{V}$ (or $\zeta \in \widehat{U}$), it must be that $\overset{-1}{\tau}\ \zeta \subset V$ (or $\overset{-1}{\tau}\ \zeta \subset U$). This is because of $E^{\perp} \subset V$ (or $E^{\perp} \subset U$).

Now we start to show $N_{B^*}(\varepsilon, V, U) = N_{\mathbb{F}^n}(\varepsilon, \widehat{V}, \widehat{U})$, where functions N are defined as in Section 2.

First we point out a fact: If G is an ε-set for V relative to U, then $\widehat{G}$ is an ε-set for $\widehat{V}$ relative to $\widehat{U}$.

Let $\zeta \in \widehat{V}$. If $f \in \overset{-1}{\tau}\,\zeta \subset V$, then there exists $g \in G$, such that $f \in g + \varepsilon U$. Hence $\hat{f} - \hat{g} \in \varepsilon\widehat{U}$ and $\zeta \in \hat{g} + \varepsilon\widehat{U}$. So $\widehat{V} \subset \bigcup_{\xi \in \widehat{G}} (\xi + \varepsilon\widehat{U})$, and the fact is true.

From the definition of functions r in Section 2, it is obvious that $r_G \geq r_{\widehat{G}}$. By this it follows that

$$\begin{aligned} N_{B^*}(\varepsilon, V, U) &= \min\{r_G \,|\, G \text{ is any one } \varepsilon\text{-set for } V \text{ relative to } \ U\} \\ &\geq \min\{r_{\widehat{G}} \,|\, G \text{ is as above}\} \\ &\geq \min\{r_Q \,|\, Q \text{ is any one } \varepsilon\text{-set for } \widehat{V} \text{ relative to } \ \widehat{U}\} \\ &= N_{\mathbb{F}^n}(\varepsilon, \widehat{V}, \widehat{U}). \end{aligned} \tag{3.1}$$

Next we point out the second fact: If Q is an ε-set for $\widehat{V}$ relative to $\widehat{U}$, then we can find a set G_Q which is an ε-set for V relative to U, such that $\widehat{G}_Q = Q$ with $r_{G_Q} = r_Q$.

Let $G_Q = \{g_\zeta \,|\, \text{arbitrarily take an element } g_\zeta \in \overset{-1}{\tau}\,\zeta \text{ for every } \zeta \in Q\}$. It is obvious that $\widehat{G}_Q = Q,\ r_{G_Q} = r_Q$. If $f \in V$, then there exists $\zeta \in Q$, such that $\hat{f} \in \zeta + \varepsilon\widehat{U}$. Hence $\tau[(f - g_\zeta)/\varepsilon] \in \widehat{U}$. By the previous claim, it follows that $(f - g_\zeta)/\varepsilon \in U$. That is, $f \in g_\zeta + \varepsilon U$ and G_Q is an ε-set for V relative to U.

From the second fact we have

$$\begin{aligned} N_{\mathbb{F}^n}(\varepsilon, \widehat{V}, \widehat{U}) &= \min\{r_Q \,|\, Q \text{ is any one } \varepsilon\text{-set for } \widehat{V} \text{ relative to } \ \widehat{U}\} \\ &= \min\{r_{G_Q} \,|\, Q \text{ is as above}\} \\ &\geq \min\{r_G \,|\, G \text{ is any one } \varepsilon\text{-set for } V \text{ relative to } \ U\} \\ &= N_{B^*}(\varepsilon, V, U). \end{aligned} \tag{3.2}$$

By (3.1) and (3.2) it is true that $N_{B^*}(\varepsilon, V, U) = N_{\mathbb{F}^n}(\varepsilon, \widehat{V}, \widehat{U})$. Here we recall that the number n is just defined by V.

After this we calculate $N_{\mathbb{F}^n}(\varepsilon, \widehat{V}, \widehat{U})$. Let

$$S = S_n = \Big\{\zeta \,\Big|\, \zeta \in \mathbb{F}^n,\ |\zeta| = \Big(\sum_{j=1}^{n} |\,\zeta_j|^2\Big)^{1/2} < 1\Big\},$$

the unit ball in $\mathbb{F}^n$. Since $S \subset \widehat{V} \subset \sqrt{n}S$ and $S \subset \widehat{V} \subset \widehat{U}$, from Propositions 2.1 and 2.2 it follows that

$$N_{\mathbb{F}^n}(\varepsilon, \widehat{V}, \widehat{U}) \leq N_{\mathbb{F}^n}(\varepsilon, \sqrt{n}S, \widehat{U}) \leq N_{\mathbb{F}^n}(\varepsilon, \sqrt{n}S, S) = N_{\mathbb{F}^n}(n^{-1/2}\varepsilon, S, S).$$

By the argument in Theorem 2.1,

$$N_{\mathbb{F}^n}(n^{-1/2}\varepsilon, S, S) \sim \begin{cases} K_n(n^{1/2}\varepsilon^{-1})^n & \text{for } \ \mathbb{F} = \mathbb{R}, \\ K_{2n}(n^{1/2}\varepsilon^{-1})^{2n} & \text{for } \ \mathbb{F} = \mathbb{C}, \end{cases}$$

where either K_n or K_{2n} is independent of ε.

On the other hand, let

$$\widehat{U}_1 = \Big\{\zeta \,\Big|\, \zeta \in \mathbb{F}^n,\ \Big|\sum_{j=1}^{n} \beta_{1j}\zeta_j\Big| < 1\Big\} = \{\zeta \,|\, \zeta \in \mathbb{F}^n,\ |\langle\zeta, \beta\rangle| < \omega^{-1}\},$$

where $\omega = \Big(\sum\limits_{j=1}^{n} |\beta_{1j}|^2\Big)^{1/2}$, and $\beta = (\bar\beta_{11}/\omega, \bar\beta_{12}/\omega, \cdots, \bar\beta_{1n}/\omega)$ is a unit vector in $\mathbb{F}^n$. Since $\widehat{U} \subset \widehat{U}_1$, from Proposition 2.1 it follows that

$$N_{\mathbb{F}^n}(\varepsilon, \widehat{V}, \widehat{U}) \geq N_{\mathbb{F}^n}(\varepsilon, \widehat{V}, \widehat{U}_1) \geq N_{\mathbb{F}^n}(\varepsilon, S, \widehat{U}_1).$$

From the definition $\varepsilon\widehat{U}_1$ is a region between the two parallel planes: $\langle\zeta, \beta\rangle = \varepsilon/\omega$, and $\langle\zeta, \beta\rangle = -\varepsilon/\omega$ with a normal vector β. In addition, the distance between the two planes is $2\varepsilon/\omega$. Translating $\varepsilon\widehat{U}_1$, we can take a cover for S, such that the number of regions in the cover is the smallest. Obviously, the smallest number is exactly $1 + [\omega/\varepsilon]$, where $[\omega/\varepsilon]$ is the greatest integer in ω/ε. And also, the smallest number is just equal to $N_{\mathbb{F}^n}(\varepsilon, S, \widehat{U}_1)$. Then we have $N_{\mathbb{F}^n}(\varepsilon, S, \widehat{U}_1) > [\omega/\varepsilon]$. Taking $\varepsilon < \omega/2$, we have

$$N_{\mathbb{F}^n}(\varepsilon, S, \widehat{U}_1) > \omega/\varepsilon - 1 = (\omega - \varepsilon)\varepsilon^{-1} > (\omega/2)\varepsilon^{-1}.$$

By the above analysis we get the following estimation

$$c_0\varepsilon^{-1} \leq N_{B^*}(\varepsilon, V, U) \leq c(n^{1/2}\varepsilon^{-1})^{2n}, \tag{3.3}$$

where $n = n(V), c = c(n), c_0 = c_0(V, U)$, and they are all independent of ε. By a simple calculation, $\sigma_{UV} = 1$ and furthermore $df B^* = 1$.

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