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# WELL-POSEDNESS FOR THE CAUCHY PROBLEM TO THE HIROTA EQUATION IN SOBOLEV SPACES OF NEGATIVE INDICES

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**Abstract**

The local well-posedness of the Cauchy problem for the Hirota equation is established for low regularity data in Sobolev spaces $H^s(s \geq -\frac{1}{4})$. Moreover, the global well-posedness for $L^2$ data follows from the local well-posedness and the conserved quantity. For data in $H^s(s > 0)$, the global well-posedness is also proved. The main idea is to use the generalized trilinear estimates, associated with the Fourier restriction norm method.

**Keywords** Fourier restriction norm, Trilinear estimates, Hirota equation, Low regularity, Global well-posedness
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## § 1 . Introduction

We study the Cauchy problem for the Hirota equation

$$\partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + \mu\partial_x(|u|^2 u) + i\gamma|u|^2 u = 0, \qquad x, t \in \mathbb{R}, \tag{1.1}$$

where $\alpha,\ \beta\ (\mu,\ \gamma)$ are real (complex) constants and $\alpha\beta \neq 0$, $u$ is complex valued function.

(**??**) is a typical model in mathematical physics, which encompasses the well-known nonlinear Schrödinger equation and the modified KdV equation, and especially contains the nonlinear derivative Schrödinger equation. Hasegawa and Kodama [3, 7] proposed (**??**) as a model for propagation of pulse in optical fiber.

The Cauchy problem of (**??**) changes as follows if $\mu = 0$,

$$\partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + i\gamma|u|^2 u = 0, \qquad x, t \in \mathbb{R}, \tag{1.2}$$

$$u(x, 0) = u_0(x). \tag{1.3}$$

Recently, Carvajal [2] has proved that the Cauchy problem (**??**)–(**??**) is locally well-posed in $H^s(s > -\frac{1}{4})$, and the mapping data-solution $u_0 \to u(t)$ for the Cauchy problem (**??**)–(**??**) is not $\mathcal{C}^3$ at origin in the case $s < -\frac{1}{4}$. Moreover, the local solution for initial data in $L^2$ is global by using the $L^2$ conservation law of (**??**). However, he did not answer the question whether the Cauchy problem (**??**)–(**??**) is locally well-posed or not for initial data in space

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$H^{-\frac{1}{4}}$, whether the Cauchy problem (**??**)–(**??**) is globally well-posed or not for initial data in $H^s(s > 0)$.

In this paper, we first prove that the Cauchy Problem (**??**)–(**??**) is locally well-posed for data in $H^s(s \geq -\frac{1}{4})$ by the Fourier restriction norm method and the contraction mapping principle. Compared to the results in [2], our results of the first part on local well-posedness of the Cauchy problem (**??**)–(**??**) in $H^s(s \geq -\frac{1}{4})$ seem difficult to be improved.

Then we have the globally well-posed results for data in $L^2$ by using the fact that the equation (**??**) preserves $L^2$ norm. For data in $H^s(0 < s \leq 1)$, we first establish the generalized trilinear estimate, which associated with the Fourier restriction norm method, to prove that the existence time of the solution in $H^s(0 < s \leq 1)$ only depends on the norm of data in $L^2$. Therefore, we can prove the global well-posedness of the Cauchy problem (**??**)–(**??**) in $H^s(0 < s \leq 1)$. Further, as to the initial value in $H^s(1 < s \leq 2)$, we can establish the analogous generalized trilinear estimate as above, then we can also obtain the global well-posedness for data in $H^s(1 < s \leq 2)$. By using induction, we are able to establish the global well-posedness of the Cauchy Problem (**??**)–(**??**) in $H^s(s > 0)$.

The Fourier restriction norm method was first introduced by J. Bourgain [1] to study the KdV and Nonlinear Schrödinger equations in the periodic case. It was simplified by Kenig, Ponce and Vega in dealing with KdV equation in [4, 5].

In order to study the Cauchy problem (**??**)–(**??**), we use its equivalent formulation

$$u(x,t) = S(t)u_0 - i\int_0^t S_1(t-t')\gamma|u|^2|u|(t')dt',$$

where $S(t) = \mathcal{F}_x^{-1}e^{it(\alpha\xi^2+\beta\xi^3)}\mathcal{F}_x$ is the unitary operator associated to the corresponding linear equation. Here the phase function is denoted by $\phi(\xi) = \alpha\xi^2 + \beta\xi^3$.

It is important to point out that the phase function $\phi(\xi)$ has non-zero singular points, which makes difference from the phase functions of the semigroup of the linear KdV equation and also makes the problem much more difficult. Therefore, we need use Fourier restriction operators

$$P^N f = \int_{|\xi|\geq N} e^{ix\xi}\hat{f}(\xi)d\xi, \quad P_N f = \int_{|\xi|\leq N} e^{ix\xi}\hat{f}(\xi)d\xi, \qquad \forall\, N > 0 \tag{1.4}$$

to eliminate the singularity of the phase function.

Moreover, the operators will be used to decompose the nonlinear term $|u|^2u$ in (**??**). To deal with the term, we first decompose it as the high frequency part and the corresponding low one as follows

$$|u|^2u = P^N\{|u|^2u\} + P_N\{|u|^2u\}. \tag{1.5}$$

Next, we continue to decompose each term in the right side of (**??**) as the summation of those products which consist of each factor acted by the Fourier restriction operators $P^N$ or $P_N$. We will estimate each resulting term with different methods to overcome the obstacles.

**Definition 1.1.** *For $s, b \in \mathbb{R}$, the space $X_{s,b}$ is defined to be the completion of the Schwartz function space on $\mathbb{R}^2$ with respect to the norm*

$$\|u\|_{X_{s,b}} = \|\langle\xi\rangle^s\langle\tau - \beta\xi^3 - \alpha\xi^2\rangle^b\mathcal{F}u\|_{L^2_\xi L^2_\tau}, \quad or \quad \|\bar{u}\|_{\overline{X}_{s,b}} = \|\langle\xi\rangle^s\langle\tau - \beta\xi^3 + \alpha\xi^2\rangle^b\mathcal{F}\bar{u}\|_{L^2_\xi L^2_\tau},$$

*where $\langle\cdot\rangle = (1 + |\cdot|)$. One can easily prove that*

$$\|u\|_{X_{s,b}} = \|\bar{u}\|_{\overline{X}_{s,b}},$$

*which will be used later without pointing out it.*

We shall use the trivial embedding $\|u\|_{X_{s_1,b_1}} \le \|u\|_{X_{s_2,b_2}}$, whenever $s_1 \le s_2$, $b_1 \le b_2$. Denote by $\hat{u} = \mathcal{F}u$ (or $\mathcal{F}_{(\cdot)}u$) the Fourier transform in $t$ and $x$ ( or $(\cdot)$, respectively) of $u$.

Let us introduce some variables for convenience

$$\sigma = \tau - \beta\xi^3 - \alpha\xi^2, \quad \sigma_j = \tau_j - \beta\xi_j^3 - \alpha\xi_j^2 \quad (j = 1, 2), \quad \sigma_3 = \tau_3 - \beta\xi_3^3 + \alpha\xi_3^2. \tag{1.6}$$

Throughout this paper, we shall denote the following notation $\int_\star \cdot d\delta$ as the convolution integral

$$\int_{\xi=\xi_1+\xi_2+\xi_3;\tau=\tau_1+\tau_2+\tau_3} \cdot d\tau_1 d\tau_2 d\tau_3 d\xi_1 d\xi_2 d\xi_3.$$

Let $\psi \in C_0^\infty(\mathbb{R})$ with $\psi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and supp$\psi \subset [-1, 1]$. We denote $\psi_\delta(\cdot) = \psi(\delta^{-1}(\cdot))$ for some non-zero $\delta \in \mathbb{R}$.

We use $A \sim B$ to denote the statement that $A \le C_1 B$ and $B \le C_1 A$ for some constant $C_1 > 0$, and use $A \ll B$ to denote the statement $A \le \frac{1}{C_2} B$ for some large enough constant $C_2 > 0$.

We give our results as follows.

**Theorem 1.1.** *Let $s \ge -\frac{1}{4}$, $\frac{1}{2} < b < \frac{7}{12}$. Then there exists a constant $T > 0$, such that* (**??**)–(**??**) *admits a unique local solution $u(x,t) \in C(0,T;H^s) \bigcap X_{s,b}$ with $u_0 \in H^s$. Moreover, given $t \in (0,T)$, the map $u_0 \to u(t)$ is Lipschitz continuous from $H^s$ to $C(0,T;H^s)$.*

The $L^2$ conservation law can be established easily for smooth solution of the equation (**??**),

$$\|u(x,t)\|_{L^2} = \|u_0(x)\|_{L^2}, \qquad \forall\, t \in \mathbb{R}. \tag{1.7}$$

Then we have the global well-posedness for data in $L^2$, that is

**Theorem 1.2.** *If $s = 0$, the solution obtained in Theorem* 1.1 *can be extended for any $T > 0$.*

Moreover, for data in $H^s(s > 0)$, the solution of (**??**)–(**??**) is globally well-posed by the generalized trilinear estimates (which are proved in Section 3).

**Theorem 1.3.** *The solution of* (**??**)–(**??**) *is globally well-posed in $H^s(s > 0)$.*

## § 2. Preliminary Estimates and Local Results

We can get the following trilinear estimates. It will be proved that the contraction argument provides the local well-posedness, once the following estimate holds for some $b \in \mathbb{R}$, namely, for some $b > \frac{1}{2}$,

$$\| u_1 u_2 \bar{u}_3 \|_{X_{s,b-1}} \le C \| u_1 \|_{X_{s,b}} \| u_2 \|_{X_{s,b}} \| u_3 \|_{X_{s,b}} . \tag{2.1}$$

In fact, we can prove the following more general theorem.

**Theorem 2.1.** *If $s \ge -\frac{1}{4}$, $\frac{1}{2} < b < \frac{7}{12}$, $b' > \frac{1}{2}$. Then*

$$\| u_1 u_2 \bar{u}_3 \|_{X_{s,b-1}} \le C \| u_1 \|_{X_{s,b'}} \| u_2 \|_{X_{s,b'}} \| u_3 \|_{X_{s,b'}} . \tag{2.2}$$

Next, we deduce some lemmas which will be used in the proof of Theorem 2.1.

First, we introduce the notations

$$a = \max\Big(1, \Big|\frac{2\alpha}{3\beta}\Big|\Big), \qquad \mathcal{F}F_\rho(\xi,\tau) = \frac{f(\xi,\tau)}{(1 + |\tau - \phi(\xi)|)^\rho},$$

and list the following notations, which will be used later,

$$D_x^s = \mathcal{F}^{-1}|\xi|^s\mathcal{F}, \quad \|f\|_{L_x^p L_t^q} = \Big(\int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} |f(x,t)|^q dt\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}, \quad \|f\|_{L_t^\infty H_x^s} = \|\|f\|_{H_x^s}\|_{L_t^\infty}.$$

**Lemma 2.1.** *The group $\{S(t)\}_{-\infty}^{+\infty}$ satisfies*

$$\|S(t)\varphi\|_{L_x^8 L_t^8} \le C\|\varphi\|_{L^2}. \tag{2.3}$$

We refer to [8] for the proof.

**Lemma 2.2.** *The group $\{S(t)\}_{-\infty}^{+\infty}$ satisfies*

$$\|D_x S(t)P^{2a}\varphi\|_{L_x^\infty L_t^2} \le C\|\varphi\|_{L^2}, \tag{2.4}$$

$$\|D_x^{-\frac{1}{4}} S(t)P^{a}\varphi\|_{L_x^4 L_t^\infty} \le C\|\varphi\|_{L^2}, \tag{2.5}$$

$$\|D_x^{\frac{1}{6}} S(t)P^{2a}\varphi\|_{L_x^6 L_t^6} \le C\|\varphi\|_{L^2}, \tag{2.6}$$

$$\|S(t)P^{2a}\varphi\|_{L_x^5 L_t^{10}} \le C\|\varphi\|_{L^2}. \tag{2.7}$$

**Proof.** We prove (**??**) first. It is clear that $\phi'(\xi) = 2\alpha\xi + 3\beta\xi^2$ has non-zero singularity, then $\phi$ is invertible if $|\xi| \ge N$ (here we use $N = 2a$). Therefore, we have

$$\begin{aligned} P^N S(t)\varphi &= \int_{|\xi|\ge N} e^{ix\xi} e^{it\phi(\xi)} \widehat{\varphi}(\xi) d\xi \\ &= \int_{|\phi^{-1}|\ge N} e^{ix\phi^{-1}} e^{it\phi} \widehat{\varphi}(\phi^{-1}) \frac{1}{\phi'} d\phi = \mathcal{F}_t^{-1}\Big(e^{ix\phi^{-1}} \chi_{\{|\phi^{-1}|\ge N\}} \widehat{\varphi}(\phi^{-1}) \frac{1}{\phi'}\Big). \end{aligned}$$

In the following steps, we will use the changed variable $\xi = \phi^{-1}$. It can be proved that

$$\begin{aligned} \|P^N S(t)\varphi\|_{L_t^2}^2 &= \Big\|\chi_{\{|\phi^{-1}|\ge N\}} \widehat{\varphi}(\phi^{-1}) \frac{1}{\phi'}\Big\|_{L_\phi^2}^2 \\ &= \int_{|\phi^{-1}|\ge N} |\widehat{\varphi}(\phi^{-1})|^2 \frac{1}{|\phi'|^2} d\phi \le C \int_{|\xi|\ge N} \frac{|\widehat{\varphi}(\xi)|^2}{|\xi|^2} d\xi \le C\|\varphi\|_{\dot{H}^{-1}}^2. \end{aligned}$$

In fact, this implies the estimate (**??**).

Let us turn to the proof of (**??**) next. The first inequality as below holds with the help of Theorem 2.5 in [6]. We show that

$$\begin{aligned} \|S(t)P^a\varphi\|_{L_x^4 L_t^\infty}^2 &\le \int |\mathcal{F}P^a\varphi(\xi)|^2 \Big|\frac{\phi'(\xi)}{\phi''(\xi)}\Big|^{\frac{1}{2}} d\xi \\ &\le \int |\mathcal{F}P^a\varphi(\xi)|^2 \Big(\frac{|3\beta\xi^2|(1 + a\frac{1}{a})}{|6\beta\xi|(1 - \frac{1}{2}a\frac{1}{a})}\Big)^{\frac{1}{2}} d\xi \le \|P^a\varphi\|_{H^{\frac{1}{4}}}^2. \end{aligned}$$

Therefore, we obtain the estimate (**??**).

Finally, (**??**) and (**??**) follow by interpolation between (**??**) and (**??**).

**Lemma 2.3.** *If $\rho > \frac{1}{2}$, for any fixed $N$ with $0 < N < +\infty$, it holds that*

$$\|P_N F_\rho\|_{L_x^2 L_t^\infty} \le C\|f\|_{L_\xi^2 L_\tau^2}. \tag{2.8}$$

The proof is similar to that of Lemma 2.3 in [4], so we omit the details here.

**Lemma 2.4.** *If $\rho > \dfrac{1}{2}\dfrac{4(q-2)}{3q}$, then for $2 \le q \le 8$,*

$$\|F_\rho\|_{L_x^q L_t^q} \le C\|f\|_{L_\xi^2 L_\tau^2}. \tag{2.9}$$

**Proof.** Changing variable $\tau = \lambda + \phi(\xi)$, we have

$$\begin{aligned} F_\rho(x,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(x\xi+t\tau)}\frac{f(\xi,\tau)}{(1+|\tau-\phi(\xi)|)^\rho}d\xi d\tau \\ &= \int_{-\infty}^{\infty} e^{it\lambda}\Big(\int_{-\infty}^{\infty} e^{i(x\xi+t\phi(\xi))}f(\xi,\lambda+\phi(\xi))d\xi\Big)\frac{d\lambda}{(1+|\lambda|)^\rho}. \end{aligned}$$

Therefore, using (**??**), Minkowski's integral inequality and taking $\rho > \frac{1}{2}$, one is able to show that

$$\|F_\rho\|_{L^8_xL^8_t} \le C\int_{-\infty}^{+\infty}\|f(\xi,\lambda+\phi(\xi))\|_{L^2_\xi}\frac{d\lambda}{(1+|\lambda|)^\rho} \le C\|f\|_{L^2_\xi L^2_\tau}. \tag{2.10}$$

By interpolation between the inequality (**??**) and

$$\|F_0\|_{L^2_xL^2_t} \le C\|f\|_{L^2_\xi L^2_\tau}, \tag{2.11}$$

we have, for $\rho > \dfrac{1}{2}\dfrac{4(q-2)}{3q}$, that $\|F_\rho\|_{L^q_xL^q_t} \le C\|f\|_{L^2_\xi L^2_\tau}$.

**Lemma 2.5.** *Let $\rho > \dfrac{\theta}{2}$ with $\theta \in [0,1]$. Then*

$$\|D_x^\theta P^{2a}F_\rho\|_{L_x^{\frac{2}{1-\theta}}L^2_t} \le C\|f\|_{L^2_\xi L^2_\tau}. \tag{2.12}$$

**Proof.** The argument in the proof of (**??**) and the inequality (**??**) shows that for $\rho > \frac{1}{2}$,

$$\|D_xP^{2a}F_\rho\|_{L^\infty_xL^2_t} \le C\|f\|_{L^2_\xi L^2_\tau},$$

which interpolated with (**??**) yields (**??**).

**Lemma 2.6.** *If $\rho > \dfrac{1}{2}$, then*

$$\|D_x^{-\frac{1}{4}}P^{2a}F_\rho\|_{L^4_xL^\infty_t} \le C\|f\|_{L^2_\xi L^2_\tau}. \tag{2.13}$$

**Proof.** From the argument in (**??**) and (**??**), it follows that (**??**) holds for $\rho > \frac{1}{2}$.

**Lemma 2.7.** *If $\rho > \dfrac{5}{12}$, then*

$$\|P^{2a}F_\rho\|_{L^4_xL^6_t} \le C\|f\|_{L^2_\xi L^2_\tau}. \tag{2.14}$$

**Proof.** Similarly to Lemma 2.6, we get the following inequality by (**??**) and the argument in (**??**) for $\rho > \frac{1}{2}$,

$$\|P^{2a}F_\rho\|_{L^5_xL^{10}_t} \le C\|f\|_{L^2_\xi L^2_\tau},$$

which interpolated with (**??**) yields (**??**).

**Lemma 2.8.** *If $\rho > \dfrac{1}{3}$, then*

$$\|D_x^{\frac{1}{4}}P^{2a}F_\rho\|_{L^4_xL^3_t} \le C\|f\|_{L^2_\xi L^2_\tau}. \tag{2.15}$$

**Proof.** For $\rho > \frac{1}{2}$, similarly with above, from (**??**) and the argument in (**??**), we can obtain

$$\|D_x^{\frac{1}{6}} P^{2a} F_\rho\|_{L_x^6 L_t^6} \le C\|f\|_{L_\xi^2 L_\tau^2}, \tag{2.16}$$

which interpolated with (**??**) shows that for $\rho > \frac{3}{8}$,

$$\|D_x^{\frac{1}{8}} P^{2a} F_\rho\|_{L_x^4 L_t^4} \le C\|f\|_{L_\xi^2 L_\tau^2}. \tag{2.17}$$

Then the inequality (**??**) follows by interpolation between (**??**) with $\theta = \frac{1}{2}$ and (**??**).

**Lemma 2.9.** *Assume that $f, f_1, f_2$ and $f_3$ belong to Schwartz space on $\mathbb{R}^2$. Then*

$$\int_\star \hat{\bar{f}}(\xi,\tau)\hat{f}_1(\xi_1,\tau_1)\hat{f}_2(\xi_2,\tau_2)\hat{f}_3(\xi_3,\tau_3)d\delta = \iint \bar{f} f_1 f_2 f_3(x,t)dxdt. \tag{2.18}$$

**Proof.** For simplicity, we only discuss the case of space variable. In fact, we can obtain

$$\begin{aligned}
&\int_{\xi=\xi_1+\xi_2+\xi_3} \hat{\bar{f}}(\xi)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)d\delta\\
&= \int_{\xi=\xi_1+\xi_2+\xi_3} \hat{\bar{f}}(-\xi)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)d\delta\\
&= \int_{\xi_1}\int_{\xi_2'}\int_{\xi_3'} \hat{\bar{f}}(-\xi_3{}')\hat{f}_1(\xi_1)\hat{f}_2(\xi_2{}'-\xi_1)\hat{f}_3(\xi_3{}'-\xi_2{}')d\xi_1 d\xi_2{}' d\xi_3{}'\\
&= \hat{\bar{f}} * \hat{f}_1 * \hat{f}_2 * \hat{f}_3(0) = \mathcal{F}\bar{f} f_1 f_2 f_3(0) = \int \bar{f} f_1 f_2 f_3(x)dx.
\end{aligned}$$

Therefore, Lemma 2.9 is proved.

We give the proof of Theorem 2.1 now.

Here we only prove the case $s < 0$. The case $s \ge 0$ is easy to be dealt with. For simplicity, we let $r = -s$.

By duality and the Plancheral identity, it suffices to show that for all $\bar{f} \in L_2, \bar{f} \ge 0$,

$$\begin{aligned}
\Upsilon &= \int_\star \frac{\bar{f}(\tau,\xi)}{\langle\xi\rangle^r\langle\sigma\rangle^{1-b}}\mathcal{F}u_1(\tau_1,\xi_1)\mathcal{F}u_2(\tau_2,\xi_2)\mathcal{F}\bar{u}_3(\tau_3,\xi_3)d\delta\\
&= \int_\star \frac{\prod_{j=1}^3 \langle\xi_j\rangle^r}{\langle\xi\rangle^r\langle\sigma\rangle^{1-b}\prod_{j=1}^3\langle\sigma_j\rangle^{b'}}\bar{f}(\tau,\xi)f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)f_3(\tau_3,\xi_3)d\delta\\
&\le C\|f\|_{L_2}\prod_{j=1}^3\|f_j\|_{L_2},
\end{aligned}$$

where

$$\xi = \xi_1+\xi_2+\xi_3, \quad \tau = \tau_1+\tau_2+\tau_3; \quad f_j = \langle\xi_j\rangle^s\langle\sigma_j\rangle^{b'}\hat{u}_j, \quad j=1,2; \quad f_3 = \langle\xi_3\rangle^s\langle\sigma_3\rangle^{b'}\hat{\bar{u}}_3.$$

It is clear that $\|f_j\|_{L_\xi^2 L_\tau^2} = \|u_j\|_{X_{s,b'}}$ $(j = 1,2,3)$, which will be used later without pointing out it.

We may assume $f_j \geq 0,\ j = 1, 2, 3$. Let

$$\mathcal{F}F^j_\rho(\xi,\tau) = \frac{f_j(\xi,\tau)}{(1+|\tau-\beta\xi^3-\alpha\xi^2|)^\rho}, \qquad j = 1, 2,$$
$$\mathcal{F}F^3_\rho(\xi,\tau) = \frac{f_3(\xi,\tau)}{(1+|\tau-\beta\xi^3+\beta\xi^2|)^\rho}.$$

We can obtain the following identity from (**??**),

$$\sigma - \sigma_1 - \sigma_2 - \sigma_3 = -3\beta(\xi-\xi_1)(\xi-\xi_2)\Big(\xi - \xi_3 + \frac{2\alpha}{3\beta}\Big),$$

which implies that, if $|\xi-\xi_1| \geq 2a$, $|\xi-\xi_2| \geq 2a$ and $|\xi-\xi_3| \geq 2a$, we have

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|, |\sigma_3|\} \geq C|\xi-\xi_1||\xi-\xi_2||\xi-\xi_3|.$$

Let

$$K(\xi,\xi_1,\xi_2,\xi_3) = \frac{\langle\xi_1\rangle^r\langle\xi_2\rangle^r\langle\xi_3\rangle^r}{\langle\xi\rangle^r}.$$

In order to obtain the boundedness of $\Upsilon$, we split the domain of integration in several pieces.

**Situation I.** Assume $|\xi| \leq 6a$, $|\xi-\xi_3| \leq 2a$ (it is easy to see that $|\xi_3| \leq 8a$).

**Case 1.** If $|\xi-\xi_1| \leq 2a$ or $|\xi-\xi_2| \leq 2a$, then

$$K(\xi,\xi_1,\xi_2,\xi_3) \leq C(a).$$

By (**??**) and (**??**), the integral $\Upsilon$ restricted to this domain is bounded by

$$\begin{aligned} &C\int_\star \frac{\bar{f}(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}}\frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\frac{f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}}d\delta \\ &\leq C\|F_{1-b}\|_{L^2_xL^2_t}\|F^1_{b'}\|_{L^6_xL^6_t}\|F^2_{b'}\|_{L^6_xL^6_t}\|F^3_{b'}\|_{L^6_xL^6_t} \\ &\leq C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau}. \end{aligned}$$

**Case 2.** $|\xi-\xi_1| \geq 2a$ and $|\xi-\xi_2| \geq 2a$.

**Subcase (1).** If $|\xi_1| \leq 2a$ or $|\xi_2| \leq 2a$, then

$$K(\xi,\xi_1,\xi_2,\xi_3) \leq C(a).$$

Therefore, similarly to Case 1, in the region $\Upsilon$ is bounded by

$$C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau}.$$

**Subcase (2).** If $|\xi_1| \geq 2a$ and $|\xi_2| \geq 2a$, we get $|\xi_1| \sim |\xi_2|$, which follows from

$$|\xi-\xi_3| = |\xi_1+\xi_2| \leq 2a.$$

Then using $r \leq \frac{1}{2}$, we have

$$K(\xi,\xi_1,\xi_2,\xi_3) \leq C(a)|\xi_2|.$$

For $b<\frac{2}{3}$, by (**??**), (**??**), (**??**) and (**??**), $\Upsilon$ restricted to this domain is bounded by

$$\begin{aligned}
&C\int_\star \frac{\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}}\frac{|\xi_2|\chi_{|\xi_2|\ge 2a}f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\frac{\chi_{|\xi_3|\le 8a}f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}}d\delta\\
&\le C\|F_{1-b}\|_{L^4_xL^4_t}\|F^1_{b'}\|_{L^4_xL^4_t}\|D_xP^{2a}F^2_{b'}\|_{L^\infty_xL^2_t}\|P_{8a}F^3_{b'}\|_{L^2_xL^\infty_t}\\
&\le C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau}.
\end{aligned}$$

**Situation II.** $|\xi|\le 6a$, $|\xi-\xi_3|\ge 2a$.

**Case 1.** If $|\xi-\xi_1|\le 2a$ or $|\xi-\xi_2|\le 2a$, without loss of generality, we can assume $|\xi-\xi_1|\le 2a$ (we have $|\xi_1|\le 8a$).

**Subcase (1).** If $|\xi_3|\le 2a$ or $|\xi_2|\le 2a$, then we can see

$$K(\xi,\xi_1,\xi_2,\xi_3)\le C(a).$$

Hence, we obtain that the contribution of this region to $\Upsilon$ is bounded by

$$C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau},$$

by an analogous argument to Case 1 in Situation I.

**Subcase (2).** If $|\xi_3|\ge 2a$ and $|\xi_2|\ge 2a$, we use $|\xi-\xi_1|=|\xi_2+\xi_3|\le 2a$ to get $|\xi_2|\sim|\xi_3|$. Then for $r\le\frac{1}{2}$, we obtain

$$K(\xi,\xi_1,\xi_2,\xi_3)\le C(a)|\xi_3|.$$

By (**??**), (**??**), (**??**) and (**??**) for $b<\frac{2}{3}$, $\Upsilon$ in this region is bounded by

$$\begin{aligned}
&C\int_\star \frac{\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\frac{|\xi_1|\chi_{|\xi_1|\le 8a}f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}}\frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\frac{\chi_{|\xi_3|\ge 2a}f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}}d\delta\\
&\le C\|F_{1-b}\|_{L^4_xL^4_t}\|P_{8a}F^1_{b'}\|_{L^2_xL^\infty_t}\|F^2_{b'}\|_{L^4_xL^4_t}\|D_xP^{2a}F^3_{b'}\|_{L^\infty_xL^2_t}\\
&\le C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau}.
\end{aligned}$$

**Case 2.** If $|\xi-\xi_1|\ge 2a$ and $|\xi-\xi_2|\ge 2a$, it follows that $\langle\xi-\xi_1\rangle\sim\langle\xi_1\rangle,\langle\xi-\xi_2\rangle\sim\langle\xi_2\rangle,\langle\xi-\xi_3\rangle\sim\langle\xi_3\rangle$. Then we get, by $r\le 1-b$, that

$$\frac{K(\xi,\xi_1,\xi_2,\xi_3)}{\langle\xi-\xi_1\rangle^{1-b}\langle\xi-\xi_2\rangle^{1-b}\langle\xi-\xi_3\rangle^{1-b}}\le C(a).$$

If $|\sigma|\ge C|\xi-\xi_1||\xi-\xi_2||\xi-\xi_3|$, by (**??**) and (**??**), $\Upsilon$ in this region is bounded by

$$\begin{aligned}
&C\int_\star \bar f(\tau,\xi)\frac{f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}}\frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\frac{f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}}d\delta\\
&\le C\|F_0\|_{L^2_xL^2_t}\|F^1_{b'}\|_{L^6_xL^6_t}\|F^2_{b'}\|_{L^6_xL^6_t}\|F^3_{b'}\|_{L^6_xL^6_t}\le C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau}.
\end{aligned}$$

If $|\sigma_1|\ge C|\xi-\xi_1||\xi-\xi_2||\xi-\xi_3|$, by (**??**) and (**??**) for $b<\frac{2}{3}$, $\Upsilon$ in this region is bounded by

$$\begin{aligned}
&C\int_\star \frac{\bar f(\tau,\xi)}{\langle\sigma\rangle^{1-b}}f_1(\tau_1,\xi_1)\frac{f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}}\frac{f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}}d\delta\\
&\le C\|F_{1-b}\|_{L^4_xL^4_t}\|F^1_0\|_{L^2_xL^2_t}\|F^2_{b'}\|_{L^8_xL^8_t}\|F^3_{b'}\|_{L^8_xL^8_t}\le C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau}.
\end{aligned}$$

For the other cases ($|\sigma_2|\ge C|\xi-\xi_1||\xi-\xi_2||\xi-\xi_3|$ and $|\sigma_3|\ge C|\xi-\xi_1||\xi-\xi_2||\xi-\xi_3|$), we obtain the desired estimates in an analogous arguments as above.

**Situation III.** $|\xi| \geq 6a$, $|\xi - \xi_3| \leq 2a$ (we have $|\xi_3| \geq 4a$).

**Case 1.** If $|\xi - \xi_1| \leq 2a$ or $|\xi - \xi_2| \leq 2a$, without loss of generality, we can assume $|\xi - \xi_1| \leq 2a$. Hence we obtain that $|\xi| \sim |\xi_1|$, $|\xi_1| \geq 4a$. By $|\xi - \xi_3| = |\xi_1 + \xi_2| \leq 2a$, we can get $|\xi_2| \geq 2a$, $|\xi_1| \sim |\xi_2|$. Then

$$K(\xi, \xi_1, \xi_2, \xi_3) \leq C(a)|\xi_2|^r|\xi_3|^r.$$

For $1 - b > \frac{5}{12}, r \leq \frac{1}{4}$, by (**??**), (**??**) and (**??**), $\Upsilon$ is bounded by

$$\begin{aligned}
&C\int_{\star} \frac{\chi_{|\xi|\geq 6a}\bar{f}(\tau,\xi)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\geq 4a}f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}} \frac{|\xi_2|^r\chi_{|\xi_2|\geq 2a}f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}} \frac{|\xi_3|^r\chi_{|\xi_3|\geq 4a}f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}} d\delta \\
&\leq C\|P^{6a}F_{1-b}\|_{L^4_xL^6_t}\|P^{4a}F^1_{b'}\|_{L^4_xL^6_t}\|D_x^{\frac{1}{4}}P^{2a}F^2_{b'}\|_{L^4_xL^3_t}\|D_x^{\frac{1}{4}}P^{4a}F^3_{b'}\|_{L^4_xL^3_t} \\
&\leq C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau}.
\end{aligned}$$

**Case 2.** If $|\xi - \xi_1| \geq 2a$ and $|\xi - \xi_2| \geq 2a$, from $|\xi - \xi_3| \leq 2a$, it is easy to see that $|\xi| \sim |\xi_3|$.

**Subcase (1).** $|\xi_1| \leq 2a$ or $|\xi_2| \leq 2a$.

If $|\xi_1| \leq 2a$ and $|\xi_2| \leq 2a$, we have

$$K(\xi, \xi_1, \xi_2, \xi_3) \leq C(a).$$

Then we obtain the desired estimate in an analogous argument to Case 1 (Situation I).

If $|\xi_1| \leq 2a$ and $|\xi_2| \geq 2a$, we have

$$K(\xi, \xi_1, \xi_2, \xi_3) \leq C(a)|\xi_2|^r.$$

By (**??**), (**??**), (**??**) and (**??**), for $r \leq 1$, we have the boundedness of $\Upsilon$ as follows:

$$\begin{aligned}
&C\int_{\star} \frac{\bar{f}(\tau,\xi)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\leq 2a}f_1(\tau_1,\xi_1)}{\langle\sigma_1\rangle^{b'}} \frac{|\xi_2|^r\chi_{|\xi_2|\geq 2a}f_2(\tau_2,\xi_2)}{\langle\sigma_2\rangle^{b'}} \frac{f_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}} d\delta \\
&\leq C\|F_{1-b}\|_{L^4_xL^4_t}\|P_{2a}F^1_{b'}\|_{L^2_xL^\infty_t}\|D_xP^{2a}F^2_{b'}\|_{L^\infty_xL^2_t}\|F^3_{b'}\|_{L^4_xL^4_t} \\
&\leq C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau}.
\end{aligned}$$

**Subcase (2).** If $|\xi_1| \geq 2a$ and $|\xi_2| \geq 2a$, we have

$$K(\xi, \xi_1, \xi_2, \xi_3) \leq C(a)|\xi_2|^r|\xi_2|^r.$$

For $1 - b > \frac{5}{12}$ and $r \leq \frac{1}{4}$, the integral $\Upsilon$ is bounded by

$$C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau},$$

in an analogous to the above Case 1 (Situation III).

**Situation IV.** $|\xi| \geq 6a$, $|\xi - \xi_3| \geq 2a$.

**Case 1.** If $|\xi - \xi_1| \leq 2a$ or $|\xi - \xi_2| \leq 2a$, without loss of generality, we can assume $|\xi - \xi_1| \leq 2a$. It is clear that $|\xi| \sim |\xi_1|, |\xi_1| \geq 4a$.

**Subcase (1).** If $|\xi_3| \leq 4a$, we get $|\xi_2| \leq 6a$ by $|\xi - \xi_1| = |\xi_2 + \xi_3| \leq 2a$. Then

$$K(\xi, \xi_1, \xi_2, \xi_3) \leq C(a).$$

Hence we obtain the contribution of this region to the integral $\Upsilon$ by

$$C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau},$$

in an analogous to Case 1 in Situation I.

**Subcase (2).** If $|\xi_3| \geq 4a$, it follows that $|\xi_2| \sim |\xi_3|, |\xi_2| \geq 2a$ from $|\xi - \xi_1| = |\xi_2 + \xi_3| \leq 2a$. Then

$$K(\xi, \xi_1, \xi_2, \xi_3) \leq C(a)|\xi_2|^r|\xi_3|^r.$$

We use $1 - b > \frac{5}{12}, r \leq \frac{1}{4}$ to bound the integral $\Upsilon$ by

$$C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau},$$

in an analogous to Case 1 in Situation III.

**Case 2.** $|\xi - \xi_1| \geq 2a$ and $|\xi - \xi_2| \geq 2a$.

**Subcase (1).** $|\xi_3| \leq 2a$. We can see that $\langle \xi - \xi_1 \rangle \sim \langle \xi_2 \rangle, \langle \xi - \xi_2 \rangle \sim \langle \xi_1 \rangle$. Then we get by $r \leq 1 - b$,

$$\frac{K(\xi, \xi_1, \xi_2, \xi_3)}{\langle \xi - \xi_1 \rangle^{1-b}\langle \xi - \xi_2 \rangle^{1-b}\langle \xi - \xi_3 \rangle^{1-b}} \leq C(a).$$

We can obtain the desired estimate by an analogous argument to Case 2 in Situation II.

**Subcase (2).** $|\xi_3| \geq 2a$.

If $|\xi_3| \leq C|\xi|$, then

$$K(\xi, \xi_1, \xi_2, \xi_3) \leq C(a)|\xi_1|^r|\xi_2|^r.$$

In the subdomain of $|\xi_1| \leq 2a$ or $|\xi_2| \leq 2a$, we can obtain the results in an analogous argument to Situation III (Case 2, Subcase (1)). In the subcase $|\xi_1| \geq 2a$ and $|\xi_2| \geq 2a$, we can obtain the results by an analogous argument to Case 1 in Situation III.

If $|\xi| \ll |\xi_3|$, we get $|\xi| \ll |\xi_3| \sim |\xi - \xi_3| \sim |\xi_1 + \xi_2| \leq C\max\{|\xi_1|, |\xi_2|\}$, which follows from the fact $\xi - \xi_3 = \xi_1 + \xi_2$. Hence if $|\xi| \ll |\xi_1|$ or $|\xi| \ll |\xi_1|$ , we have $|\xi_1| \sim |\xi - \xi_1|$ or $|\xi_1| \sim |\xi - \xi_1|$ respectively. Using that $r \leq 1 - b$, we get

$$\frac{K(\xi, \xi_1, \xi_2, \xi_3)}{\langle \xi - \xi_1 \rangle^{1-b}\langle \xi - \xi_2 \rangle^{1-b}\langle \xi - \xi_3 \rangle^{1-b}} \leq C(a).$$

Therefore, we can obtain the desired estimate by an analogous argument to Case 2 in Situation II. This completes the proof.

**Lemma 2.10.** *Let* $s \in \mathbb{R}$, $\dfrac{1}{2} < b < b' < 1$, $0 < \delta \leq 1$. *Then we have*

$$\|\psi_\delta(t)S(t)\varphi\|_{X_{s,b}} \leq C\delta^{\frac{1}{2}-b}\|\varphi\|_{H^s},$$

$$\|\psi_\delta(t)F\|_{X_{s,b-1}} \leq C\delta^{b'-b}\|F\|_{X_{s,b'-1}}$$

$$\left\|\psi_\delta(t)\int_0^t S(t-\tau)F(\tau)d\tau\right\|_{X_{s,b}} \leq C\delta^{\frac{1}{2}-b}\|F\|_{X_{s,b-1}},$$

$$\left\|\psi_\delta(t)\int_0^t S(t-\tau)F(\tau)d\tau\right\|_{L^\infty_t H^s_x} \leq C\delta^{\frac{1}{2}-b}\|F\|_{X_{s,b-1}}.$$

The proof can be found in [4, 5].

Now we turn to the proof of Theorem 1.1. For $u_0 \in H^s (s \geq -\frac{1}{4})$, we define the operator

$$\Phi(u) = \psi_1(t)S(t)u_0 - \psi_1(t)i\int_0^t S(t-t')\psi_\delta(t')\gamma|u|^2u(t')dt',$$

and the set

$$\mathcal{B} = \{u \in X_{s,b} : \|u\|_{X_{s,b}} \le 2C\|u_0\|_{H^s}\}.$$

In order to show that $\Phi$ is a contraction on $\mathcal{B}$, we first prove that $\Phi(\mathcal{B}) \subset \mathcal{B}$.

By Theorem 2.1 and Lemma 2.10, we have the next chain of inequalities for $b < b' < \frac{7}{12}$,

$$\begin{aligned}\|\Phi(u)\|_{X_{s,b}} &\le C\|u_0\|_{H^s} + C\delta^{b'-b}|\gamma|\|u\|^2_{X_{s,b}}\|u\|_{X_{s,b}} \\ &\le C\|u_0\|_{H^s} + C\delta^{b'-b}\|u_0\|^2_{H^s}\|u\|_{X_{s,b}}.\end{aligned}$$

Therefore, if we fix $\delta$ such that $C\delta^{b'-b}\|u_0\|^2_{H^s} \le \frac{1}{2}$, then $\Phi(\mathcal{B}) \subset \mathcal{B}$.

Let $(u_1, u_2) \in \mathcal{B}$. In an analogous way as above, we obtain

$$\|\Phi(u_1) - \Phi(u_2)\|_{X_{s,b}} \le \frac{1}{2}\|u_1 - u_2\|_{X_{s,b}}.$$

Therefore, $\Phi$ is a contraction map on $\mathcal{B}$. Thus we can obtain a unique fixed point which solves the Cauchy problem (**??**)–(**??**) for $T < \frac{\delta}{2}$.

The solution obtained above is also in $C(0, T; H^s)$ due to the inequality $\|u\|_{L^\infty H^s} \le \|u\|_{X_{s,b}}$.

Moreover, given $t \in (0, T)$, by the definition of Lipschitz continuous and Lemma 2.10, one can easily prove that the map $u_0 \to u(t)$ is Lipschitz continuous from $H^s$ to $C(0, T; H^s)$.

This completes the proof of Theorem 1.1.

## § 3. Global Well-Posedness in $H^s(s > 0)$

In this section, we will first prove the generalized trilinear estimate as follows.

**Theorem 3.1.** *Let* $0 < s \le 1$, $\dfrac{1}{2} < b < \dfrac{2}{3}$, $b' > \dfrac{1}{2}$. *Then*

$$\| u_1 u_2 \bar{u}_3 \|_{X_{s,b-1}} \le C \| u_1 \|_{X_{0,b'}} \| u_2 \|_{X_{0,b'}} \| u_3 \|_{X_{s,b'}} . \tag{3.1}$$

**Remark 3.1.** One can prove similarly when it also holds that the left side of (**??**) is replaced by $\|u_1\bar{u}_2u_3\|_{X_{s,b-1}}$ or $\|\bar{u}_1u_2u_3\|_{X_{s,b-1}}$. We only give the proof for the case (**??**) here.

**Proof.** By duality and the Plancheral identity, it suffices to show that for all $\bar{f} \in L_2,\ \bar{f} \ge 0,\ f_j = \langle\sigma_j\rangle^{b'}\hat{u}_j,\ j = 1,\ 2;\ f_3 = \langle\xi_3\rangle^s\langle\sigma_3\rangle^{b'}\hat{\bar{u}}_3$, we have

$$\begin{aligned}\Lambda &= \int_\star \frac{\langle\xi\rangle^s \bar{f}(\tau,\xi)}{\langle\sigma\rangle^{1-b}} \mathcal{F}u_1(\tau_1,\xi_1)\mathcal{F}u_2(\tau_2,\xi_2)\mathcal{F}\bar{u}_3(\tau_3,\xi_3)d\delta \\ &= \int_\star \frac{\langle\xi\rangle^s}{\langle\xi_3\rangle^s\langle\sigma\rangle^{1-b}\prod_{j=1}^{3}\langle\sigma_j\rangle^{b'}} \bar{f}(\tau,\xi)f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)f_3(\tau_3,\xi_3)d\delta \\ &\le C\|f\|_{L_2}\prod_{j=1}^{3}\|f_j\|_{L_2}.\end{aligned}$$

We may assume $f_j \ge 0,\ j = 1, 2, 3$. Let $K(\xi, \xi_1, \xi_2, \xi_3) = \langle\xi\rangle^s/\langle\xi_3\rangle^s$.

We also split the domain of integration similarly as the proof of Theorem 2.1.

**Situation I.** $|\xi| \le 6a$. We have

$$K(\xi, \xi_1, \xi_2, \xi_3) \le C(a).$$

By (**??**) and (**??**), the contribution of the above region to the integral $\Lambda$ is bounded by

$$\|F_{1-b}\|_{L^2_xL^2_t}\|F^1_{b'}\|_{L^6_xL^6_t}\|F^2_{b'}\|_{L^6_xL^6_t}\|F^3_{b'}\|_{L^6_xL^6_t} \leq C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau}.$$

**Situation II.** $|\xi| \geq 6a$.

**Case 1.** $|\xi| \leq C|\xi_3|$ for some constants $C$. We have

$$K(\xi,\xi_1,\xi_2,\xi_3) \leq C(a).$$

Hence we obtain the desired estimate by an analogous argument to Case 1 in Situation I.

**Case 2.** $|\xi_3| \ll |\xi|$.

**Subcase (1).** $|\xi_3| \leq 4a$. Then $|\xi| \leq 3|\xi_1|$ or $|\xi| \leq 3|\xi_2|$ (without loss of generality, we can assume $2a \leq \frac{1}{3}|\xi| \leq 3|\xi_1|$). By $s \leq 1$, we get

$$K(\xi,\xi_1,\xi_2,\xi_3) \leq C(a)|\xi_1|.$$

Then for $b < \frac{2}{3}$, by (**??**), (**??**), (**??**) and (**??**), $\Lambda$ is bounded by

$$\begin{aligned}&\|F_{1-b}\|_{L^4_xL^4_t}\|D_xP^{2a}F^1_{b'}\|_{L^\infty_xL^2_t}\|F^2_{b'}\|_{L^4_xL^4_t}\|P_{4a}F^3_{b'}\|_{L^2_xL^\infty_t}\\ &\leq C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau}.\end{aligned}$$

**Subcase (2).** $|\xi_3| \geq 4a$. Then $|\xi| \leq 3|\xi_1|$ or $|\xi| \leq 3|\xi_2|$ (without loss of generality, we assume $2a \leq \frac{1}{3}|\xi| \leq |\xi_1|$).

If $|\xi_2| \leq 2a$, then the integral $\Lambda$ is bounded by, similarly to the above,

$$\begin{aligned}&\|F_{1-b}\|_{L^4_xL^4_t}\|D_xP^{2a}F^1_{b'}\|_{L^\infty_xL^2_t}\|P_{2a}F^2_{b'}\|_{L^2_xL^\infty_t}\|F^3_{b'}\|_{L^4_xL^4_t}\\ &\leq C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau},\end{aligned}$$

due to (**??**), (**??**), (**??**) and (**??**), for $b < \frac{2}{3}$ and $s < 1$.

If $|\xi_2| \geq 2a$, it is easy to see that $|\xi - \xi_3| \geq 2a$ from $|\xi_3| \ll |\xi|$. we will split this region into the following two parts.

(i) $|\xi - \xi_1| \leq 2a$ or $|\xi - \xi_2| \leq 2a$ (without loss of generality,we can assume $|\xi - \xi_1| \leq 2a$, so we have $|\xi| \sim |\xi_1|$ and $|\xi_2| \sim |\xi_3|$).

If $0 < s \leq \frac{1}{4}$, then

$$K(\xi,\xi_1,\xi_2,\xi_3) \leq C(a)|\xi_1|^{\frac{1}{4}}|\xi|^{\frac{1}{4}}.$$

Then, for $b < \frac{2}{3}$, by (**??**), (**??**) and (**??**), the integral $\Lambda$ is bounded by

$$\begin{aligned}&\|D_x^{\frac{1}{4}}P^{6a}F_{1-b}\|_{L^4_xL^3_t}\|D_x^{\frac{1}{4}}P^{4a}F^1_{b'}\|_{L^4_xL^3_t}\|P^{2a}F^2_{b'}\|_{L^4_xL^6_t}\|P^{4a}F^3_{b'}\|_{L^4_xL^6_t}\\ &\leq C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau}.\end{aligned}$$

If $\frac{1}{4} < s \leq \frac{1}{2}$, then

$$K(\xi,\xi_1,\xi_2,\xi_3) \leq C(a)|\xi_1|^{\frac{1}{2}}/|\xi_3|^{\frac{1}{4}}.$$

Then we use $b < \frac{2}{3}$ to bound the integral $\Lambda$ by

$$\begin{aligned}&\|F_{1-b}\|_{L^4_xL^4_t}\|D_x^{\frac{1}{2}}P^{4a}F^1_{b'}\|_{L^4_xL^2_t}\|F^2_{b'}\|_{L^4_xL^4_t}\|D_x^{-\frac{1}{4}}P^{4a}F^3_{b'}\|_{L^4_xL^\infty_t}\\ &\leq C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}\|f_3\|_{L^2_\xi L^2_\tau},\end{aligned}$$

due to (**??**), (**??**), (**??**) and (**??**).

If $\frac{1}{2} < s \le 1$, then

$$K(\xi,\xi_1,\xi_2,\xi_3) \le C(a)\frac{|\xi_1|}{|\xi_3|^{\frac{1}{4}}|\xi_2|^{\frac{1}{4}}}.$$

Then, for $b < \frac{2}{3}$, by (**??**), (**??**), (**??**) and (**??**), the integral $\Lambda$ is bounded by

$$\begin{aligned}&\|F_{1-b}\|_{L_x^2L_t^2}\|D_xP^{4a}F_{b'}^1\|_{L_x^\infty L_t^2}\|D_x^{-\frac{1}{4}}P^{2a}F_{b'}^2\|_{L_x^4L_t^\infty}\|D_x^{-\frac{1}{4}}P^{4a}F_{b'}^3\|_{L_x^4L_t^\infty}\\ &\le C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f_3\|_{L_\xi^2L_\tau^2}.\end{aligned}$$

(ii) $|\xi-\xi_1| \ge 2a$ and $|\xi-\xi_2| \ge 2a$. We get

$$|\xi_3| \ll |\xi| \sim |\xi-\xi_3| \sim |\xi_1+\xi_2| \le C\max\{|\xi_1|,|\xi_2|\},$$

which follows from $\xi-\xi_3=\xi_1+\xi_2$ and $|\xi_3|\ll|\xi|$. Here, $A\sim B$ means $B\frac{9}{10} \le A \le \frac{10}{9}B$.

Hence, we have $|\xi-\xi_1| \ge C|\xi|$ or $|\xi-\xi_2| \ge C|\xi|$ for some constants $C$. Then we have

$$\frac{K(\xi,\xi_1,\xi_2,\xi_3)}{\langle\xi-\xi_1\rangle^{1-b}\langle\xi-\xi_2\rangle^{1-b}\langle\xi-\xi_3\rangle^{1-b}} \le C(a)|\xi|^{s-2(1-b)}.$$

Without loss of generality, we can assume $|\xi| \le 3|\xi_1|$ and $|\sigma| \ge C|\xi-\xi_1||\xi-\xi_2||\xi-\xi_3|$. Then we use $s \le 2(1-b)+\frac{1}{6}$ to bound the integral $\Lambda$ by

$$\|F_0\|_{L_x^2L_t^2}\|D_x^{\frac{1}{6}}P^{4a}F_{b'}^1\|_{L_x^6L_t^6}\|F_{b'}^2\|_{L_x^6L_t^6}\|F_{b'}^3\|_{L_x^6L_t^6} \le C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f_3\|_{L_\xi^2L_\tau^2},$$

due to (**??**), (**??**) and (**??**). This completes the proof.

Now we turn to the proof of Theorem 1.3.

We use the contraction mapping principle to prove theorem.

For $u_0 \in H^s(0<s\le 1)$, we define the operator

$$\Phi(u)=\psi_1(t)S(t)u_0-\psi_1(t)i\int_0^t S(t-t')\psi_\delta(t')\gamma|u|^2u(t')dt',$$

and the sets

$$\mathcal{C}=\{u\in X_{0,b}:\|u\|_{X_{0,b}}\le 2C\|u_0\|_{L^2}\},\quad \mathcal{D}=\{u\in X_{s,b}:\|u\|_{X_{s,b}}\le 2C\|u_0\|_{H^s}\}.$$

First, we prove that there exists a unique solution $u(t)\in C(0,T;L^2)$ to the Cauchy problem (**??**)–(**??**) for initial data in $H^s(s>0)$.

By Theorem 2.1 and Lemma 2.10, for $b<b'<\frac{7}{12}$, we have the next chain of inequalities

$$\|\Phi(u)\|_{X_{0,b}} \le C\|u_0\|_{L^2}+C\delta^{b'-b}|\gamma|\|u\|_{X_{0,b}}^2\|u\|_{X_{0,b}} \le C\|u_0\|_{L^2}+C\delta^{b'-b}\|u_0\|_{L^2}^2\|u\|_{X_{0,b}}.$$

Hence, we fix $\delta$ such that $C\delta^{b'-b}\|u_0\|_{L^2}^2 \le \frac{1}{2}$. We get $\Phi(\mathcal{C})\subset\mathcal{C}$.

In an analogous way as above, for $u_1,u_2\in\mathcal{C}$, we obtain

$$\|\Phi(u_1)-\Phi(u_2)\|_{X_{0,b}} \le \frac{1}{2}\|u_1-u_2\|_{X_{0,b}}.$$

Then $\Phi$ is a contraction map on $\mathcal{C}$. Thus we obtain that there exists a unique solution $u(t)\in C(0,T;L^2)$ for $T<\frac{\delta}{2}$.

Next, for the time interval $[0,T]$, we will prove that the solution $u(t)$ obtained above belongs to $C(0,T;H^s)$ for initial data in $H^s(0<s\leq 1)$.

In fact, it suffices to prove that the existence time of solution $u(t)$ for data in $H^s(0<s\leq 1)$ only depends on the $L^2$ norm of initial data.

For $b<b'<\frac{7}{12}$, by Theorem 3.1 and Lemma 2.10, we have the next chain of inequalities

$$\|\Phi(u)\|_{X_{s,b}}\leq C\|u_0\|_{H^s}+C\delta^{b'-b}\gamma\|u\|^2_{X_{0,b}}\|u\|_{X_{s,b}}\leq C\|u_0\|_{H^s}+C\delta^{b'-b}\|u_0\|^2_{L^2}\|u\|_{X_{s,b}}.$$

Then, we have $C\delta^{b'-b}\|u_0\|^2_{L^2}\leq\frac{1}{2}$ by the above arguments. Therefore, we have $\Phi(\mathcal{D})\subset\mathcal{D}$.

In an analogous way as above, for $u_1,u_2\in\mathcal{D}$, we obtain

$$\|\Phi(u_1)-\Phi(u_2)\|_{X_{s,b}}\leq\frac{1}{2}\|u_1-u_2\|_{X_{s,b}}.$$

Then $\Phi$ is a contraction map on $\mathcal{D}$. Therefore, there exists a unique solution $u(t)$ of the Cauchy problem (**??**)–(**??**) for $T<\delta$ with data in $H^s(0<s\leq 1)$.

Furthermore, we take $u(T)$ as initial value and then obtain that the solution of the Cauchy problem (**??**)–(**??**) exists on $t\in[T,2T]$ similarly as above arguments. We can continue this process to obtain the global solution in $H^s(0<s\leq 1)$ by $L^2$ conservation law (**??**).

**Remark 3.2.** For data in $H^s(1<s\leq 2)$, we can obtain the following generalized trilinear estimate for some $b>\frac{1}{2}$ similarly to the above arguments.

$$\|\, u_1u_2\bar{u}_3\,\|_{X_{s,b-1}}\leq C\,\|\,u_1\,\|_{X_{1,b'}}\|\,u_2\,\|_{X_{1,b'}}\|\,u_3\,\|_{X_{s,b'}}\,. \tag{3.2}$$

One can prove similarly that it also holds if the left side of (**??**) is replaced by $\|u_1\bar{u}_2u_3\|_{X_{s,b-1}}$ or $\|\bar{u}_1u_2u_3\|_{X_{s,b-1}}$.

Hence, we can get global well-posedness in $H^s(1<s\leq 2)$ (**??**)–(**??**) by global well-posedness in $H^1$. By using induction, we can prove the global well-posedness in $H^s(s>0)$ for Cauchy problem (**??**)–(**??**).

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