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GLOBAL EXISTENCE AND ASYMPTOTICS BEHAVIOR OF SOLUTIONS FOR A RESONANT KLEIN-GORDON SYSTEM IN TWO SPACE DIMENSIONS***

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Abstract

The authors study a resonant Klein-Gordon system with convenient nonlinearities in two space dimensions, prove that such a system has global solutions for small, smooth, compactly supported Cauchy data, and find that the asymptotic profile of the solution is quite different from that of the free solution.

Keywords Klein-Gordon equation, Asymptotic behavior, Global existence
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§ 1 . Introduction

This paper deals with the problem of global existence and asymptotic behavior for a class of Klein-Gordon systems in two space dimensions, $j = 1, 2$,

$$\begin{aligned} \Box u_1 + m_1^2 u_1 &= F_1(u, \partial_t u, \partial_x u, \partial_t\partial_x u, \partial_x^2 u), \\ \Box u_2 + m_2^2 u_2 &= F_2(u, \partial_t u, \partial_x u, \partial_t\partial_x u, \partial_x^2 u) \end{aligned} \tag{1.1}$$

with small, smooth, compactly supported Cauchy data, where (t, x) denotes coordinates on $\mathbb{R} \times \mathbb{R}^2$, $u = (u_1, u_2)$, $\Box = \partial_t^2 - \Delta$ is the wave operator defined on $\mathbb{R} \times \mathbb{R}^2$, F_1 and F_2 are nonlinearities vanishing at order 2 at the origin. We say (**??**) is a resonant Klein-Gordon system if $m_1 = 2m_2$ or $2m_1 = m_2$.

The problem of global existence for quasi-linear Klein-Gordon equations in two space dimensions with small, smooth Cauchy data has been considered by many authors. For the scalar Klein-Gordon equation the global existence has been proved by Ozawa, Tsutaya and Tsutsumi [7] in the semi-linear case and Ozawa, Tsutaya and Tsutsumi [8] in the quasi-linear case, after partial results of Georgiev and Popivanov [2], Kosecki [4] and Simon and Taflin [9]. Under a non resonance assumption on the masses, Sunagawa [**?**] studied systems of Klein-Gordon equations for quadratic nonlinearities in two space dimensions and got the

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global existence for small data. He found that the global solution tends to a free solution as $t \to \infty$. In [?], Delort, Fang and Xue considered the resonant case, gave the global existence for quasi-linear systems when the nonlinearity satisfies a convenient structure condition (i.e. a null condition), and found that the solution have a linear behavior at infinity. Is there a global solution for the resonant Klein-Gordon equations if the nonlinearity does not satisfy the null condition? How does the solution behave as $t \to +\infty$? The only result due to Sunagawa [?] said that the behave of the solution for the following special case

$$\begin{aligned} &\Box u_1 + u_1 = 0, \\ &\Box u_2 + u_2 = 0, \\ &\Box u_3 + 4u_3 = u_1 u_2 \end{aligned} \tag{1.2}$$

with small, smooth, compactly supported Cauchy data, is quite different from that of a free solution, and said that the energy of the solution of (**??**) behaves like $C \ln t$ as $t \to +\infty$. In this paper we consider the global existence and asymptotic behavior of the solution for the resonant Klein-Gordon system whose nonlinearity does not satisfy the null condition. We consider the following special Klein-Gordon system:

$$\begin{aligned} &\Box u_1 + 4u_1 = Q_1(u_2, \partial u_2) + P_1((\partial^\alpha u_1)_{|\alpha|\le 1}, (\partial^\beta u_2)_{|\beta|\le 1}), \\ &\Box u_2 + u_2 = Q_2(u_2, \partial u_2) + P_2((\partial^\alpha u_1)_{|\alpha|\le 1}, (\partial^\beta u_2)_{|\beta|\le 1}), \\ &(u_1, u_2)|_{t=0} = \epsilon(f_1, f_2), \\ &(\partial_t u_1, \partial_t u_2)|_{t=0} = \epsilon(g_1, g_2), \end{aligned} \tag{1.3}$$

where $\alpha = (\alpha_0, \alpha') \in \mathbb{N} \times \mathbb{N}^2$ with $\alpha' = (\alpha_1, \alpha_2)$ denotes a multi-index in $\mathbb{N}^3$, $Q_i(u_2, \partial u_2)$ and $P_i((\partial^\alpha u_1)_{|\alpha|\le 1}, (\partial^\beta u_2)_{|\beta|\le 1})$ denote, respectively, a real quadratic form and a real cubic form depending on the derivatives up to order 1, f_i and g_i are two C_0^∞ functions defined on $\mathbb{R}^2$, and $\epsilon > 0$ small. We will prove that the system (**??**) possesses a global solution, whose large time asymptotic profile is modulated in the logarithmic order. Our main results are

Theorem 1.1. *Fix $B > 0$. There is $\sigma_0 \in \mathbb{N}$ and for any integer $\sigma \ge \sigma_0$, there is $\epsilon_0 > 0$ such that for any $\epsilon \in]0, \epsilon_0[$, any (f, g) in the unit ball of $H^\sigma(\mathbb{R}^2) \times H^{\sigma-1}(\mathbb{R}^2)$, $\mathbb{R}^2$ valued, supported inside $\{x \in \mathbb{R}^2; |x| \le B\}$, the problem* (**??**) *has a unique global solution $(u_1, u_2) \in C^0(\mathbb{R}, H^\sigma(\mathbb{R}^2)) \cap C^1(\mathbb{R}, H^{\sigma-1}(\mathbb{R}^2))$.*

For (t, x) satisfying $|x| \le t$, write $\varphi(t, x) = \sqrt{t^2 - x^2}$.

Theorem 1.2. *In Theorem* **??** *assume that f and g are in C_0^∞. There are C^∞ functions $y \to a_{\epsilon,j}(y)$, $j = 1, 2, 3$ defined on $\mathbb{R}^2$, supported inside the unit ball, such that if we set*

$$\begin{aligned} u_{1,\epsilon}(t, x) &= \frac{1}{t} \mathrm{Re}\,(a_{\epsilon,1}(x/t) \ln t + a_{\epsilon,2}(x/t)) e^{2i\varphi}, \\ u_{2,\epsilon}(t, x) &= \frac{1}{t} \mathrm{Re}\, a_{\epsilon,3}(x/t) e^{i\varphi}, \end{aligned} \tag{1.4}$$

then there is, for any $p > 0$, a constant C_p such that

$$\begin{aligned}|u_1(t,x) - u_{1,\epsilon}(t,x)| &\le C_p \frac{\epsilon \ln t}{t^2}\Big[\Big(1 - \frac{|x|}{t}\Big)_+ + \frac{1}{t}\Big]^p,\\ |u_2(t,x) - u_{2,\epsilon}(t,x)| &\le C_p \frac{\epsilon}{t^2}\Big[\Big(1 - \frac{|x|}{t}\Big)_+ + \frac{1}{t}\Big]^p.\end{aligned} \tag{1.5}$$

Remark 1.1. When $Q_1 = u_2^2$, we have

$$|a_{\epsilon,1}(y)| = \frac{1}{64}\mathbb{1}_{|y|<1}(|D_T w_2 + 2w_2|^2 - |D_T w_2 - 2w_2|^2)_{T=T_0}(\cosh(\kappa|X|))^{-1},$$

where $X = \frac{1}{2y}\ln\frac{1+|y|}{1-|y|}$, w_2, T and T_0 are defined in (**??**) and (**??**). Moreover, $(|D_T w_2 + 2w_2|^2 - |D_T w_2 - 2w_2|^2)_{T=T_0}$ depends only on the restriction of u_2 and its derivatives to the hyperboloid $H_{T_0} = \{(t,x) : (t+2B)^2 - x^2 = T_0^2\}$, and we can choose f_2 and g_2 such that $(|D_T w_2 + 2w_2|^2 - |D_T w_2 - 2w_2|^2)_{T=T_0} \neq 0$.

Remark 1.2. Let $m_1 = 2$ and $m_2 = 1$ and consider the equation (**??**) with the special nonlinearity $F_j(u)$ which is dependent only on u. Note that $F_j(u)$ satisfies the "null condition" if and only if $F_1(u)$ is independent of the term u_1^2 and $F_2(u)$ independent of the term u_1u_2. In [**?**] we proved that the equation (**??**) with small, smooth, compactly supported Cauchy data has a global solution if $F_j(u)$ satisfies the "null condition". The nonlinearity we consider in this paper is the special form $F_1(u) = u_2^2$ and $F_2(u) = 0$, which does not satisfy the "null condition". However, our approach used in this paper is not suitable for one to obtain the asymptotic profile of the solution to the equation (**??**) with the nonlinearity such as $F_1(u) = 0$ and $F_2(u) = u_1u_2$ or $F_1(u) = u_2^2$ and $F_2(u) = u_1u_2$.

§ 2. Reduction of the Problem

Let u_3 be the solution satisfying the following Cauchy problem

$$\begin{aligned}&\Box u_3 + 4u_3 = Q_1(u_2, \partial u_2),\\ &u_3|_{t=0} = 0, \qquad \partial_t u_3|_{t=0} = 0.\end{aligned} \tag{2.1}$$

Denote $v = (v_1, v_2, v_3) = (u_1 - u_3, u_2, u_3)$, and

$$G_i((\partial^\alpha v)_{|\alpha|\le 1}) = P_i((\partial^\alpha(v_1+v_3))_{|\alpha|\le 1}, ((\partial^\alpha v_2)_{|\alpha|\le 1})).$$

Then the following system is equivalent to the system (**??**):

$$\begin{aligned}&\Box v_1 + 4v_1 = G_1((\partial^\alpha v)_{|\alpha|\le 1}),\\ &\Box v_2 + v_2 = Q_2(v_2, \partial v_2) + G_2((\partial^\alpha v)_{|\alpha|\le 1}),\\ &\Box v_3 + 4v_3 = Q_1(v_2, \partial v_2),\\ &(v_1, v_2, v_3)|_{t=0} = \epsilon(f_1, f_2, 0),\\ &(\partial_t v_1, \partial_t v_2, \partial_t v_3)|_{t=0} = \epsilon(g_1, g_2, 0).\end{aligned} \tag{2.2}$$

As in [?], it is sufficient to study the problem (**??**) in the domain $\{(t,x)\in[0,+\infty)\times\mathbb{R}^2 : (t+2B)^2-|x|^2\geq T_0^2\}$ with the data defined on $\{(t+2B)^2-|x|^2=T_0^2\}$, where T_0 is a positive constant. We introduce new coordinates (T,X) defined by

$$t+2B=T\cosh|X|,\qquad x=TX\frac{\sinh|X|}{|X|},\tag{2.3}$$

and a new function w associated with a solution v of (**??**)

$$v_\ell(t,x)=\frac{1}{T}(\cosh(\kappa|X|))^{-1}w_\ell(T,X),\tag{2.4}$$

where κ is a positive number to be chosen later. Denote $D_t=\frac{1}{i}\frac{\partial}{\partial t},\ D_{x_j}=\frac{1}{i}\frac{\partial}{\partial x_j},\ j=1,2$ and let $(m_1,m_2,m_3)=(2,1,2)$. By (**??**) and (**??**) we have

$$(D_t^2-D_x^2-{m_\ell}^2)v_\ell=\frac{1}{T}(\cosh(\kappa|X|))^{-1}(P_\kappa^\ell w_\ell),\tag{2.5}$$

while $\Xi_\kappa(\rho)=\kappa^2(1-2(\tanh(\kappa\rho))^2)+\kappa\frac{\tanh(\kappa\rho)}{\tanh\rho}$,

$$P_\kappa^\ell=P^\ell-\frac{2i}{T}\kappa\tanh(\kappa|X|)\Big(\frac{X}{|X|}\cdot\frac{D_X}{T}\Big)-\frac{1}{T^2}\Xi_\kappa(|X|),\tag{2.6}$$

$$P^\ell=D_T^2-\sum_{1\leq i,j\leq 2}a_{ij}(X)\frac{D_{X_i}}{T}\frac{D_{X_j}}{T}-\frac{i}{T}\frac{\psi(|X|)}{\sinh|X|}\Big(X\cdot\frac{D_X}{T}\Big)-m_\ell^2\tag{2.7}$$

with $(a_{ij}(X))_{i,j}$ the coefficients of the matrix defined by

$$A(X)=\begin{bmatrix}1-X_2^2\phi(|X|) & X_1X_2\phi(|X|)\\ X_1X_2\phi(|X|) & 1-X_1^2\phi(|X|)\end{bmatrix}\tag{2.8}$$

and

$$\phi(\rho)=\frac{1}{\rho^2}-\frac{1}{\sinh^2\rho},\qquad \psi(\rho)=\frac{1}{\sinh\rho}-\frac{\cosh\rho}{\rho}.\tag{2.9}$$

Let us define the Klainerman vector fields

$$\begin{aligned}
Z_0&=X_1\partial_{X_2}-X_2\partial_{X_1},\\
Z_1&=\Big(\frac{X_1^2}{|X|^2}+\frac{X_2^2}{|X|}\frac{\cosh|X|}{\sinh|X|}\Big)\partial_{X_1}+\frac{X_1X_2}{|X|^2}\Big(1-\frac{|X|\cosh|X|}{\sinh|X|}\Big)\partial_{X_2},\\
Z_2&=\frac{X_1X_2}{|X|^2}\Big(1-\frac{|X|\cosh|X|}{\sinh|X|}\Big)\partial_{X_1}+\Big(\frac{X_2^2}{|X|^2}+\frac{X_1^2}{|X|}\frac{\cosh|X|}{\sinh|X|}\Big)\partial_{X_2},\\
Z_3&=\partial_T.
\end{aligned}\tag{2.10}$$

If $I=\{i_1,\cdots,i_N\}$ is a family of indices between 0 and 3, we set $|I|=N$ and denote $Z^I=Z_{i_1}\cdots Z_{i_N}$.

Definition 2.1. *For $r\in\mathbb{R}$ we denote by $\mathcal{E}^r$ the space of C^∞ functions $(T,X)\to a(T,X)$ defined on $[T_0,+\infty[\times\mathbb{R}^2$, such that for any family I of indices between 0 and 2, and for any $m\in\mathbb{N}$, there is $C_{I,m}>0$ with*

$$|\partial_T^m Z^I a(T,X)|\leq C_{I,m}T^{-m}e^{r|X|}$$

for any $X\in\mathbb{R}^2,\ T\geq T_0$.

Using a similar argument as that of Proposition 1.2.4 in [**?**], we deduce the following proposition.

Proposition 2.1. (i) *There are two quadratic polynomials*

$$\Big(w_2, D_T w_2, \frac{D_X}{T} w_2\Big) \to \widetilde{Q}_j\Big(T, X, w_2, D_T w_2, \frac{D_X}{T} w_2\Big), \qquad j = 2, 3 \tag{2.11}$$

which are, for real w_2*, valued in* $\mathbb{R}$*, with* $\mathcal{E}^{-\kappa+3}$ *dependence in* (T, X)*. There are two cubic polynomials*

$$\Big(w, D_T w, \frac{D_X}{T} w\Big) \to \widetilde{P}_j\Big(T, X, w, D_T w, \frac{D_X}{T} w\Big), \qquad j = 1, 2 \tag{2.12}$$

which are, for real $w = (w_1, w_2, w_3)$*, valued in* $\mathbb{R}$*, with* $\mathcal{E}^{-2\kappa+3}$ *dependence in* (T, X)*, such that* $v = (v_1, v_2, v_3)$ *is a solution to the equation* (**??**)*, defined on a domain* $T_0 \le T \le T_1$*, if and only if* w *given by* (**??**) *satisfies*

$$\begin{aligned}
P_\kappa^1 w_1 &= \frac{1}{T^2} \widetilde{P}_1\Big(T, X, w, D_T w, \frac{D_X}{T} w\Big), \\
P_\kappa^2 w_2 &= \frac{1}{T} \widetilde{Q}_2\Big(T, X, w_2, D_T w_2, \frac{D_X}{T} w_2\Big) + \frac{1}{T^2} \widetilde{P}_2\Big(T, X, w, D_T w, \frac{D_X}{T} w\Big), \\
P_\kappa^3 w_3 &= \frac{1}{T} \widetilde{Q}_3\Big(T, X, w_2, D_T w_2, \frac{D_X}{T} w_2\Big).
\end{aligned} \tag{2.13}$$

(ii) *Denote by* $\widetilde{Q}_j^0$*,* $j = 2, 3$ *the expressions obtained when one replaces* $\frac{D_X}{T} w_2$ *by* 0 *in* $\widetilde{Q}_j$*. Then* $\widetilde{Q}_j^0$ *may be written as*

$$q_j(X, w_2, D_T w_2) + \frac{1}{T} r_j(T, X, w_2, D_T w_2),$$

where q_j *is defined by*

$$(q_2, q_3) = (\cosh \kappa |X|)^{-1} (Q_2(w_2, (\omega^\alpha D_T w_2)_{|\alpha|=1}), Q_1(w_2, (\omega^\alpha D_T w_2)_{|\alpha|=1})), \tag{2.14}$$

$\omega(y) = (\omega_0(y), \omega_1(y), \omega_2(y))$ *is the function defined by*

$$\omega_0(y) = \frac{1}{\sqrt{1 - y^2}}, \quad \omega_j(y) = \frac{-y_j}{\sqrt{1 - y^2}}, \qquad j = 1, 2, \tag{2.15}$$

q_j *and* r_j *are quadratic forms in* $(w_2, D_T w_2)$ *with coefficients in* $\mathcal{E}^{-\kappa+3}$*.*

Proposition **??** implies that Theorem **??** follows from

Theorem 2.1. *Let* $A > 0$*,* $B > 0$ *be given constants. There is* $\sigma_0 \in \mathbb{N}$ *such that for any* $\sigma \in \mathbb{N}$*,* $\sigma \ge \sigma_0$*, there is* $\epsilon_0 > 0$ *and for any* $\epsilon \in]0, \epsilon_0[$*, any couple* $(w^0, w^1) \in H^\sigma \times H^{\sigma-1}$*, real valued, supported inside* $\{|x| \le B\}$*, satisfying* $\|w^0\|_{H^\sigma} + \|w^1\|_{H^{\sigma-1}} \le A\epsilon$*, the system*

$$\begin{aligned}
P_\kappa^1 w_1 &= \frac{1}{T^2} \widetilde{P}_1\Big(T, X, w, D_T w, \frac{D_X}{T} w\Big), \\
P_\kappa^2 w_2 &= \frac{1}{T} \widetilde{Q}_2\Big(T, X, w_2, D_T w_2, \frac{D_X}{T} w_2\Big) + \frac{1}{T^2} \widetilde{P}_2\Big(T, X, w, D_T w, \frac{D_X}{T} w\Big), \\
P_\kappa^3 w_3 &= \frac{1}{T} \widetilde{Q}_3\Big(T, X, w_2, D_T w_2, \frac{D_X}{T} w_2\Big), \\
(w_1, w_2, w_3)|_{T=T_0} &= w^0, \\
\partial_T (w_1, w_2, w_3)|_{T=T_0} &= w^1
\end{aligned} \tag{2.16}$$

has a unique global solution $w \in C^0([T_0, +\infty[, H^\sigma) \cap C^1([T_0, +\infty[, H^{\sigma-1})$.

§ 3 . Global Existence

Let $(m_1, m_2, m_3) = (2, 1, 2)$. Introduce the energy at time T,

$$E(T, w) = E_1(T, w_1) + E_2(T, w_2) + E_3(T, w_3) \tag{3.1}$$

with

$$E_\ell(T, w_\ell) = \int_{\mathbb{R}^2} \Big[|\partial_T w_\ell(T, X)|^2 + {}^t\Big(\frac{\nabla_X w_\ell}{T}\Big) A(X) \Big(\overline{\frac{\nabla_X w_\ell}{T}}\Big) + m_\ell^2 |w_\ell|^2 \Big] \frac{\sinh |X|}{|X|} dX.$$

We take $\kappa \geq 5$ and $\kappa \in \frac{1}{8} + \mathbb{N}$. Denote by $j_0(\kappa)$ the integer defined by $j_0(\kappa) < 2\kappa < j_0(\kappa) + 1$, and denote by N' the largest integer smaller or equal to $\min(N, N/2 + 1)$. Fix $N \in \mathbb{N}$ so large that $N - j_0(\kappa) \geq N' + 6$ (i.e. $N \geq 14 + 2j_0(\kappa)$). For any $q \in \mathbb{N}$, $q \leq N$ we introduce the following notation with $w = (w_1, w_2, w_3)$,

$$\begin{aligned} &W_j^q(T, X) = (Z^I w_j(T, X))_{|I| \leq q} \in (\mathbb{R})^q, \\ &W^q(T, X) = (Z^I w(T, X))_{|I| \leq q} \in (\mathbb{R}^3)^q, \\ &E(\widehat{W}^q, T) = \sum_{|I| \leq q, j=1,2,3} E(Z^I w_j, T). \end{aligned} \tag{3.2}$$

Set $\sigma = N + 1$ in Theorem **??**. We have to find a global solution to (**??**) when the Cauchy data

$$(w_1, w_2, w_3)|_{T=T_0} = w^0, \qquad (\partial_T w_1, \partial_T w_2, \partial_T w_3)|_{T=T_0} = w^1$$

satisfy

$$\|w^0\|_{H^{N+1}} + \|w^1\|_{H^N} \leq A\epsilon \tag{3.3}$$

for some $\epsilon > 0$ small enough. Since w^0 and w^1 are compactly supported, there is a constant $\widetilde{A}$ depending only on A such that

$$E(Z^I w^0, T_0) + E(Z^I w^1, T_0) < \widetilde{A}^2 \epsilon^2 < +\infty, \qquad |I| \leq N. \tag{3.4}$$

Using a similar argument as that of Lemma 2.2.1 in [**?**], we obtain

Lemma 3.1. *For any set of indices* I *with* $|I| \leq N$, *there is a bilinear map* $(p, p') \to B_I^j(T, X, p, p')$ $(j = 2, 3)$, *defined on* $\mathbb{C}^{8|I|} \times \mathbb{C}^{8|I'|}$, *with values in* $\mathbb{C}^2$, *with coefficients in* $\mathcal{E}^{-\kappa+3}$; *there is a trilinear map* $(p, p', p'') \to C_I^j(T, X, p, p', p'')$ $(j = 1, 2)$, *defined on* $\mathbb{C}^{8|I|} \times \mathbb{C}^{8|I'|} \times \mathbb{C}^{8|I''|}$, *with values in* $\mathbb{C}^2$, *with coefficients in* $\mathcal{E}^{-2\kappa+3}$; *there is a linear map* $p \to L_I^j(T, X, p)$, *defined on* $\mathbb{C}^{8|I|}$, *with values in* $\mathbb{C}^2$, *with coefficients belonging to* $\mathcal{E}^{-\kappa+3}$, *such that*

$$\begin{aligned} P_\kappa^1 Z^I w_1 &= \frac{1}{T^2} C_I^1(T, X, \widehat{D}^{|I|} W, \widehat{D}^{|I'|} W, \widehat{D}^{|I''|} W)) + \frac{1}{T^2} L_I^1(T, X, \widehat{D}^{|I|} W_1), \\ P_\kappa^2 Z^I w_2 &= \frac{1}{T} B_I^2(T, X, \widehat{D}^{|I|} W_2) + \frac{1}{T^2} C_I^2(T, X, \widehat{D}^{|I|} W, \widehat{D}^{|I'|} W, \widehat{D}^{|I''|} W)) \\ &\quad + \frac{1}{T^2} L_I^2(T, X, \widehat{D}^{|I|} W_2), \\ P_\kappa^3 Z^I w_3 &= \frac{1}{T} B_I^3(T, X, \widehat{D}^{|I|} W_2) + \frac{1}{T^2} L_I^3(T, X, \widehat{D}^{|I|} W_3), \end{aligned} \tag{3.5}$$

where $|I'| + |I''| \leq \frac{|I|}{2}$, *and*

$$\widehat{D}^{|I|}W = \Big(W^{|I|}, D_T W^{|I|}, \frac{D_X}{T} W^{|I|}\Big),$$
$$\widehat{D}^{|I|}W_j = \Big(W_j^{|I|}, D_T W_j^{|I|}, \frac{D_X}{T} W_j^{|I|}\Big).$$

Moreover the terms in the right hand side of (**??**) *are real valued when* $W^{|I|}$ *is.*

We shall assume from now on that we have a solution w to (**??**), defined on some interval $[T_0, T^*[$, and that w satisfies with a fixed constant $\mu' > 0$, for $T \in [T_0, T^*[$, an estimate

$$\sum_{j=1}^{2} \|W_j^{N'+1}(T,\cdot)\|_{L^\infty} \leq \mu'\epsilon, \qquad \|W_3^{N'+1}(T,\cdot)\|_{L^\infty} \leq \mu'\epsilon \ln T. \tag{3.6}$$

Our objective is to push forward this estimate to get a global solution. In the sequel we denote by $C(\mu')$ the different constants depending on μ', and any constant that does not depend on μ' is called absolute constant. Denote $E(T, W^{N-\ell}) = \sum_{j=1}^{3} E(T, W_j^{N-\ell})$.

Proposition 3.1. *There are constants* $C(\mu')$, $\epsilon_0 = \epsilon_0(\mu')$ *depending on* μ', *and for* $\ell = 0, \cdots, j_0(\kappa) + 1$, *absolute constants* ν_ℓ *such that, for any* w *satisfying* (**??**), *we have inequalities*

$$E(T, W^{N-\ell}) \leq \nu_\ell E(T_0, W^N) T^{\kappa + C(\mu')\epsilon - \ell/2}, \qquad \ell = 0, \cdots, j_0(\kappa), \tag{3.7}$$
$$E(T, W^{N-\ell}) \leq \nu_\ell E(T_0, W^N) T^{C(\mu')\epsilon}, \qquad \ell = j_0(\kappa) + 1. \tag{3.8}$$

Proof. We deduce from (**??**) and (**??**) the estimate

$$\begin{aligned}|P_\kappa^j(Z^I w_j)| &\leq \delta_{2j} \frac{C(\mu')\epsilon}{T} |\widehat{D}^{|I|}W_j(T,X)| + \Big\{\frac{C(\mu')\epsilon \ln^3 T}{T^2} + \frac{C}{T^2}\Big\} |\widehat{D}^{|I|}W(T,X)| \\ &\leq \Big\{\frac{C(\mu')\epsilon}{T} + \frac{C}{T^{3/2}}\Big\} |\widehat{D}^{|I|}W(T,X)|\end{aligned}$$

when $|I| \leq N$. Choosing ϵ so small that $C(\mu')\epsilon \leq 1$, we get

$$\begin{aligned}E(T, Z^I w) \leq E(T_0, Z^I w) &+ 2\kappa \int_{T_0}^{T} \int_{\mathbb{R}^2} |\partial_T Z^I w| \Big| \frac{X}{|X|} \cdot \frac{\nabla_X}{\tau}(Z^I w)\Big| (\tau, X) \frac{\sinh|X|}{|X|} dX \frac{d\tau}{\tau} \\ &+ C(\mu')\epsilon \int_{T_0}^{T} E(\tau, W^{|I|}) \frac{d\tau}{\tau} + C \int_{T_0}^{T} E(\tau, W^{|I|}) \frac{d\tau}{\tau^{3/2}}\end{aligned} \tag{3.9}$$

for an absolute constant $C > 0$ and a constant $C(\mu') > 0$ depending on μ'. If we bound in the right hand side of (**??**) the second term by $\kappa \int_{T_0}^{T} E(\tau, Z^I w) \frac{d\tau}{\tau}$, using Cauchy-Schwarz inequality, and summing (**??**) for $|I| \leq N$, we get

$$E(T, W^N) \leq E(T_0, W^N) + (\kappa + C(\mu')\epsilon) \int_{T_0}^{T} E(\tau, W^N) \frac{d\tau}{\tau} + C \int_{T_0}^{T} E(\tau, W^N) \frac{d\tau}{\tau^{3/2}}. \tag{3.10}$$

Next we use (**??**) for $|I| \le q \le N-1$. To bound the second term in the right hand side, we write, using the fact that ∇_X can be expressed in terms of the family Z_k,

$$\begin{aligned}
&\int |\partial_T Z^I w| \Big| \frac{X}{|X|} \cdot \frac{\nabla_X}{\tau} (Z^I w) \Big| (\tau, X) \frac{\sinh |X|}{|X|} dX \\
&\le \frac{1}{\tau} \int |\partial_T Z^I w| |\nabla_X Z^I w| \frac{\sinh |X|}{|X|} dX \le \frac{C}{\tau} E(W^{q+1}, \tau).
\end{aligned}$$

Summing for $|I| \le q$, one has

$$E(T, W^q) \le E(T_0, W^q) + C(\mu')\epsilon \int_{T_0}^T E(\tau, W^q) \frac{d\tau}{\tau} + C \int_{T_0}^T E(\tau, W^{q+1}) \frac{d\tau}{\tau^{3/2}}. \tag{3.11}$$

Using the Gronwall's inequality we get from (**??**) that, $K = \kappa + C(\mu')\epsilon$,

$$\begin{aligned}
E(T, W^N) &\le E(T_0, W^N) \Big(1 + C \int_{T_0}^T \Big(\frac{C}{s^{3/2}} + \frac{K}{s} \Big) \Big(\frac{T}{s} \Big)^K ds \Big) \\
&\le \nu_0 E(T_0, W^N) T^K
\end{aligned}$$

for an absolute constant ν_0, since $K > 0,\ T_0 \ge 1$. This gives (**??**) at rank $\ell = 0$.

We shall assume from now on that $\epsilon \in (0, \epsilon_0(\mu'))$ with $\epsilon_0(\mu')$ so small that $C(\mu')\epsilon < 1/8$. Let us assume that we have proved (**??**) for some index ℓ with $\ell < j_0(\kappa)$. Apply (**??**) with $q = N - \ell - 1$. We deduce from this inequality and the induction hypothesis

$$\begin{aligned}
E(T, W^{N-\ell-1}) \le E(T_0, W^{N-\ell-1}) + C(\mu')\epsilon \int_{T_0}^T E(\tau, W^{N-\ell-1}) \frac{d\tau}{\tau} \\
+ C \int_{T_0}^T \nu_\ell E(T_0, W^N) \tau^{K-\ell/2-3/2} \, d\tau.
\end{aligned} \tag{3.12}$$

The fact that $\ell + 1 \le j_0(\kappa) < 2\kappa$ and $\kappa \in \mathbb{N} + 1/8$ implies $K = \kappa + C(\mu')\epsilon \ge \ell/2 + 1/2$. Applying the Gronwall's inequality, we get

$$E(T, W^{N-\ell-1}) \le C' E(T_0, W^N) \Big[T^{K-\ell/2-1/2} + \int_{T_0}^T C(\mu')\epsilon s^{K-\ell/2-3/2} (T/s)^{C(\mu')\epsilon} ds \Big].$$

Since we assumed $C(\mu')\epsilon < 1/8$, $K - \ell/2 - 3/2 - C(\mu')\epsilon \ge -1$, whence we get an estimate

$$E(T, W^{N-\ell-1}) \le 2C' E(T_0, W^N) T^{K-\ell/2-1/2},$$

which gives (**??**) at rank $\ell + 1$.

We are left with proving the inequality (**??**). We have proved (**??**) when $\ell = j_0(\kappa)$. Since then $\kappa < j_0(\kappa) + 1 = \ell + 1$, we have $2\kappa = j(\kappa_0) + \frac{1}{4}$ and $K - \ell/2 - 3/2 = \kappa + C(\mu')\epsilon - \ell/2 - 3/2 \le -5/4$. Then (**??**) gives

$$E(T, W^{N-\ell-1}) \le C' E(T_0, W^N) + C(\mu')\epsilon \int_{T_0}^T E(\tau, W^{N-\ell-1}) \frac{d\tau}{\tau}$$

for some absolute constant $C' > 0$. The Gronwall's inequality implies that

$$\begin{aligned}
E(T, W^{N-\ell-1}) &\le C' E(T_0, W^N) \Big[1 + C(\mu')\epsilon \int_{T_0}^T (T/s)^{C(\mu')\epsilon} \frac{ds}{s} \Big] \\
&\le C' E(T_0, W^N) [1 + (T/T_0)^{C(\mu')\epsilon}],
\end{aligned}$$

which gives (**??**) since $T_0 \ge 1$.

Corollary 3.1. *Let $\delta > 0$. There is an absolute constant $C > 0$ and $\epsilon_0(\mu') > 0$ such that for any w satisfying (**??**) and any $\epsilon \in]0, \epsilon_0(\mu')[$, we have*

$$\|W^{N-j_0(\kappa)-3}(T,\cdot)\|_{L^\infty} \le CE(T_0, W^N)^{1/2}T^\delta, \qquad T \in [T_0, T^*[. \tag{3.13}$$

Proof. By (**??**), we have for any $j, k \in \{0, 1, 2\}$, since $\sinh|X|/|X| \ge 1$,

$$\|Z_j Z_k W^{N-j_0(\kappa)-3}(T,\cdot)\|^2_{L^2(dX)} \le \nu_\ell E(T_0, W^N)T^{C(\mu')\epsilon}.$$

If we take $\epsilon_0(\mu')$ such that $C(\mu')\epsilon < \delta$, (**??**) follows from this inequality and Sobolev embedding.

To obtain a uniform L^∞ estimate for the solution to (**??**) and its derivatives, we need the following lemma.

Lemma 3.2. *Let w be a solution to (**??**) satisfying (**??**). There is $\epsilon_0(\mu') > 0$, depending on μ', a quadratic map $(w_2, D_T w_2) \to q(T, X, w_2, D_T w_2) = (q^2, q^3)$ whose coefficients belong to $\mathcal{E}^{-\kappa+3}$, and a function $(T, X) \to R(T, X) = (R_1, R_2, R_3)$ satisfying*

$$\sum_{|I|\le N'+1} \|Z^I R(T,\cdot)\|^2_{L^\infty} \le CE(T_0, W^N)(1 + E^2(T_0, W^N)) \tag{3.14}$$

for an absolute constant C, such that, for $\epsilon \in]0, \epsilon_0(\mu')[$, $w = (w_1, w_2, w_3)$ satisfies the ordinary differential equation

$$\begin{aligned}
(D_T^2 - 4)w_1 &= \frac{1}{T^{3/2}}R_1,\\
(D_T^2 - 1)w_2 &= \frac{1}{T}q^2(X, w_2, D_T w_2) + \frac{1}{T^{3/2}}R_2,\\
(D_T^2 - 4)w_3 &= \frac{1}{T}q^3(X, w_2, D_T w_2) + \frac{1}{T^{3/2}}R_3.
\end{aligned} \tag{3.15}$$

Proof. By (**??**) we have, for $|I| \le N' + 1 \le N - j_0(\kappa) - 5$,

$$\begin{aligned}
&\Big|Z^I\Big(\frac{1}{T^2}P_j\Big(T, X, w, D_T w, \frac{D_X}{T}w\Big)\Big)\Big|\\
&\le CT^{-2+3\delta}E(T_0, W^N)^{3/2} \le CT^{-3/2}E(T_0, W^N)^{3/2},
\end{aligned} \tag{3.16}$$

which implies that $\frac{1}{T^2}P_j\big(T, X, w, D_T w, \frac{D_X}{T}w\big)$ is of form $\frac{1}{T^{3/2}}R_j$ with R_j satisfying (**??**). By Corollary **??** we have for any index I with $|I| \le N - j_0(\kappa) - 5$,

$$\|Z^I D_T^{\alpha_0}(D_X/T)^{\alpha'}w\|_{L^\infty} \le \frac{C}{T^{|\alpha'|}}E(T_0, W^N)^{1/2}T^\delta \tag{3.17}$$

when $\alpha_0 + |\alpha'| \le 2$. Consequently

$$\frac{1}{T}\left[Q_j(T, X, w_2, D_T w_2, (D_X/T)w_2) - Q(T, X, w_2, D_T w_2, 0)\right]$$

and $\left(P^j_\kappa - (D_T^2 - m_j^2)\right)w_j$ are of form $CT^{-2+2\delta}R_j$, with R_j satisfying (**??**).

Let us introduce some more notations. Set

$$u_j^{\pm} = (D_T \pm m_j)w_j, \qquad u^{\pm} = (u_1^{\pm}, u_2^{\pm}, u_3^{\pm}). \tag{3.18}$$

Denote by $H(T, X, u^+, u^-) = (0, H^2, H^3)$ the quadratic term $q = (0, q^2, q^3)$ of the right hand side of (**??**) in which we substitute

$$w_j = \frac{u_j^+ - u_j^-}{2m_j}, \qquad D_T w_j = \frac{u_j^+ + u_j^-}{2} \tag{3.19}$$

with $(m_1, m_2, m_3) = (2, 1, 2)$. Then H is a quadratic expression in u^+, u^-, with coefficients in $\mathcal{E}^{-\kappa+3}$, and we can write (**??**) as

$$\begin{aligned}
(D_T \mp 2)u_1^{\pm} &= \frac{1}{T^{3/2}} R_1, \\
(D_T \mp 1)u_2^{\pm} &= \frac{1}{T} H^2(X, u_2^+, u_2^-) + \frac{1}{T^{3/2}} R_2, \\
(D_T \mp 2)u_3^{\pm} &= \frac{1}{T} H^3(X, u_2^+, u_2^-) + \frac{1}{T^{3/2}} R_3,
\end{aligned} \tag{3.20}$$

where H^3 is given by $q^3\big(X, (u_2^+ - u_2^-)/2, (u_2^+ + u_2^-)/2\big)$ with coefficients in $\mathcal{E}^{-\kappa+3}$, the coefficients of $(u_2^+)^2$ and $(u_2^-)^2$ in H^3 are given by $\frac{1}{4}q^3(X, 1, 1)$ and $\frac{1}{4}q^3(X, 1, -1)$, q_3 is defined in (**??**). Remember that by (**??**) we have an estimate $E(Z^I w, T_0)^{1/2} \le \widetilde{A}\epsilon$. The main remaining step to prove global existence will be

Proposition 3.2. *There is an absolute constant $C_1 > 0$, and for any $\mu' > 0$, there is $\epsilon_0(\mu') \in (0, 1)$ such that for any $\epsilon \in (0, \epsilon_0(\mu'))$, any solution w to (**??**) satisfying (**??**) at $T = T_0$ and the estimate (**??**) on an interval $[T_0, T^*[$, we have, for any $T \in [T_0, T^*[$,*

$$\sum_{j=1}^{2} \|W_j^{N'+1}(T, \cdot)\|_{L^\infty} \le C_1 \widetilde{A}\epsilon, \qquad \|W_3^{N'+1}(T, \cdot)\|_{L^\infty} \le C_1 \widetilde{A}\epsilon \log T. \tag{3.21}$$

Proof. By (**??**) it is enough to control $\|Z^I u_j^{\pm}(T, \cdot)\|_{L^\infty}$ for $j = 1, 2, 3$, $|I| \le N' + 1$. Define

$$g(T) = \sum_{|I| \le N'+1} \sum_{k=1}^{2} (\|Z^I u_k^+(T, \cdot)\|_{L^\infty} + \|Z^I u_k^-(T, \cdot)\|_{L^\infty}). \tag{3.22}$$

We shall denote

$$\widetilde{E}_0 = E(T_0, W^N)^{1/2}(1 + E^2(T_0, W^N))^{1/2}. \tag{3.23}$$

Denote $(m_1, m_2, m_3) = (2, 1, 2)$. Consider a quadratic polynomial $\widetilde{Q}(Y_2^+, Y_2^-)$ in $2q$ indeterminates Y_2^+, Y_2^- and consider for $k = 2, 3$ the map $\mathcal{I}_{\mp}^k$ sending $\widetilde{Q}$ to

$$[Y_2^+(\partial\widetilde{Q}/\partial Y_2^+) - Y_2^-(\partial\widetilde{Q}/\partial Y_2^-)] \mp m_k \widetilde{Q}. \tag{3.24}$$

The action of $\mathcal{I}_{\delta''}^k$ $(\delta'' = \mp)$ on a monomial $Y_{2,a}^{\delta} \cdot Y_{2,a'}^{\delta'}$ with $\delta, \delta' \in \{+, -\}$, a, a' denoting the indices of the coordinates of $Y_2^{\delta}, Y_{j2}^{\delta'}$, is given by

$$(\delta m_2 + \delta' m_2 + \delta'' m_k) Y_{2,a}^{\delta} Y_{2,a'}^{\delta'}. \tag{3.25}$$

When $k=2$, one checks immediately that the coefficient in (**??**) never vanishes. This means that in this case the map is surjective on the space of quadratic polynomials. When $k=3$, the coefficient $\delta m_2+\delta' m_2'+\delta'' m_3$ vanishes only if $\delta=\delta'=-\delta''$. So the range of $\mathcal{I}^3_{\mp}$ contains all monomials except $Y^{\delta}_{2,a}Y^{\delta}_{2,a'}$ with $\delta=\pm$.

We now take $Y_j^{\pm}=(Z^J u_j^{\pm})_{|J|\le|I|}$, $j=1,2$. By the above properties of $\mathcal{I}^2_{\mp}$, we can choose $\widetilde{H}^{2,\pm}_I$ such that $\mathcal{I}^2_{\mp}\widetilde{H}^{2,\pm}_I=H^2_I$. Moreover, the coefficients of $\widetilde{H}^{2,\pm}_I$ belong to $\mathcal{E}^0$ since the coefficients of H^2_I are of $\mathcal{E}^0$. Then we have

$$
\begin{aligned}
&(D_T\mp 1)\Big[Z^I u_2^{\pm}-\frac{1}{T}\widetilde{H}^{2,\pm}_I(T,X,(Z^J u_2^{\pm})_{|J|\le|I|})\Big]\\
&=\frac{1}{iT^2}\widetilde{H}^{2,\pm}_I(T,X,(Z^J u_2^{\pm})_{|J|\le|I|})\\
&\quad-\frac{1}{T^2}\Big(\frac{\partial\widetilde{H}^2_{\mp}}{\partial Y_2^+}+\frac{\partial\widetilde{H}^2_{\mp}}{\partial Y_2^-}\Big)(H^2+T^{-1/2}R_I^{\pm,2})+\frac{1}{T^{3/2}}R_I^{\pm,2}\\
&=\frac{1}{T^2}S_I^{\mp,2}+\frac{1}{T^{3/2}}R_I^{\pm,2},
\end{aligned}
\tag{3.26}
$$

where $R_I^{\pm,2}$ satisfies (**??**), $S_I^{\mp,2}$ satisfies

$$
|S_I^{\mp,2}|\le Cg(T)(\widetilde{E}_0+g(T)+g^2(T)).
\tag{3.27}
$$

Now we rewrite H^3 as

$$
\begin{aligned}
Z^I H^3&=H^3_+(T,X,(Z^J u_2^+)_{|J|\le|I|})+H^3_-(T,X,(Z^J u_2^+)_{|J|\le|I|})\\
&\quad+\widehat{H}^3(T,X,(Z^J u_2^+,Z^{J'}u_2^+)_{|J|+|J'|\le|I|}).
\end{aligned}
$$

As above, there exists $\widetilde{H}^{3,\pm}_I(T,X,(Z^J u_2^{\pm})_{|J|\le|I|})$ such that

$$
\begin{aligned}
&(D_T\mp 2)\Big[Z^I u_3^{\pm}-\frac{1}{T}\widetilde{H}^{3,\pm}_I(T,X,(Z^J u_2^{\pm})_{|J|\le|I|})\Big]\\
&=\frac{1}{T}H^{3,\pm}_I(T,X,(Z^J u_2^{\pm})_{|J|\le|I|})+\frac{1}{T^2}S_I^{\mp,3}+\frac{1}{T^{3/2}}R_I^{\pm,3},
\end{aligned}
\tag{3.28}
$$

where $R_I^{\pm,3}$ satisfies (**??**), $S_I^{\mp,3}$ satisfies (**??**). Obviously we have

$$
(D_T\mp 2)[Z^I u_1^{\pm}]=\frac{1}{T^{3/2}}R_I^{\pm,1},
\tag{3.29}
$$

where $R_I^{\pm,1}$ satisfies (**??**).

If we conjugate (**??**) with $e^{\mp iT}$, (**??**) with $e^{\mp 2iT}$, and integrate, we get the estimate

$$
|Z^I u_1^{\pm}-e^{\mp 2i(T-T_0)}(Z^I u_1^{\pm})_{T=T_0}|\le\int_{T_0}^{T}|R_I^{\pm,1}|\frac{d\tau}{\tau^{3/2}}\le\widetilde{E}_0,
\tag{3.30}
$$

$$
\begin{aligned}
&\Big|\Big(Z^I u_2^{\pm}-\frac{1}{T}\widetilde{H}^2_{\pm}\Big)-e^{\mp 2i(T-T_0)}\Big(Z^I u_2^{\pm}-\frac{1}{T}\widetilde{H}^2_{\pm}\Big)_{T=T_0}\Big|\\
&\le\int_{T_0}^{T}|R_I^{\pm,2}|\frac{d\tau}{\tau^{3/2}}+\int_{T_0}^{T}|S_I^{\pm,2}|\frac{d\tau}{\tau^2}\\
&\le\widetilde{E}_0+\int_{T=T_0}^{T}g(\tau)(\widetilde{E}_0+g(\tau)+g^2(\tau))\frac{d\tau}{\tau^2}.
\end{aligned}
\tag{3.31}
$$

Note that $\widetilde{H}^2_\pm$ is controlled by $g(T)$. We deduce from (**??**) and (**??**) that

$$g(T) \le C\Big[\widetilde{E}_0 + g(T_0) + g^2(T_0) + g^2(T) + \int_{T_0}^{T} (\widetilde{E}_0 + g(\tau) + g^2(\tau))\frac{d\tau}{\tau^2}\Big]. \tag{3.32}$$

To estimate $g(T)$ in the right hand side of (**??**), we make use of the estimates deduced from the energy inequality in Corollary **??**: taking $\delta \le 1/4$ we have for $|I| \le N - j_0(\kappa) - 3$,

$$\|Z^I w(T,\cdot)\|_{L^\infty} \le CE(T_0, W^N)^{1/4} T^{1/2}. \tag{3.33}$$

If we take $N' + 2 \le N - j_0(\kappa) - 3$, we have a similar estimate for $\|Z^I u_k^\pm(T,\cdot)\|_{L^\infty}$ when $|I| \le N'+1$, and thus of $g(T)$. We use this inequality to control the term $(\widetilde{E}_0+1)\int_{T_0}^{T} g(\tau)\,\frac{d\tau}{\tau^{3/2}}$ in the right hand side of (**??**). We get

$$g(T) \le C\Big[g(T_0) + g(T_0)^2 + g(T)^2 + \sup_{[T_0,T]} g(\tau)^3 + \widetilde{E}_0(\widetilde{E}_0 + 1)\Big]. \tag{3.34}$$

By (**??**), $\widetilde{E}_0 \le \widetilde{A}\epsilon(1 + \widetilde{A}\epsilon)$, and by (**??**) at time $T = T_0$, $g(T_0) \le C\widetilde{A}\epsilon$ for an absolute constant $C > 0$. We deduce from (**??**) that there is a new absolute constant C such that when $\epsilon \in (0, \epsilon_0(\mu'))$,

$$g(T) \le C\Big[\widetilde{A}\epsilon + g(T)^2 + \sup_{[T_0,T]} g(\tau)^3\Big]$$

for any $T \in [T_0, T^*[$. This implies, taking $\epsilon_0(\mu')$ small enough and C_1 large enough with respect to C, that the first inequality in (**??**) is satisfied.

Denote $A_I^\pm(X,T_0) = e^{\mp iT_0}\big(Z^I u_2^\pm - \frac{1}{T}\widetilde{H}^2_\pm\big)_{T=T_0}$. Then $|A_I^\pm(X,T_0)| \le \widetilde{A}\epsilon(1 + \widetilde{A}\epsilon)$. (**??**), (**??**) and the first inequality in (**??**) imply that

$$|Z^I u_2^\pm - e^{\mp iT} A_I^\pm(X,T_0)| \le C_1\widetilde{A}\epsilon,$$

and

$$\Big|Z^I u_3^\pm - \frac{1}{T}\widetilde{H}_I^{3,\pm}(T, X, (Z^J u_2^\pm)_{|J|\le|I|}) - \int_{T_0}^{T} H_I^{3,\pm}(\tau, X, (A_J^\pm(X,T_0))_{|J|\le|I|})\frac{d\tau}{\tau}\Big| \le C_1\widetilde{A}\epsilon$$

for some positive constant C_1 large enough. Using the fact that the coefficients of $\widetilde{H}_I^{3,\pm}$ belong to $\mathcal{E}^0$, we deduce that

$$|Z^I u_3^\pm| \le g(T) + C\int_{T_0}^{T}\frac{d\tau}{\tau} + C_1\widetilde{A}\epsilon \le C_1\widetilde{A}\epsilon\ln T$$

for $|I| \le N' + 1$. This completes the proof of the second inequality in (**??**).

Proof of Theorem ??. The constants C_1 and $\widetilde{A}$ of Proposition **??** are independent of μ' of estimate (**??**). Consequently we may fix $\mu' = 2C_1\widetilde{A}$. Then Proposition **??** asserts that if ϵ is small enough and if we have on some interval $[T_0, T^*[$,

$$\sum_{j=1,2}\|W_j^{N'+1}(T,\cdot)\|_{L^\infty} \le 2C_1\widetilde{A}\epsilon, \qquad \|W_3^{N'+1}(T,\cdot)\|_{L^\infty} \le 2C_1\widetilde{A}\epsilon\ln T, \tag{3.35}$$

we get, on the same interval,

$$\sum_{j=1,2} \|W_j^{N'+1}(T,\cdot)\|_{L^\infty} \le C_1 \widetilde{A}\epsilon, \qquad \|W_3^{N'+1}(T,\cdot)\|_{L^\infty} \le C_1 \widetilde{A}\epsilon \ln T. \tag{3.36}$$

Since, by (**??**), (**??**) can be assumed to be valid at $T = T_0$ if C_1 has been fixed large enough, we deduce from the above property that (**??**) holds true on the whole interval of existence of the solution. Since $N'+1 \ge 2$, the classical blowing up criterion for solutions to quasi-linear wave equations implies the global existence.

§ 4. Asymptotic Behavior

In this section we shall prove Theorem **??**. We assume here that the Cauchy data are in $C_0^\infty(\mathbb{R}^2)$. Consequently, we can define $v = (v_1, v_2, v_3)$ in terms of (u_1, u_2) and define w in terms of v by (**??**) using a κ as large as we want. We go back to the equation (**??**). We remark that we have now a uniform estimate for $g(T)$ in (**??**). This shows that (**??**) is true with the remainders $T^{-3/2}R_j$ with $R_j = O(\epsilon)$ in L^∞. (**??**), (**??**) and (**??**) can be written now as

$$\begin{aligned}
&\Big|(D_T \mp 1)\Big[u_2^\pm - \frac{1}{T}\widetilde{H}_0^{2,\pm}\Big]\Big| \le \frac{C}{T^{3/2}}\epsilon,\\
&|(D_T \mp 2)u_1^\pm| \le \frac{C}{T^{3/2}}\epsilon,\\
&\Big|(D_T \mp 2)\Big[u_3^\pm - \frac{1}{T}\widetilde{H}_0^{3,\pm}(X,u_2^\pm)\Big] - \frac{1}{T}H^{3,\pm}(X,u_2^\pm)\Big| \le \frac{C}{T^{3/2}}\epsilon.
\end{aligned} \tag{4.1}$$

The first two equations in (**??**) imply that

$$\Big|\partial_T\Big[e^{\mp iT}\Big(u_2^\pm - \frac{1}{T}\widetilde{H}_0^{\pm,2}\Big)\Big]\Big| \le \frac{C}{T^{3/2}}\epsilon, \qquad |\partial_T(e^{\mp 2iT}u_1^\pm)| \le \frac{C}{T^{3/2}}\epsilon,$$

which show that there are L^∞ functions $a_1^\pm(X)$ and $a_2^\pm(X)$ such that

$$\|u_2^\pm(T,X) - e^{\pm iT}a_2^\pm(X)\|_{L^\infty(dX)} = O\Big(\frac{\epsilon}{T^{1/2}}\Big), \qquad T \to +\infty, \tag{4.2}$$

$$\|u_1^\pm(T,X) - e^{\pm 2iT}a_1^\pm(X)\|_{L^\infty(dX)} = O\Big(\frac{\epsilon}{T^{1/2}}\Big), \qquad T \to +\infty. \tag{4.3}$$

The last equation in (**??**) can be written as

$$\Big|\partial_T\Big[e^{\mp 2iT}\Big(u_3^\pm - \frac{1}{T}\widetilde{H}_0^{\pm,3}(X,u_2^\pm)\Big)\Big] - \frac{1}{4T}e^{\mp 2iT}q_3(X,1,\pm 1)(u_2^\pm)^2\Big| \le \frac{C}{T^{3/2}}\epsilon. \tag{4.4}$$

Combining (**??**) with (**??**) and (**??**) implies that there is an L^∞ function $a_3^\pm(X)$ such that, as $T \to +\infty$,

$$\Big\|u_3^\pm(T,X) - e^{\pm 2iT}a_3^\pm(X) + \frac{1}{4}e^{\pm 2iT}q_3(X,1,\pm 1)[a_2^\pm(X)]^2 \ln T\Big\|_{L^\infty(dX)} = O\Big(\frac{\epsilon}{T^{1/2}}\Big). \tag{4.5}$$

Reasoning in the same way on derivatives, we get that $a_j^\pm$, $j = 1,2,3$ are C^∞ with bounded derivatives. The conclusion will follow from the expression (**??**) of v in terms of w and so in terms of u, if we can write

$$\begin{aligned}&\frac{1}{T}e^{imT}b^{\pm}(X)(\cosh(\kappa|X|))^{-1}\ln T\\&=\frac{\ln t}{t}e^{im\sqrt{t^2-x^2}}\widetilde{b}_1^{\pm}(x/t)+\frac{1}{t}e^{im\sqrt{t^2-x^2}}\widetilde{b}_2^{\pm}(x/t)+\frac{\epsilon\ln t}{t^2}\rho(t,x)\end{aligned}\tag{4.6}$$

and

$$\frac{1}{T}e^{imT}a^{\pm}(X)(\cosh(\kappa|X|))^{-1}=\frac{1}{t}e^{im\sqrt{t^2-x^2}}\tilde{a}^{\pm}(x/t)+\frac{\epsilon}{t^2}\rho(t,x)\tag{4.7}$$

with $\tilde{a}^{\pm}$, $\tilde{b}_1^{\pm}$ and $\tilde{b}_2^{\pm}$ being C^∞ supported inside the closed unit ball and $\rho(t,x)$ satisfying

$$|\rho(t,x)|\le C\Big[\Big(1-\frac{|x|}{t}\Big)_+ +\frac{1}{t}\Big]^p,$$

where $\tilde{b}_1^{\pm}(x/t)=1\!\!1_{|x|<t}b^{\pm}(Y)(\cosh(\kappa|Y|))^{-1}C(x/t)$, $|C(x/t)|=1$, $Y=\frac{t}{|x|}\ln\frac{t+|x|}{t-|x|}$.

Remember that our change of coordinates is

$$t+2B=T\cosh|X|,\qquad x=TX\frac{\sinh|X|}{|X|},$$

whence

$$T=\sqrt{(t+2B)^2-|x|^2},\qquad X=\frac{x}{t}g\Big(\frac{x}{t},\frac{2B}{t}\Big)$$

with

$$g(y,s)=\frac{1}{2|y|}\ln\frac{1+s+|y|}{1+s-|y|}.$$

Moreover our solution v is supported for $|x|\le t+B$, i.e. $|y|\le 1+\frac{s}{2}$ if $y=x/t, s=2B/t$. It follows that on the support

$$\frac{T}{t}=[(1+s)^2-y^2]^{1/2}\sim[(1-|y|)_+ +s]^{1/2},\tag{4.8}$$

so

$$\cosh(\kappa|X|)^{-1}\le C\Big[\Big(1-\frac{|x|}{t}\Big)_+ +\frac{1}{t}\Big]^{\kappa/2}.\tag{4.9}$$

This shows that the contribution $\frac{1}{T}e^{imT}b^{\pm}(X)(\ln T)(\cosh(\kappa|X|))^{-1}1\!\!1_{|x|>t}$ to the left hand side of (**??**) can be incorporated to $\frac{\epsilon\ln t}{t^2}\rho$, if κ is large enough with respect to p. The fact that

$$|yg(y,s)-yg(y,0)|\le Cs[(1-|y|)_+ +s]^{-1}$$

when $|y|\le 1+s/2$ and

$$|\nabla_\theta[a^{\pm}(\theta)(\cosh(\kappa|\theta|))^{-1}]|\le C\epsilon(\cosh(\kappa|\theta|))^{-1}\le C\epsilon\Big[\Big(1-\frac{|x|}{t}\Big)_+ +\frac{1}{t}\Big]^{\kappa/2}$$

when $\theta\in[\frac{x}{t}g(\frac{x}{t},0),\frac{x}{t}g(\frac{x}{t},\frac{2B}{t})]$ implies that

$$\begin{aligned}&1\!\!1_{|x|<t}\Big|b^{\pm}(X)(\cosh(\kappa|X|))^{-1}-b^{\pm}\Big(\frac{x}{t}g\Big(\frac{x}{t},0\Big)\Big)\Big(\cosh\Big(\kappa\Big|\frac{x}{t}\Big|\Big|g\Big(\frac{x}{t},0\Big)\Big|\Big)\Big)^{-1}\Big|\\&\le\frac{C\epsilon}{t}\Big[\Big(1-\frac{|x|}{t}\Big)_+ +\frac{1}{t}\Big]^{-1+\kappa/2}.\end{aligned}\tag{4.10}$$

Using the expression

$$\frac{\ln T}{T}\frac{t}{\ln t}\sim\Big[\Big(1-\frac{|x|}{t}\Big)_+ +\frac{1}{t}\Big]^{-1/2}\Big\{1-\frac{1}{\ln t}\ln\Big[\Big(1-\frac{|x|}{t}\Big)_+ +\frac{1}{t}\Big]\Big\}$$

we deduce from (**??**) that the term

$$\frac{\ln T}{T}\mathbb{1}_{|x|<t}\Big|b^{\pm}(X)(\cosh(\kappa|X|))^{-1}-b^{\pm}\Big(\frac{x}{t}g\Big(\frac{x}{t},0\Big)\Big)\Big(\cosh\Big(\kappa\Big|\frac{x}{t}\Big|\Big|g\Big(\frac{x}{t},0\Big)\Big|\Big)\Big)^{-1}\Big|$$

gives also a contribution to the remainder in (**??**). If we set

$$\underline{b^{\pm}}=b^{\pm}(X)(\cosh(\kappa|X|))^{-1}|_{X=\frac{x}{t}g(\frac{x}{t},0)},$$

we are thus left with the term

$$\frac{\ln T}{T}e^{imT}\mathbb{1}_{|x|<t}\underline{b^{\pm}}(x/t). \tag{4.11}$$

Write, when $|y|<1$,

$$T=t\sqrt{1-y^2+2s+s^2}=t\Big[\sqrt{1-y^2}+\frac{s}{\sqrt{1-y^2}}+s^2h(s,y)\Big]$$

with $|h(s,y)|\le C(1-|y|)^{-3/2}$. We thus write (**??**) as

$$\frac{\ln t}{t}e^{im\sqrt{t^2-|x|^2}}\Big[\mathbb{1}_{|x|<t}\underline{b^{\pm}}(x/t)e^{im\frac{2B}{\sqrt{1-(|x|/t)^2}}}\Big]\frac{t\ln T}{T\ln t}e^{i\frac{4B^2}{t}h(2B/t,x/t)}. \tag{4.12}$$

The term between brackets is by definition of $\underline{b^{\pm}}$ smaller than $C\epsilon(1-\frac{|x|}{t})_+^{\kappa/2}$. The last exponential may be written as $1+O\big(\frac{1}{t}(1-\frac{|x|}{t})_+^{-3/2}\big)$. We also have

$$\frac{\ln T}{T}\frac{t}{\ln t}=\frac{1}{\sqrt{1-(|x|/t)^2}}\Big\{1+\frac{1}{\ln t}\ln\sqrt{1-(|x|/t)^2}\Big\}+O\Big(\frac{1}{t}(1-|x|/t)^{-5/2}\Big).$$

Plugging these expressions in (**??**) we get the principal part in the right hand side of (**??**) plus a contribution to the remainder. Using the similar approach we can prove (**??**).

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