

ON EFFECTIVE NON-VANISHING OF WEIL DIVISORS ON ALGEBRAIC SURFACES

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Abstract

The author gives a counterexample and some conclusions for effective non-vanishing of Weil divisors on algebraic surfaces.

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The basepoint-free theorem plays a crucial role in the Minimal Model Program since it yields a basepoint-free complete linear system, hence an extremal contraction morphism. More precisely, it says that if D is a nef Cartier divisor such that the difference between D and the log canonical divisor is nef and big, then the linear system $|mD|$ is basepoint-free for sufficiently large integer m (cf. [6, 7]). Obviously, it is natural and interesting to investigate the non-emptiness of the linear system $|D|$. Therefore, the following so-called Effective Non-vanishing Conjecture has been put forward formally by Yujiro Kawamata (cf. [5]).

Conjecture 1. *Let X be a complete normal variety, B an effective \mathbb{R} -divisor on X such that the pair (X, B) is Kawamata log terminal, and D a Cartier divisor on X . Assume that D is nef and that $D - (K_X + B)$ is nef and big. Then $H^0(X, D) \neq 0$.*

By the Kawamata-Viehweg vanishing theorem (cf. [6]), we have $H^p(X, D) = 0$ for any positive integer p . Thus the condition $H^0(X, D) \neq 0$ is equivalent to that $\chi(X, D) \neq 0$. Since the Riemann-Roch formula of the line bundle $\mathcal{O}_X(D)$ depends only on the intersections of Chern classes of X and D , we can say that the Effective Non-vanishing Conjecture is indeed a topological problem in a sense of complex geometry. This conjecture is of great importance by at least two reasons. At first, it provides a good tool to study the geometry of Fano varieties (cf. [1]). Secondly, it is closely related to the general elephant problem (cf. [?]).

Kawamata has proven that the Effective Non-vanishing Conjecture holds for all log surfaces with only Kawamata log terminal singularities (cf. [5]). In dimension three, this conjecture is more complicated, only a few results are known.

It is also interesting to consider a new problem by replacing Cartier with \mathbb{Q} -Cartier.

Problem 1. *Notations are the same as in Conjecture 1. Assume that D is a nef \mathbb{Q} -Cartier Weil divisor on X , such that $D - (K_X + B)$ is nef and big. Does $H^0(X, D) \neq 0$ hold?*

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Unfortunately, this problem does not hold generally. In dimension three, Fletcher has given the following famous example (cf. [4]).

Example 1. A general weighted complete intersection $X_{12,14}$ in $\mathbb{P}(2, 3, 4, 5, 6, 7)$ is a terminal \mathbb{Q} -Fano threefold of Picard number one with $H^0(-K_X) = 0$. This variety has the following isolated singularities: one of type $\frac{1}{5}(4, 1, 2)$, two of type $\frac{1}{3}(2, 1, 1)$, and seven of type $\frac{1}{2}(1, 1, 1)$.

In dimension two, such a counterexample of Problem 1 also exists, which is Example 2 given later in this paper. At first, we give some notations and definitions for the convenience of the reader.

Definition 1. A normal surface X is said to be log terminal (resp. canonical), if the following conditions are satisfied.

- (1) K_X is a \mathbb{Q} -Cartier divisor, i.e., there exists some positive integer r such that rK_X is Cartier, and
- (2) for any resolution $f : Y \rightarrow X$, if we write $K_Y = f^*K_X + \sum \alpha_i E_i$, where E_i are irreducible exceptional curves of f , then $\alpha_i > -1$ (resp. $\alpha_i \geq 0$) for all i .

The index of X is, by definition, the smallest positive integer r such that rK_X is Cartier. If the index of X is one, then X is said to be Gorenstein. If the resolution f is minimal, i.e., K_Y is f -nef, then we have that $\alpha_i \leq 0$ for all i since the intersection matrix $(E_i \cdot E_j)$ is negative definite.

Definition 2. A normal surface X is called a log del Pezzo surface, if X has only log terminal singularities and $-K_X$ is ample.

For a \mathbb{Q} -divisor $D = \sum a_i D_i$, the round up (resp. round down) of D is defined to be $\lceil D \rceil = \sum \lceil a_i \rceil D_i$ (resp. $\lfloor D \rfloor = \sum \lfloor a_i \rfloor D_i$), where $\lceil a \rceil$ (resp. $\lfloor a \rfloor$) is the smallest integer not less (resp. the largest integer not greater) than a .

Proposition 1. Let X be a log terminal surface, $f : Y \rightarrow X$ the minimal resolution of X . Assume that $K_Y = f^*K_X + \sum \alpha_i E_i$, where $-1 < \alpha_i \leq 0$ for all i . Then for each integer $n \geq 0$, $\chi(-nK_X) := \chi(X, \mathcal{O}_X(-nK_X))$ is equal to

$$\frac{n(n+1)}{2} K_Y^2 - \frac{2n+1}{2} K_Y \cdot \left(\sum \lceil (n+1)\alpha_i \rceil E_i \right) + \frac{1}{2} \left(\sum \lceil (n+1)\alpha_i \rceil E_i \right)^2 + \chi(\mathcal{O}_Y).$$

Proof. Let $D = -nK_X$. Then we have

$$\begin{aligned} f^*(D - K_X) &= -(n+1)f^*K_X = -(n+1)\left(K_Y - \sum \alpha_i E_i\right) \\ &= \left(-nK_Y + (n+1) \sum \alpha_i E_i\right) - K_Y. \end{aligned}$$

We define \tilde{D}, \tilde{B} as follows:

$$\begin{aligned} \tilde{D} &= \left[-nK_Y + (n+1) \sum \alpha_i E_i\right] = -nK_Y + \sum \lceil (n+1)\alpha_i \rceil E_i, \\ \tilde{B} &= \sum (\lceil (n+1)\alpha_i \rceil - (n+1)\alpha_i) E_i. \end{aligned}$$

It is easy to see that $f^*(D - K_X) = \tilde{D} - (K_Y + \tilde{B})$ is an f -nef and f -big divisor, and \tilde{B} has simple normal crossing support. By the Kawamata-Viehweg vanishing theorem, we have $R^j f_* \mathcal{O}_Y([\tilde{D} - \tilde{B}]) = R^j f_* \mathcal{O}_Y(\tilde{D}) = 0$ for any positive integer j .

Since we have

$$f^* D = -nK_Y + \sum n\alpha_i E_i = \tilde{D} + \sum (n\alpha_i - \lceil (n+1)\alpha_i \rceil) E_i,$$

$$n\alpha_i - \lceil (n+1)\alpha_i \rceil < (n+1)\alpha_i - \lceil (n+1)\alpha_i \rceil + 1 \leq 1,$$

there exists a \mathbb{Q} -divisor $H = \sum h_i E_i$ on Y such that $0 \leq h_i < 1$ and $h_i - (n\alpha_i - \lceil (n+1)\alpha_i \rceil) \geq 0$ for each i . For a \mathbb{Q} -divisor M on Y , the sheaf $\mathcal{O}_Y(M)$ is defined to be $\mathcal{O}_Y(\lfloor M \rfloor)$. Thus $f_* \mathcal{O}_Y(\tilde{D}) = f_* \mathcal{O}_Y(\tilde{D} + H) = \mathcal{O}_X(D)$ by the projection formula. It follows from [3] that we have the Leray spectral sequence

$$E_2^{i,j} = H^i(X, R^j f_* \mathcal{O}_Y(\tilde{D})) \Rightarrow H^{i+j}(Y, \mathcal{O}_Y(\tilde{D})).$$

Since $E_2^{i,j} = 0$ for all $i \geq 0$ and $j > 0$, $E_2^{i,0} \cong E_\infty^{i,0}$ implies that $h^i(X, \mathcal{O}_X(D)) = h^i(Y, \mathcal{O}_Y(\tilde{D}))$ for all $i \geq 0$, hence $\chi(X, \mathcal{O}_X(D)) = \chi(Y, \mathcal{O}_Y(\tilde{D}))$. Therefore we have

$$\begin{aligned} & \chi(-nK_X) \\ &= \chi(X, \mathcal{O}_X(D)) = \chi(Y, \mathcal{O}_Y(\tilde{D})) = \frac{1}{2} \tilde{D}(\tilde{D} - K_Y) + \chi(\mathcal{O}_Y) \\ &= \frac{1}{2} \left(-nK_Y + \sum \lceil (n+1)\alpha_i \rceil E_i \right) \left(-(n+1)K_Y + \sum \lceil (n+1)\alpha_i \rceil E_i \right) + \chi(\mathcal{O}_Y) \\ &= \frac{n(n+1)}{2} K_Y^2 - \frac{2n+1}{2} K_Y \cdot \left(\sum \lceil (n+1)\alpha_i \rceil E_i \right) + \frac{1}{2} \left(\sum \lceil (n+1)\alpha_i \rceil E_i \right)^2 + \chi(\mathcal{O}_Y). \end{aligned}$$

Example 2. There exists a log del Pezzo surface S of Picard number one such that $H^0(S, -K_S) = 0$.

Let B be a smooth conic in \mathbb{P}^2 , D a tangent line to B at d , and A a secant line intersecting B at $\{a, b\}$. Let $g_1 : S_1 \rightarrow \mathbb{P}^2$ be the blow-up of \mathbb{P}^2 along $d \in D$ with an exceptional curve L_1 . By abuse of notation, we always denote the strict transform of a curve by the same symbol. Let $g_i : S_{i+1} \rightarrow S_i$ be the blow-up of S_i along $d \in D$ with exceptional curves L_{i+1} for $i = 1, 2$. It is easy to see that $L_1^2 = L_2^2 = -2$, $L_3^2 = -1$. In a similar way, let $g_4 : S_4 \rightarrow S_3$ be the blow-up of S_3 5 times along $b \in B$. Assume that the exceptional curves of g_4 are denoted by M_1, M_2, M_3, M_4, M_5 respectively with $M_1^2 = M_2^2 = M_3^2 = M_4^2 = -2$, and $M_5^2 = -1$. Let $g_5 : \tilde{S} \rightarrow S_4$ be the blow-up of S_4 5 times along $a \in A$. Assume that the exceptional curves of g_5 are denoted by N_1, N_2, N_3, N_4, N_5 respectively with $N_1^2 = N_2^2 = N_3^2 = N_4^2 = -2$, and $N_5^2 = -1$. Denote by $g : \tilde{S} \rightarrow \mathbb{P}^2$ the composition of all g_i .

We consider two reduced connected curves $X_1 := D \cup A \cup M_1 \cup M_2 \cup M_3 \cup M_4$ and $X_2 := L_1 \cup L_2 \cup B \cup N_1 \cup N_2 \cup N_3 \cup N_4$ on \tilde{S} . It is easy to verify that the intersection matrix associated with X_i is negative definite, and the arithmetic genus of any positive cycle with support on X_i is non-positive for $i = 1, 2$. Thus $X_1 \cup X_2$ is contractible, and there is a contraction $f : \tilde{S} \rightarrow S$ such that S is a normal projective surface of Picard number one with only two singular points x_1, x_2 , which is the contraction point of X_1, X_2 respectively (cf. [2]).

It is easy to see that \tilde{S} is just the minimal resolution of S , and the dual graph Γ consists of two connected components corresponding to x_1, x_2 respectively.

$$\Gamma: \begin{array}{cccccccccccccccc} 2^D & 5^A & 2^{M_1} & 2^{M_2} & 2^{M_3} & 2^{M_4} & 2^{L_1} & 2^{L_2} & 4^B & 2^{N_1} & 2^{N_2} & 2^{N_3} & 2^{N_4} \end{array}$$

where 2^D denotes the (-2) -curve on \tilde{S} which is the strict transform of D , and so on.

By an elementary calculation, we have

$$\begin{aligned} K_{\tilde{S}} = f^* K_S - \frac{15}{37} D - \frac{30}{37} A - \frac{24}{37} M_1 - \frac{18}{37} M_2 - \frac{12}{37} M_3 - \frac{6}{37} M_4 \\ - \frac{5}{19} L_1 - \frac{10}{19} L_2 - \frac{15}{19} B - \frac{12}{19} N_1 - \frac{9}{19} N_2 - \frac{6}{19} N_3 - \frac{3}{19} N_4. \end{aligned}$$

It follows easily that $K_{\tilde{S}}^2 = 8/(37 \times 19) > 0$, then $-K_{\tilde{S}}$ is ample since S is rational and $\rho(S) = 1$. Furthermore, we have $H^0(S, -nK_S) = 0$ for $1 \leq n \leq 5$, but $H^0(S, -6K_S) \cong \mathbb{C}$ by Proposition 1.

Remark 1. For a log del Pezzo surface X , we define $\tau(X)$ to be the smallest positive integer τ such that $H^0(X, -\tau K_X) \neq 0$. It is easy to see that the index $r(X)$ of X is not less than $\tau(X)$. The study of complements on surfaces is helpful to solve the non-vanishing of $H^0(X, -nK_X)$ for log del Pezzo surfaces. Shokurov proved that there exists a positive integer N such that $\tau(X) \leq N$ for all log del Pezzo surfaces (cf. [?]). In particular, Prokhorov used the same method to yield that $H^0(X, -nK_X) \neq 0$ for some $n \in \mathcal{R}_2 := \{1, 2, 3, 4, 6\}$, if $K_X^2 > 4$ (cf. [9]).

Example 2 illustrates that Problem 1 does not hold generally for surfaces. But, when some reasonable condition is assumed, we hope that the effective non-vanishing of Weil divisors holds. We give the following theorem as a special example.

Theorem 1. *Let X be a Gorenstein log del Pezzo surface of Picard number one, D a nef and big \mathbb{Q} -Cartier Weil divisor on X . Then $H^0(X, D) \neq 0$.*

It is easy to see that X has only canonical singularities. The condition $\rho(X) = 1$ enables us to list all types of its singularities.

Lemma 1. *Let X be a Gorenstein log del Pezzo surface with $\rho(X) = 1$. Then its singularity type is one of the following:*

$A_1, A_1 + A_2, A_4, 2A_1 + A_3, D_5, A_1 + A_5, 3A_2, E_6, 3A_1 + D_4, A_7, A_1 + D_6, E_7, A_1 + 2A_3, A_2 + A_5, D_8, 2A_1 + D_6, E_8, A_1 + E_7, A_1 + A_7, 2A_4, A_8, A_1 + A_2 + A_5, A_2 + E_6, A_3 + D_5, 4A_2, 2A_1 + 2A_3, 2D_4.$

Proof. X is said to be of type, for example, $2A_1 + A_3$, if X has three singular points, two of type A_1 , and one of type A_3 . We refer to [8] for the proof.

Definition 3. *The quotient $X = \mathbb{A}^n / \mu_r$ is said to be of cyclic quotient singularity of type $\frac{1}{r}(a_1, \dots, a_n)$, if μ_r acts on \mathbb{A}^n by*

$$\mu_r \ni \varepsilon : (x_1, \dots, x_n) \mapsto (\varepsilon^{a_1} x_1, \dots, \varepsilon^{a_n} x_n).$$

Let $\pi : \mathbb{A}^n \rightarrow X$ be the quotient morphism. Then the group μ_r acts on $\pi_* \mathcal{O}_{\mathbb{A}^n}$, and so decomposes it into r eigensheaves $\mathcal{L}_i = \{f | \varepsilon(f) = \varepsilon^i \cdot f \text{ for all } \varepsilon \in \mu_r\}$ for $i = 0, \dots, r-1$.

A singularity $p \in X$ with a Weil divisor D is said to be of cyclic quotient singularity of type $i(\frac{1}{r}(a_1, \dots, a_n))$, if locally, $p \in X$ is isomorphic to a point of type $\frac{1}{r}(a_1, \dots, a_n)$, and $\mathcal{O}_X(D) \cong \mathcal{L}_i$.

There are two keypoints in the proof of Theorem 1. First, one canonical singularity can be considered as the cyclic quotient of another canonical singularity by using the local cyclic cover. Second, we can calculate the Euler characteristic of a Weil divisor, thanks to the Singular Riemann-Roch Formula.

The following lemma is helpful, which can be found in [?].

Lemma 2. *Let $Q \in Y : (f = 0) \subset \mathbb{A}^3$ be a surface germ. Assume that there is an action of μ_r on Y free outside Q . Let $p \in X = (Q \in Y)/\mu_r$ be the surface germ and $\pi : Y \rightarrow X$ the corresponding cyclic cover. If $p \in X$ is canonical, then $Q \in Y$ is also canonical and $\pi : Y \rightarrow X$ is exactly one of the following types.*

	r	Type	f	Description
(1)	any	$\frac{1}{r}(1, -1, 0)$	$xy + z^n$	$A_{n-1} \xrightarrow{r:1} A_{rn-1}$
(2)	4	$\frac{1}{4}(1, 3, 2)$	$x^2 + y^2 + z^{2n-1}$	$A_{2n-2} \xrightarrow{4:1} D_{2n+1}$
(3)	2	$\frac{1}{2}(0, 1, 1)$	$x^2 + y^2 + z^{2n}$	$A_{2n-1} \xrightarrow{2:1} D_{n+2}$
(4)	3	$\frac{1}{3}(0, 1, 2)$	$x^2 + y^3 + z^3$	$D_4 \xrightarrow{3:1} E_6$
(5)	2	$\frac{1}{2}(1, 1, 0)$	$x^2 + y^2 z + z^n$	$D_{n+1} \xrightarrow{2:1} D_{2n}$
(6)	2	$\frac{1}{2}(1, 0, 1)$	$x^2 + y^3 + z^4$	$E_6 \xrightarrow{2:1} E_7$

Proof of Theorem 1. If D is a Cartier divisor, then the non-vanishing follows from [?]. Now assume that D is not Cartier. For the reflexive sheaf $\mathcal{O}_X(D)$, we have the following Singular Riemann-Roch Formula (cf. [?]):

$$\chi(X, \mathcal{O}_X(D)) = \frac{1}{2}D \cdot (D - K_X) + \chi(\mathcal{O}_X) + \sum_{p \in X} c_p(D),$$

where $c_p(D) \in \mathbb{Q}$ is a contribution due to the singularity of $\mathcal{O}_X(D)$ at p , the sum is taken over the points $p \in X$ at which D is not Cartier.

If $p \in X$ and D is of type $i(\frac{1}{r}(1, -1))$, then $c_p(D) = -i(r - i)/2r$. For other cases, we use the suitable Q -smoothing of X , then there is a flat deformation $\{X_\lambda, D_\lambda\}$ of $\{X, D\}$ such that X_λ has only cyclic quotient singularities. Thus $c_p(D)$ is equal to the sum of the contributions of cyclic quotient singularities of X_λ . The concrete conclusions are listed below.

	X_λ	Q -smoothing points
(1)	$f + \lambda z$	n
(2)	$f + \lambda z$	$2n + 1$
(3)	$f + \lambda x$	2
(4)	$f + \lambda x$	2
(5)	$f + \lambda z$	n
(6)	$f + \lambda y$	3

Since $-K_X$ is an ample Cartier divisor and D is a nef and big Weil divisor, it is easy to show that $\chi(\mathcal{O}_X) = 1$ and $D \cdot (-K_X) \geq 1$. Thus $h^0(X, D) \geq D^2/2 + 3/2 + \sum_{p \in X} c_p(D)$.

Next, we will estimate $c_p(D)$ for a given germ $p \in X$ at which D is not Cartier. Since the cyclic covers in Lemma 2 can be constructed locally, and the contribution $c_p(D)$ depends only on the local analytic type of $p \in X$ and D , we can take $\pi : Y \rightarrow X$ to be a suitable cyclic cover in Lemma 2 and estimate $c_p(D)$ as follows:

Let $p \in X$ be of type A_{n-1} . Then $p \in X$ is a cyclic quotient singularity of type $\frac{1}{n}(1, -1)$. By definition, we have $c_p(D) = -i_0(n - i_0)/2n$ for some integer $i_0 \in [0, n]$. If $n = 2k$, then $c_p(D) \geq -k/4$. If $n = 2k + 1$, then $c_p(D) \geq -k(k + 1)/2(2k + 1)$.

Let $p \in X$ be of type D_{n+2} . We may consider $p \in X$ as a cyclic quotient of type (3). Then $c_p(D) \geq -1/4 \times 2 = -1/2$.

Let $p \in X$ be of type E_6 . We may consider $p \in X$ as a cyclic quotient of type (4). Then $c_p(D) \geq -1/3 \times 2 = -2/3$.

Let $p \in X$ be of type E_7 . We may consider $p \in X$ as a cyclic quotient of type (6). Then $c_p(D) \geq -1/4 \times 3 = -3/4$.

Since E_8 does not appear in the table of Lemma 2, $p \in X$ could not be of type E_8 provided that D is not Cartier at p .

It is easy to check for all cases in Lemma 1 that $\sum_{p \in X} c_p(D) \geq -3/2$. Hence $h^0(X, D) \geq D^2/2 > 0$.

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