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SELF-CANCELLATION OF MODULES HAVING THE FINITE EXCHANGE PROPERTY

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Abstract

Self-cancellation of modules having the finite exchange property is introduced. If a right R-module M has the finite exchange property, it is shown that M has self-cancellation if and only if $\mathrm{End}_R(M)$ is a strongly separative ring. Using this result, some new characterizations of strong separativity are obtained.

Keywords Self-cancellation, Strong separativity
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§ 1. Introduction

A right R-module M has the finite exchange property if for every right R-module A and any decompositions $A = M' \oplus N = \bigoplus\limits_{i\in I} A_i$, where $M' \cong M$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M' \oplus \Big(\bigoplus\limits_{i\in I} A_i'\Big)$. If a ring R as a right R-module has the finite exchange property, we say that R is an exchange ring (see [10]). It is well known that a right R-module M has the finite exchange property if and only if $\mathrm{End}_R(M)$ is an exchange ring. Following Ara et al. (see [3]), a ring R is said to be strongly separative if for all finitely generated projective right R-modules A, B we have $A \oplus A \cong A \oplus B \Longrightarrow A \cong B$. Strong separativity is very useful in a number of various cancellation problems for modules over exchange rings.

An abelian group A has self-cancellation if $A \oplus A \cong A \oplus B$ implies that $A \cong B$ (see [5]). By [5, Corollary 8.19], every almost completely decomposable torsion free group of finite rank has self-cancellation. In this paper, we extend this concept to modules and introduce self-cancellation for modules having the finite exchange property. If a right R-module M has the finite exchange property, it is shown that M has self-cancellation if and only if $\mathrm{End}_R(M)$ is a strongly separative ring. Using this fact, we get some new characterizations of strong separativity.

Throughout, all rings are associative with identity and all modules are unitary right modules. The symbol $M \lesssim^{\oplus} N$ means that M is isomorphic to a direct summand of a module N and nM means that the direct sum of n copies of the R-module M. Let $\mathrm{add}(M_R)$ denote the full subcategory of Mod-R whose objects are all the modules isomorphic to direct summands of direct sums nM for a finite number of copies of M.

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§ 2 . Self-cancellation of Modules

In [4], Ara et al. investigated regular rings having small projectives. Let R be a regular ring. We say that R has cancellation of small projectives if for all finitely generated projective right R-modules $A, B, C,\ A\oplus C\cong B\oplus C$ and $C\lesssim^{\oplus} nA$ for some $n\in\mathbb{N}\Longrightarrow A\cong B$. By [3, Lemma 5.1], a regular ring R has cancellation of small projectives if and only if it is an strongly separative ring. We now extend this concept to modules having the finite exchange property.

Definition 2.1. *Let M be a right R-module with the finite exchange property. We say that M has self-cancellation if for all $A, B, C\in \mathrm{add}_R(M),\ A\oplus C\cong B\oplus C$ and $C\lesssim^{\oplus} nA$ for some $n\in\mathbb{N}\Longrightarrow A\cong B$.*

Clearly, an exchange ring R is an strongly separative ring if and only if it has self-cancellation as a right R-module.

Lemma 2.1. *Let M be a right R-module with the finite exchange property. For any right R-modules B and C, if $\psi: M\oplus B\cong M\oplus C$, then we have a refinement matrix*

$$\begin{array}{cc} & \begin{array}{cc} M & B \end{array} \\ \begin{array}{c} M \\ C \end{array} & \begin{pmatrix} M_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \end{array}.$$

That is, $M\cong M_1\oplus C_1\cong M_1\oplus B_1,\ B\cong B_1\oplus D_1$ and $C\cong C_1\oplus D_1$.

Proof. The result follows analogously to [14, Theorem 3.1].

Theorem 2.1. *Let M be a right R-module with the finite exchange property. Then the following statements are equivalent:*

(1) *M has self-cancellation.*

(2) *For any $A, B, C\in \mathrm{add}_R(M),\ A\oplus C\cong B\oplus C$ with $C\lesssim^{\oplus} A\Longrightarrow A\cong B$.*

(3) *For any $A, B\in \mathrm{add}_R(M),\ A\oplus A\cong A\oplus B\Longrightarrow A\cong B$.*

Proof. (1)⇒(3) is clear.

(3)⇒(2). Suppose that $A\oplus C\cong B\oplus C,\ C\lesssim^{\oplus} A$ and $A, B\in \mathrm{add}_R(M)$. Then we have $A\cong C\oplus D$ for a right R-module D. Hence

$$2(C\oplus D)\cong (A\oplus C)\oplus D\cong B\oplus (C\oplus D),$$

and then $A\cong C\oplus D\cong B$.

(2)⇒(1). Suppose that $A\oplus C\cong B\oplus C$ and $C\lesssim^{\oplus} nA$ $(n\in\mathbb{N}),\ A, B\in \mathrm{add}_R(M)$. Since M is a right R-module with the finite exchange property, so is C. From $C\lesssim^{\oplus} nA$, there exists a right R-module D such that $C\oplus D\cong A\oplus (n-1)A$. By Lemma 2.1, we have a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} A \\ (n-1)A \end{array} & \begin{pmatrix} C_1 & D_1 \\ B_1 & E_1 \end{pmatrix} \end{array}.$$

So we have $B_1\oplus E_1\cong A\oplus (n-2)A$. Similarly, we have a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} B_1 & E_1 \end{array} \\ \begin{array}{c} A \\ (n-2)A \end{array} & \begin{pmatrix} C_2 & D_2 \\ B_2 & E_2 \end{pmatrix} \end{array}.$$

Furthermore, we get a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} B_{n-2} & E_{n-2} \end{array} \\ \begin{array}{c} A \\ A \end{array} & \left(\begin{array}{cc} C_{n-1} & D_{n-1} \\ B_{n-1} & E_{n-1} \end{array} \right) \end{array}.$$

From these refinement matrices, it follows that

$$C \cong C_1 \oplus B_1 \cong C_1 \oplus (C_2 \oplus B_2) \cong \cdots \cong C_1 \oplus C_2 \oplus \cdots \oplus C_{n-1} \oplus B_{n-1}.$$

Set $C_n = B_{n-1}$. So $C = C_1 \oplus C_2 \oplus \cdots \oplus C_n$ with $C_1, \cdots, C_n \lesssim^{\oplus} A$; hence

$$C_1 \oplus \cdots \oplus C_n \oplus A \cong C_1 \oplus \cdots \oplus C_n \oplus B.$$

As $C_1 \lesssim^{\oplus} A$, we deduce that

$$C_2 \oplus \cdots \oplus C_n \oplus A \cong C_2 \oplus \cdots \oplus C_n \oplus B.$$

Furthermore, we get $A \cong B$, as required.

Lemma 2.2. *Let M be a right R-module with the finite exchange property. For any right R-modules B and C, if $M \oplus B \cong M \oplus C$ with $M \lesssim^{\oplus} B$ then we have a refinement matrix*

$$\begin{array}{cc} & \begin{array}{cc} M & B \end{array} \\ \begin{array}{c} M \\ C \end{array} & \left(\begin{array}{cc} M_1 & B_1 \\ C_1 & D_1 \end{array} \right) \end{array}$$

with $M_1 \lesssim^{\oplus} B_1$.

Proof. Suppose that $M \oplus B \cong M \oplus C$ with $M \lesssim^{\oplus} B$. By Lemma 2.1, we have a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} M & B \end{array} \\ \begin{array}{c} M \\ C \end{array} & \left(\begin{array}{cc} M_1 & B_1 \\ C_1 & D_1 \end{array} \right) \end{array}.$$

Clearly

$$M_1 \lesssim^{\oplus} M \lesssim^{\oplus} B \cong B_1 \oplus D_1.$$

So M_1 has the finite exchange property, and $M_1 \oplus D \cong B_1 \oplus D_1$ for a right R-module D. By Lemma 2.1 again, we have a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} M_1 & D \end{array} \\ \begin{array}{c} B_1 \\ D_1 \end{array} & \left(\begin{array}{cc} M_2 & B_1' \\ M_2' & D_1' \end{array} \right) \end{array}.$$

Hence $M_1 \cong M_2 \oplus M_2'$ with $M_2 \lesssim^{\oplus} B_1$ and $M_2' \lesssim^{\oplus} D_1$. So we have a right R-module D_2 such that $D_1 \cong M_2' \oplus D_2$. Thus we have a new refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} M & B \end{array} \\ \begin{array}{c} M \\ C \end{array} & \left(\begin{array}{cc} M_2 & B_2 \\ C_2 & D_2 \end{array} \right) \end{array},$$

where $B_2 = M_2' \oplus B_1$ and $C_2 = M_2' \oplus C_1$. In addition, $M_2 \lesssim^{\oplus} B_1 \lesssim^{\oplus} B_2$, as asserted.

Lemma 2.3. *Let M be a right R-module with the finite exchange property. If $A \in \mathrm{add}_R(M)$, then there exist idempotents $e_1, \cdots, e_n \in \mathrm{End}_R(M)$ such that $A \cong e_1M \oplus \cdots \oplus e_nM$.*

Proof. Since $A \in \mathrm{add}(M_R)$, we can find a right R-module B such that $A \oplus B \cong nM$ for some $n \in \mathbb{N}$. Clearly, A also has the finite exchange property. Similarly to Lemma 2.1, we have decompositions $A = A_1 \oplus \cdots \oplus A_n, B = B_1 \oplus \cdots \oplus B_n$ and $A_i \oplus B_i \cong M$ for $i = 1, \cdots, n$. Thus, there exists $e_i = e_i^2 \in \mathrm{End}_R(M)$ such that $A_i \cong e_iM$ for $i = 1, \cdots, n$. Hence $A \cong e_1M \oplus \cdots \oplus e_nM$, as asserted.

Theorem 2.2. *Let M be a right R-module with the finite exchange property. Then the following statements are equivalent:*

(1) *M has self-cancellation.*

(2) *For any $C \in \mathrm{add}_R(M)$, $A \oplus C \cong B \oplus C$ with $C \lesssim^{\oplus} A \Longrightarrow A \cong B$ for any right R-modules A and B.*

Proof. (2)$\Rightarrow$(1) is clear by Theorem 2.1.

(1)$\Rightarrow$(2). Suppose that $C \in \mathrm{add}_R(M)$ and $C \oplus A \cong C \oplus B$ with $C \lesssim^{\oplus} A$. In view of Lemma 2.3, we have idempotents $e_1, \cdots, e_n \in \mathrm{End}_R(M)$ such that $C \cong e_1M \oplus \cdots \oplus e_nM$. So

$$e_1M \oplus (e_2M \oplus \cdots \oplus e_nM \oplus A) \cong e_1M \oplus (e_2M \oplus \cdots \oplus e_nM \oplus B).$$

Set

$$A_1 = e_2M \oplus \cdots \oplus e_nM \oplus A \quad \text{and} \quad B_1 = e_2M \oplus \cdots \oplus e_nM \oplus B.$$

Then

$$e_1M \oplus A_1 \cong e_1M \oplus B_1$$

with $e_1M \lesssim^{\oplus} A_1$. Clearly, e_1M has the finite exchange property. Using Lemma 2.2, we have a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} e_1M & A_1 \end{array} \\ \begin{array}{c} e_1M \\ B_1 \end{array} & \begin{pmatrix} M_2 & A_2 \\ B_2 & C_2 \end{pmatrix} \end{array}$$

with $M_2 \lesssim^{\oplus} A_2$. Clearly

$$M_2 \oplus A_2 \cong M_2 \oplus B_2 \cong e_1M \lesssim^{\oplus} M.$$

It follows by Theorem 2.1 that $A_2 \cong B_2$, hence $A_1 \cong A_2 \oplus C_2 \cong B_2 \oplus C_2 \cong B_1$. That is,

$$e_2M \oplus \cdots \oplus e_nM \oplus A \cong e_2M \oplus \cdots \oplus e_nM \oplus B.$$

Likewise, we claim that

$$e_3M \oplus \cdots \oplus e_nM \oplus A \cong e_3M \oplus \cdots \oplus e_nM \oplus B.$$

Furthermore, we conclude that $A \cong B$, as required.

Corollary 2.1. *Let M be a right R-module with the finite exchange property. Then the following statements are equivalent:*

(1) *M has self-cancellation.*

(2) *For any $C \in \mathrm{add}(M_R)$, $C \oplus A \cong C \oplus B$ and $C \lesssim^{\oplus} nA$ for some $n \in \mathbb{N} \Longrightarrow A \cong B$ for any right R-modules A and B.*

Proof. (2)⇒(1) is trivial.

(1)⇒(2). Suppose that $A \in \mathrm{add}_R(M)$ and $C \oplus A \cong C \oplus B$ and $C \lesssim^{\oplus} nA$ for some $n \in \mathbb{N}$. Clearly, $C \in \mathrm{add}_R(A)$. By virtue of Lemma 2.3, there exists a decomposition $C \cong C_1 \oplus \cdots \oplus C_m$ with all $C_i \lesssim^{\oplus} A$. Therefore

$$\Big(\bigoplus_{1\leq i\leq m} C_i\Big) \oplus A \cong \Big(\bigoplus_{1\leq i\leq m} C_i\Big) \oplus B$$

with all $C_i \lesssim^{\oplus} A$. By applying Theorem 2.2, we conclude that $A \cong B$.

Let M have the finite exchange property. It follows from Corollary 2.1 that if M has self-cancellation then $M \oplus M \cong M \oplus B \Longrightarrow M \cong B$ for any right R-module B.

Corollary 2.2. *Let M be a right R-module with the finite exchange property. Then the following statements are equivalent:*

(1) *M has self-cancellation.*

(2) *For any right R-modules A and B, $A \oplus C \cong B \oplus C \lesssim^{\oplus} M$ with $C \lesssim^{\oplus} A \Longrightarrow A \cong B$.*

Proof. (1)⇒(2) is trivial by Theorem 2.2.

(2)⇒(1). Suppose that $A \oplus C \cong B \oplus C$ and $C \lesssim^{\oplus} mA$ $(m \in \mathbb{N}), A, B \in \mathrm{add}_R(M)$. Clearly, C is a right R-module with the finite exchange property. From $C \lesssim^{\oplus} mA$, there exists a right R-module D such that $C \oplus D \cong mA$. Analogously to Theorem 2.1, we get $C = C_1 \oplus C_2 \oplus \cdots \oplus C_m$ with $C_1, \cdots, C_m \lesssim^{\oplus} A$. Hence

$$C_1 \oplus \cdots \oplus C_n \oplus A \cong C_1 \oplus \cdots \oplus C_n \oplus B.$$

As $C_1 \lesssim^{\oplus} A$ and $A \in \mathrm{add}_R(M)$, we have $C_1 \lesssim^{\oplus} n_1 M$. Similarly, we have $C_{11}, \cdots, C_{1n_1} \lesssim^{\oplus} M$ such that $C_1 \cong \bigoplus_{j=1}^{n_1} C_{1j}$. Likewise, we have $C_i \cong \bigoplus_{j=1}^{n_i} C_{ij}$ for $i = 2, \cdots, m$. Therefore

$$\Big(\bigoplus_{1\leq i\leq m, 1\leq j\leq n_i} C_{ij}\Big) \oplus B \cong \Big(\bigoplus_{1\leq i\leq m, 1\leq j\leq n_i} C_{ij}\Big) \oplus C$$

with all $C_{ij} \lesssim^{\oplus} A, M$. Set

$$A_1 = \Big(\bigoplus_{\substack{1\leq i\leq m, 1\leq j\leq n_i \\ i\neq 1 \text{ or } j\neq 1}} C_{ij}\Big) \oplus A, \qquad B_1 = \Big(\bigoplus_{\substack{1\leq i\leq m, 1\leq j\leq n_i \\ i\neq 1 \text{ or } j\neq 1}} C_{ij}\Big) \oplus B.$$

Then $C_{11} \oplus A_1 \cong C_{11} \oplus B_2$ with $C_{11} \lesssim^{\oplus} A_1$. Clearly, C_{11} has the finite exchange property as well. According to Lemma 2.2, there exists a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} C_{11} & A_1 \end{array} \\ \begin{array}{c} C_{11} \\ B_1 \end{array} & \begin{pmatrix} C_2 & A_2 \\ B_2 & D_2 \end{pmatrix} \end{array}$$

with $C_2 \lesssim^{\oplus} A_2$. Hence, $C_2 \oplus A_2 \cong C_2 \oplus B_2 \cong C_{11} \lesssim^{\oplus} M$. So we get $A_2 \cong B_2$; whence, $A_1 \cong B_1$. Analogously, we deduce that $A \cong B$, as required.

Theorem 2.3. *Let M be a right R-module with the finite exchange property. Then the following statements are equivalent:*

(1) *M has self-cancellation.*

(2) *For any right R-modules A and B, $A \oplus A \cong A \oplus B \lesssim^{\oplus} M \Longrightarrow A \lesssim^{\oplus} B$.*

Proof. (1)⇒(2) is obvious by Corollary 2.2.

(2)⇒(1). Suppose that $A\oplus C\cong B\oplus C\lesssim^{\oplus} M$ with $C\lesssim^{\oplus} A$. By Lemma 2.2, we have a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} A & C \end{array} \\ \begin{array}{c} B \\ C \end{array} & \begin{pmatrix} D_1 & B_1 \\ A_1 & C_1 \end{pmatrix} \end{array}$$

with $C_1\lesssim^{\oplus} A_1$. Hence $A_1\oplus C_1\cong C\cong B_1\oplus C_1\lesssim^{\oplus} M$. we may assume that $A_1\cong C_1\oplus D$ for a right R-module D. One checks that

$$2(C_1\oplus D)\cong A_1\oplus C_1\oplus D\cong (C_1\oplus D)\oplus B_1\cong A_1\oplus B_1\lesssim^{\oplus} C\oplus B\lesssim^{\oplus} M.$$

Thus, we get $A_1\cong C_1\oplus D\lesssim^{\oplus} B_1$.

Since $A_1\oplus C_1\cong B_1\oplus C_1\lesssim^{\oplus} M$ with $C_1\lesssim^{\oplus} A_1$, by Lemma 2.2 again, we have a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} A_1 & C_1 \end{array} \\ \begin{array}{c} B_1 \\ C_1 \end{array} & \begin{pmatrix} D_2 & B_2 \\ A_2 & C_2 \end{pmatrix} \end{array}$$

with $C_2\lesssim^{\oplus} A_2$. By the consideration above, we have $A_2\lesssim^{\oplus} B_2$. Assume now that $B_2\cong A_2\oplus E$ for a right R-module E. It is easy to check that $B_1\cong B_2\oplus D_2\cong A_2\oplus D_2\oplus E\cong A_1\oplus E$. This infers that $C\cong B_1\oplus C_1\cong A_1\oplus E\oplus C_1\cong C\oplus E$. As $C\lesssim^{\oplus} A$, we have a right R-module F such that $A\cong C\oplus F$. Thus $A\oplus E\cong C\oplus F\oplus E\cong C\oplus F\cong A$. Therefore $B\cong D_1\oplus B_1\cong D_1\oplus A_1\oplus E\cong A\oplus E\cong A$. So the result follows by Corollary 2.2.

Let M be a right R-module with the finite exchange property. We say that M satisfies general comparability in case that for any $N\lesssim^{\oplus} M$, if $N\cong N_1\oplus N_2$ then either $N_1\lesssim^{\oplus} N_2$ or $N_2\lesssim^{\oplus} N_1$.

Corollary 2.3. *Let M be a directly finite right R-module with the finite exchange property. If M satisfies general comparability, then it has self-cancellation.*

Proof. Suppose that $A\oplus A\cong A\oplus B\lesssim^{\oplus} M$. Since M satisfies general comparability, we have either $A\lesssim^{\oplus} B$ or $B\lesssim^{\oplus} A$. If $B\lesssim^{\oplus} A$, then $A\cong B\oplus C$ for a right R-module C. Hence $2A\oplus C\cong A\oplus B\oplus C\cong 2A$. Since M is directly finite and $2A\lesssim^{\oplus} M$, $2A$ is directly finite. So we deduce that $C=0$, and then $A\cong B$. Therefore we conclude that $A\lesssim^{\oplus} B$. In view of Theorem 2.3, we complete the proof.

Theorem 2.4. *Let M be a right R-module with the finite exchange property. Then the following statements are equivalent:*

(1) *M has self-cancellation.*

(2) $\mathrm{End}_R(M)$ *is a strongly separative ring.*

(3) *Every $N\lesssim^{\oplus} M$ has self-cancellation.*

Proof. (1)⇒(2). By Corollary 2.1 and [14, Lemma 3.3], M is a strongly separative right R-module. It follows from [14, Lemma 3.1] that $\mathrm{End}_R(M)$ is a strongly separative ring.

(2)⇒(3). For any $N\lesssim^{\oplus} M$, we have an idempotent $e\in \mathrm{End}_R(M)$ such that $N\cong eM$. So $\mathrm{End}_R(N)\cong e\mathrm{End}_R(M)e$ is strongly separative. It follows by [14, Lemma 3.1] that N is a strongly separative right R-module. Using [14, Lemma 3.3] and Corollary 2.2, we prove that N has self-cancellation.

(3)⇒(1) is trivial.

Theorem 2.4 and Lemma 3.1 show that the concepts of self-cancellation of modules and strongly separative right modules coincide with each other. Let M be a right R-module with the finite exchange property, and let $n\in\mathbb{N}$. As a result, we prove that M has self-cancellation if and only if so has nM.

§ 3. Strongly Separative Exchange Ideals

Following Ara et al. [3], we say that an ideal of a ring R is strongly separative in case for all $A, B \in FP(I)$, $A \oplus A \cong A \oplus B \Longrightarrow A \cong B$. In this section, we investigate strongly separative exchange ideals of a ring.

Lemma 3.1. *Let I be an exchange ideal of a ring R. Then eRe is an exchange ring for all idempotents $e \in I$.*

Proof. Let $e \in I$ be an idempotent. Given any $x \in eRe$, we have $x \in I$. Since I is an exchange ideal of R, we have an idempotent $f \in I$ such that $f \in Rx$ and $1 - f \in R(1 - x)$. Hence $fe = f$, and then $(efe)^2 = efe \in (eRe)x$. In addition, we have $e - efe = e(1 - f)e \in (eRe)(e - x)$. Consequently, eRe is an exchange ring.

Lemma 3.2. *Let I be an exchange ideal of a ring R. If $A \in FP(I)$, then $A \cong e_1R \oplus \cdots \oplus e_nR$ for some idempotents $e_1, \cdots, e_n \in I$.*

Proof. Suppose that A is a finitely generated projective right R-module such that $A = AI$. Then we have a right R-module B such that $A \oplus B \cong nR$ for some $n \in \mathbb{N}$. Let $e : nR \to A$ be the projection onto A. Then $A \cong e(nR)$, whence $\mathrm{End}_R(A) \cong eM_n(R)e$. It follows from $A = AI$ that $e(nR) = e(nR)I \subseteq nI$. Set $e = (\alpha_1, \cdots, \alpha_1) \in M_n(R)$. Then $e(1, 0, \cdots, 0)^T \in nI$, and hence $\alpha_1 \in nI$. Likewise, $\alpha_2, \cdots, \alpha_n \in nI$. Therefore $e \in M_n(I)$. Since I is an exchange ideal of R, $M_n(I)$ is an exchange ideal of $M_n(R)$ from [1, Theorem 1.4]. According to Lemma 3.1, $\mathrm{End}_R(A)$ is an exchange ring. That is, A has the finitely exchange property. Set $M = A \oplus B$. Then $M = A \oplus B = \bigoplus\limits_{i=1}^{n} R_i$ with all $R_i \cong R$. By the finite exchange property of A, we have $B_i \lesssim^{\oplus} R_i$ $(1 \leq i \leq n)$ such that $M = A \oplus \Big(\bigoplus\limits_{i=1}^{n} B_i\Big)$. Assume that $B_i \oplus A_i = R_i$ $(1 \leq i \leq n)$. Then $A \oplus \Big(\bigoplus\limits_{i=1}^{n} B_i\Big) = \Big(\bigoplus\limits_{i=1}^{n} A_i\Big) \oplus \Big(\bigoplus\limits_{i=1}^{n} B_i\Big)$. Hence $A \cong A_1 \oplus \cdots \oplus A_n$. Clearly, we have idempotents e_i such that $A_i \cong e_iR$ $(1 \leq i \leq n)$. It follows from $A = AI$ that $A \bigotimes\limits_R (R/I) = 0$; hence, $A_i \bigotimes\limits_R (R/I) = 0$ $(1 \leq i \leq n)$. That is, each $(e_iR) \bigotimes\limits_R (R/I) = 0$; hence, $e_i \in e_iR = e_iRI \subseteq I$. Therefore $A \cong e_1R \oplus \cdots \oplus e_nR$ with all $e_i \in I$.

Lemma 3.3. *Let I be an exchange ideal of a ring R. Then the following statements are equivalent:*

(1) *I is strongly separative.*

(2) *For any idempotent $e \in I$, eR has self-cancellation.*

Proof. Let $e \in I$ be an idempotent. By Lemma 3.1, $\mathrm{End}_R(eR) \cong eRe$ is an exchange ring. Thus eR has the finite exchange property.

Suppose that I is strongly separative. Given $A \oplus A \cong A \oplus B$ as right eRe-modules, then $A \bigotimes\limits_{eRe} eR \oplus A \bigotimes\limits_{eRe} eR \cong A \bigotimes\limits_{eRe} eR \oplus B \bigotimes\limits_{eRe} eR$ as right R-modules. Clearly, $A \bigotimes\limits_{eRe} eR, B \bigotimes\limits_{eRe} eR \in FP(I)$. Hence $A \bigotimes\limits_{eRe} eR \cong B \bigotimes\limits_{eRe} eR$, and then $A \bigotimes\limits_{eRe} eR \bigotimes\limits_R Re \cong B \bigotimes\limits_{eRe} eR \bigotimes\limits_R Re$. Since $eR \bigotimes\limits_R Re \cong eRe$ as right eRe-modules, we deduce that $A \cong B$. Thus eRe is a strongly separative ring by [3, Lemma 5.1]. That is, $\mathrm{End}_R(eR)$ is a strongly separative ring. Using Theorem 2.4, eR has self-cancellation.

Conversely, assume now that $A\oplus A\cong A\oplus B$ with $A,B\in FP(I)$. By Lemma 3.2, we have idempotents $e_1,\cdots,e_n\in I$ such that $A\cong e_1R\oplus\cdots\oplus e_nR$. Then we have

$$e_1R\oplus(e_2R\oplus e_nR\oplus A)\cong e_1R\oplus(e_2R\oplus e_nR\oplus B).$$

Since e_1R has self-cancellation, it follows by Theorem 2.2 that

$$e_2R\oplus(e_3R\oplus\cdots\oplus e_nR\oplus A)\cong e_2R\oplus(e_3R\oplus\cdots\oplus e_nR\oplus B).$$

Likewise, we have $e_3R\oplus\cdots\oplus e_nR\oplus A\cong e_3R\oplus\cdots\oplus e_nR\oplus B$. Furthermore, we deduce that $A\cong B$. Therefore I is strongly separative.

Theorem 3.1. *Let I be an exchange ideal of a ring R. Then the following statements are equivalent:*

(1) *I is strongly separative.*

(2) *Every $A\in FP(I)$ has self-cancellation.*

Proof. (1)$\Rightarrow$(2). Let $A\in FP(I)$. According to Lemma 3.2, we have idempotents $e_1,\cdots,e_n\in I$ such that $A\cong e_1R\oplus\cdots\oplus e_nR$. It follows by Lemma 3.3, Theorem 2.4 and [14, Lemma 3.1 and Theorem 3.2] that $\mathrm{End}_R(A)$ is a strongly separative ring.

(2)$\Rightarrow$(1). For any idempotent $e\in I$, eR has self-cancellation. Therefore we complete the proof by Lemma 3.3.

Let I be a strongly separative exchange ideal of a ring R, and let $n\in\mathbb{N}$. As a consequence of Theorem 3.1, one can prove that $M_n(I)$ is a strongly separative exchange ideal of the ring $M_n(R)$.

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