SELF-CANCELLATION OF MODULES HAVING
THE FINITE EXCHANGE PROPERTY

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Abstract

Self-cancellation of modules having the finite exchange property is introduced. If a right \( R \)-module \( M \) has the finite exchange property, it is shown that \( M \) has self-cancellation if and only if \( \text{End}_R(M) \) is a strongly separative ring. Using this result, some new characterizations of strong separativity are obtained.

Keywords  Self-cancellation, Strong separativity  
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§ 1. Introduction

A right \( R \)-module \( M \) has the finite exchange property if for every right \( R \)-module \( A \) and any decompositions \( A = M' \oplus N = \bigoplus_{i \in I} A_i \), where \( M' \cong M \) and the index set \( I \) is finite, there exist submodules \( A'_i \subseteq A_i \) such that \( A = M' \oplus \left( \bigoplus_{i \in I} A'_i \right) \). If a ring \( R \) as a right \( R \)-module has the finite exchange property, we say that \( R \) is an exchange ring (see [10]). It is well known that a right \( R \)-module \( M \) has the finite exchange property if and only if \( \text{End}_R(M) \) is an exchange ring. Following Ara et al. (see [3]), a ring \( R \) is said to be strongly separative if for all finitely generated projective right \( R \)-modules \( A, B \) we have \( A \oplus A \cong A \oplus B \implies A \cong B \). Strong separativity is very useful in a number of various cancellation problems for modules over exchange rings.

An abelian group \( A \) has self-cancellation if \( A \oplus A \cong A \oplus B \implies A \cong B \) (see [5]). By [5, Corollary 8.19], every almost completely decomposable torsion free group of finite rank has self-cancellation. In this paper, we extend this concept to modules and introduce self-cancellation for modules having the finite exchange property. If a right \( R \)-module \( M \) has the finite exchange property, it is shown that \( M \) has self-cancellation if and only if \( \text{End}_R(M) \) is a strongly separative ring. Using this fact, we get some new characterizations of strong separativity.

Throughout, all rings are associative with identity and all modules are unitary right modules. The symbol \( M \lesssimop n N \) means that \( M \) is isomorphic to a direct summand of a module \( N \) and \( nM \) means that the direct sum of \( n \) copies of the \( R \)-module \( M \). Let \( \text{add}(M_R) \) denote the full subcategory of Mod-\( R \) whose objects are all the modules isomorphic to direct summands of direct sums \( nM \) for a finite number of copies of \( M \).

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§ 2. Self-cancellation of Modules

In [4], Ara et al. investigated regular rings having small projectives. Let \( R \) be a regular ring. We say that \( R \) has cancellation of small projectives if for all finitely generated projective right \( R \)-modules \( A, B, C, A \oplus C \cong B \oplus C \) and \( C \preceq nA \) for some \( n \in \mathbb{N} \implies A \cong B \). By [3, Lemma 5.1], a regular ring \( R \) has cancellation of small projectives if and only if it is an strongly separative ring. We now extend this concept to modules having the finite exchange property.

**Definition 2.1.** Let \( M \) be a right \( R \)-module with the finite exchange property. We say that \( M \) has self-cancellation if for all \( A, B, C \in \text{add}_R(M) \), \( A \oplus C \cong B \oplus C \) and \( C \preceq nA \) for some \( n \in \mathbb{N} \implies A \cong B \).

Clearly, an exchange ring \( R \) is an strongly separative ring if and only if it has self-cancellation as a right \( R \)-module.

**Lemma 2.1.** Let \( M \) be a right \( R \)-module with the finite exchange property. For any \( R \) modules \( B \) and \( C \), if \( \psi : M \oplus B \cong M \oplus C \), then we have a refinement matrix

\[
\begin{pmatrix}
M & B \\
C & (M_1 & B_1) \\
& (C_1 & D_1)
\end{pmatrix}
\]

That is, \( M \cong M_1 \oplus C_1 \cong M_1 \oplus B_1 \), \( B \cong B_1 \oplus D_1 \) and \( C \cong C_1 \oplus D_1 \).

**Proof.** The result follows analogously to [14, Theorem 3.1].

**Theorem 2.1.** Let \( M \) be a right \( R \)-module with the finite exchange property. Then the following statements are equivalent:

1. \( M \) has self-cancellation.
2. For any \( A, B, C \in \text{add}_R(M) \), \( A \oplus C \cong B \oplus C \) with \( C \preceq A \implies A \cong B \).
3. For any \( A, B \in \text{add}_R(M) \), \( A \oplus A \cong A \oplus B \implies A \cong B \).

**Proof.** (1)\( \Rightarrow \) (3) is clear.

(3)\( \Rightarrow \) (2). Suppose that \( A \oplus C \cong B \oplus C \) and \( C \preceq nA \) \((n \in \mathbb{N})\), \( A, B \in \text{add}_R(M) \). Then we have \( A \cong C \oplus D \) for a right \( R \)-module \( D \). Hence

\[
2(C \oplus D) \cong (A \oplus C) \oplus D \cong B \oplus (C \oplus D),
\]

and then \( A \cong C \oplus D \cong B \).

(2)\( \Rightarrow \) (1). Suppose that \( A \oplus C \cong B \oplus C \) and \( C \preceq nA \) \((n \in \mathbb{N})\), \( A, B \in \text{add}_R(M) \). Since \( M \) is a right \( R \)-module with the finite exchange property, so is \( C \). From \( C \preceq nA \), there exists a right \( R \)-module \( D \) such that \( C \oplus D \cong A \oplus (n-1)A \). By Lemma 2.1, we have a refinement matrix

\[
\begin{pmatrix}
A & (n-1)A \\
C & (C_1 & D_1) \\
D & (B_1 & E_1)
\end{pmatrix}
\]

So we have \( B_1 \oplus E_1 \cong A \oplus (n-1)A \). Similarly, we have a refinement matrix

\[
\begin{pmatrix}
A & (n-2)A \\
B & (C_2 & D_2) \\
E & (B_2 & E_2)
\end{pmatrix}
\]
Furthermore, we get a refinement matrix

\[
A \begin{pmatrix}
B_{n-2} & E_{n-2} \\
C_{n-1} & D_{n-1} \\
B_{n-1} & E_{n-1}
\end{pmatrix}
\]

From these refinement matrices, it follows that

\[
C \cong C_1 \oplus B_1 \cong C_1 \oplus (C_2 \oplus B_2) \cong \cdots \cong C_1 \oplus C_2 \oplus \cdots \oplus C_{n-1} \oplus B_{n-1}.
\]

Set \(C_n = B_{n-1}\). So \(C \cong C_1 \oplus C_2 \oplus \cdots \oplus C_n\) with \(C_1, \ldots, C_n \preceq A\); hence

\[
C_1 \oplus \cdots \oplus C_n \oplus A \cong C_1 \oplus \cdots \oplus C_n \oplus B.
\]

As \(C_1 \preceq A\), we deduce that

\[
C_2 \oplus \cdots \oplus C_n \oplus A \cong C_2 \oplus \cdots \oplus C_n \oplus B.
\]

Furthermore, we get \(A \cong B\), as required.

**Lemma 2.2.** Let \(M\) be a right \(R\)-module with the finite exchange property. For any right \(R\)-modules \(B\) and \(C\), if \(M \oplus B \cong M \oplus C\) with \(M \preceq B\) then we have a refinement matrix

\[
\begin{pmatrix}
M & B \\
C & D
\end{pmatrix}
\]

with \(M_1 \preceq B_1\).

**Proof.** Suppose that \(M \oplus B \cong M \oplus C\) with \(M \preceq B\). By Lemma 2.1, we have a refinement matrix

\[
\begin{pmatrix}
M & B \\
C & D
\end{pmatrix}
\]

Clearly

\(M_1 \preceq M \preceq B \cong B_1 \oplus D_1\).

So \(M_1\) has the finite exchange property, and \(M_1 \oplus D \cong B_1 \oplus D_1\) for a right \(R\)-module \(D\). By Lemma 2.1 again, we have a refinement matrix

\[
\begin{pmatrix}
M_1 & D \\
M_2 & B_1
\end{pmatrix}
\]

Hence \(M_1 \cong M_2 \oplus M'_2\) with \(M_2 \preceq B_1\) and \(M'_2 \preceq D_1\). So we have a right \(R\)-module \(D_2\) such that \(D_1 \cong M'_2 \oplus D_2\). Thus we have a new refinement matrix

\[
\begin{pmatrix}
M & B \\
C & D
\end{pmatrix}
\]

where \(B_2 = M'_2 \oplus B_1\) and \(C_2 = M'_2 \oplus C_1\). In addition, \(M_2 \preceq B_1 \preceq B_2\), as asserted.
Lemma 2.3. Let $M$ be a right $R$-module with the finite exchange property. If $A \in \text{add}_R(M)$, then there exist idempotents $e_1, \ldots, e_n \in \text{End}_R(M)$ such that $A \cong e_1 M \oplus \cdots \oplus e_n M$.

Proof. Since $A \in \text{add}(M_R)$, we can find a right $R$-module $B$ such that $A \oplus B \cong nM$ for some $n \in \mathbb{N}$. Clearly, $A$ also has the finite exchange property. Similarly to Lemma 2.1, we have decompositions $A = A_1 \oplus \cdots \oplus A_n$, $B = B_1 \oplus \cdots \oplus B_n$ and $A_i \oplus B_i \cong M$ for $i = 1, \ldots, n$. Thus, there exists $e_i = e_i^2 \in \text{End}_R(M)$ such that $A_i \cong e_i M$ for $i = 1, \ldots, n$. Hence $A \cong e_1 M \oplus \cdots \oplus e_n M$, as asserted.

Theorem 2.2. Let $M$ be a right $R$-module with the finite exchange property. Then the following statements are equivalent:

(1) $M$ has self-cancellation.

(2) For any $C \in \text{add}_R(M)$, $A \oplus C \cong B \oplus C$ with $C \cong^\oplus A \Rightarrow A \cong B$ for any right $R$-modules $A$ and $B$.

Proof. (2)$\Rightarrow$(1) is clear by Theorem 2.1.

(1)$\Rightarrow$(2). Suppose that $C \in \text{add}_R(M)$ and $C \oplus A \cong C \oplus B$ with $C \cong^\oplus A$. In view of Lemma 2.3, we have idempotents $e_1, \ldots, e_n \in \text{End}_R(M)$ such that $C \cong e_1 M \oplus \cdots \oplus e_n M$. So

$$e_1 M \oplus (e_2 M \oplus \cdots \oplus e_n M \oplus A) \cong e_1 M \oplus (e_2 M \oplus \cdots \oplus e_n M \oplus B).$$

Set

$$A_1 = e_2 M \oplus \cdots \oplus e_n M \oplus A \quad \text{and} \quad B_1 = e_2 M \oplus \cdots \oplus e_n M \oplus B.$$ 

Then

$$e_1 M \oplus A_1 \cong e_1 M \oplus B_1$$

with $e_1 M \cong^\oplus A_1$. Clearly, $e_1 M$ has the finite exchange property. Using Lemma 2.2, we have a refinement matrix

$$
e_1 M \begin{pmatrix} e_1 M & A_1 \\ M_2 & A_2 \end{pmatrix} \\
B_1 \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}$$

with $M_2 \cong^\oplus A_2$. Clearly

$$M_2 \oplus A_2 \cong M_2 \oplus B_2 \cong e_1 M \cong^\oplus M.$$

It follows by Theorem 2.1 that $A_2 \cong B_2$, hence $A_1 \cong A_2 \oplus C_2 \cong B_2 \oplus C_2 \cong B_1$. That is,

$$e_2 M \oplus \cdots \oplus e_n M \oplus A \cong e_2 M \oplus \cdots \oplus e_n M \oplus B.$$

Likewise, we claim that

$$e_3 M \oplus \cdots \oplus e_n M \oplus A \cong e_3 M \oplus \cdots \oplus e_n M \oplus B.$$

Furthermore, we conclude that $A \cong B$, as required.

Corollary 2.1. Let $M$ be a right $R$-module with the finite exchange property. Then the following statements are equivalent:

(1) $M$ has self-cancellation.

(2) For any $C \in \text{add}(M_R)$, $C \oplus A \cong C \oplus B$ and $C \cong^\oplus nA$ for some $n \in \mathbb{N} \Rightarrow A \cong B$ for any right $R$-modules $A$ and $B$. 

Proof. \((2)\Rightarrow(1)\) is trivial.

\((1)\Rightarrow(2)\). Suppose that \(A \in \text{add}_R(M)\) and \(C \oplus A \cong C \oplus B\) and \(C \lesssim nA\) for some \(n \in \mathbb{N}\). Clearly, \(C \in \text{add}_R(A)\). By virtue of Lemma 2.3, there exists a decomposition \(C \cong C_1 \oplus \cdots \oplus C_m\) with all \(C_i \lesssim A\). Therefore

\[
\left( \bigoplus_{1 \leq i \leq m} C_i \right) \oplus A \cong \left( \bigoplus_{1 \leq i \leq m} C_i \right) \oplus B
\]

with all \(C_i \lesssim A\). By applying Theorem 2.2, we conclude that \(A \cong B\).

Let \(M\) have the finite exchange property. It follows from Corollary 2.1 that if \(M\) has self-cancellation then \(M \oplus M \cong M \oplus B \Rightarrow M \cong B\) for any right \(R\)-module \(B\).

**Corollary 2.2.** Let \(M\) be a right \(R\)-module with the finite exchange property. Then the following statements are equivalent:

1. \(M\) has self-cancellation.
2. For any right \(R\)-modules \(A\) and \(B\), \(A \oplus C \cong B \oplus C \lesssim M\) with \(C \lesssim A \Rightarrow A \cong B\).

**Proof.** \((1)\Rightarrow(2)\) is trivial by Theorem 2.2.

\((2)\Rightarrow(1)\). Suppose that \(A \oplus C \cong B \oplus C\) and \(C \lesssim mA\) \((m \in \mathbb{N})\), \(A, B \in \text{add}_R(M)\). Clearly, \(C\) is a right \(R\)-module with the finite exchange property. From \(C \lesssim mA\), there exists a right \(R\)-module \(D\) such that \(C \oplus D \cong mA\). Analogously to Theorem 2.1, we get \(C = C_1 \oplus C_2 \oplus \cdots \oplus C_m\) with \(C_1, \ldots, C_m \lesssim A\). Hence

\[
C_1 \oplus \cdots \oplus C_n \oplus A \cong C_1 \oplus \cdots \oplus C_n \oplus B.
\]

As \(C_1 \lesssim A\) and \(A \in \text{add}_R(M)\), we have \(C_1 \lesssim n_1 M\). Similarly, we have \(C_{11}, \ldots, C_{1n_1} \lesssim M\) such that \(C_1 \cong \bigoplus_{j=1}^{n_1} C_{1j}\). Likewise, we have \(C_i \cong \bigoplus_{j=1}^{n_i} C_{ij}\) for \(i = 2, \ldots, m\). Therefore

\[
\left( \bigoplus_{1 \leq i \leq m, 1 \leq j \leq n_i} C_{ij} \right) \oplus B \cong \left( \bigoplus_{1 \leq i \leq m, 1 \leq j \leq n_i} C_{ij} \right) \oplus C
\]

with all \(C_{ij} \lesssim A, M\). Set

\[
A_1 = \left( \bigoplus_{1 \leq i \leq m, 1 \leq j \leq n_i \atop i \neq 1 \text{ or } j \neq 1} C_{ij} \right) \oplus A, \quad B_1 = \left( \bigoplus_{1 \leq i \leq m, 1 \leq j \leq n_i \atop i \neq 1 \text{ or } j \neq 1} C_{ij} \right) \oplus B.
\]

Then \(C_{11} \oplus A_1 \cong C_{11} \oplus B_2\) with \(C_{11} \lesssim A_1\). Clearly, \(C_{11}\) has the finite exchange property as well. According to Lemma 2.2, there exists a refinement matrix

\[
\begin{pmatrix}
C_{11} & A_1 \\
B_1 & C_2 & A_2 \\
& B_2 & D_2
\end{pmatrix}
\]

with \(C_2 \lesssim A_2\). Hence, \(C_2 \oplus A_2 \cong C_2 \oplus B_2 \cong C_{11} \lesssim M\). So we get \(A_2 \cong B_2\); whence, \(A_1 \cong B_1\). Analogously, we deduce that \(A \cong B\), as required.

**Theorem 2.3.** Let \(M\) be a right \(R\)-module with the finite exchange property. Then the following statements are equivalent:

1. \(M\) has self-cancellation.
2. For any right \(R\)-modules \(A\) and \(B\), \(A \oplus A \cong A \oplus B \lesssim M \Rightarrow A \lesssim B\).
Proof. (1)⇒(2) is obvious by Corollary 2.2.  
(2)⇒(1). Suppose that $A \oplus C \cong B \oplus C \cong M$ with $C \cong A$. By Lemma 2.2, we have a refinement matrix

$$
\begin{pmatrix}
A & C \\
B & D_1 \\
C & A_1 \\
\end{pmatrix}
$$

with $C_1 \cong A_1$. Hence $A_1 \oplus C_1 \cong B_1 \oplus C_1 \cong A$. We may assume that $A_1 \cong D$ for a right $R$-module $D$. One checks that

$$2(C_1 \oplus D) \cong A_1 \oplus C_1 \oplus D \cong (C_1 \oplus D) \oplus B_1 \cong A_1 \oplus B_1 \cong C \oplus B \cong M.$$ 

Thus, we get $A_1 \cong C \oplus D \cong B_1$. Since $A_1 \oplus C_1 \cong B_1 \oplus C_1 \cong M$ with $C_1 \cong A_1$, by Lemma 2.2 again, we have a refinement matrix

$$
\begin{pmatrix}
A_1 & C_1 \\
B_1 & D_2 \\
C_1 & A_2 \\
\end{pmatrix}
$$

with $C_2 \cong A_2$. By the consideration above, we have $A_2 \cong B_2$. Assume now that $B_2 \cong A_2 \oplus E$ for a right $R$-module $E$. It is easy to check that $B_1 \cong B_2 \oplus D_2 \cong A_2 \oplus D_2 \oplus E \cong A \oplus E$. This infers that $C \cong C_1 \cong A_1 \oplus E \cong C \cong E$. As $C \cong A$, we have a right $R$-module $F$ such that $C \cong C_1 \cong A_1 \oplus E \cong C \cong F \cong A$. Therefore $B \cong D_1 \oplus B_1 \cong D_1 \oplus A_1 \oplus E \cong A \oplus E \cong A$. So the result follows by Corollary 2.2.

Let $M$ be a right $R$-module with the finite exchange property. We say that $M$ satisfies general comparability in case that for any $N \cong M$, if $N \cong N_1 \oplus N_2$ then either $N_1 \cong N_2$ or $N_2 \cong N_1$.

Corollary 2.3. Let $M$ be a directly finite right $R$-module with the finite exchange property. If $M$ satisfies general comparability, then it has self-cancellation.

Proof. Suppose that $A \oplus A \cong A \oplus B \cong M$. Since $M$ satisfies general comparability, we have either $A \cong B$ or $B \cong A$. If $B \cong A$, then $A \cong B \cong C$ for a right $R$-module $C$. Hence $2A \cong A \cong A \oplus B \cong C \cong 2A$. Since $M$ is directly finite and $2A \cong M$, $2A$ is directly finite. So we deduce that $C = 0$, and then $A \cong B$. Therefore we conclude that $A \cong B$. In view of Theorem 2.3, we complete the proof.

Theorem 2.4. Let $M$ be a right $R$-module with the finite exchange property. Then the following statements are equivalent:

1. $M$ has self-cancellation.
2. $\text{End}_R(M)$ is a strongly separative ring.
3. Every $N \cong M$ has self-cancellation.

Proof. (1)⇒(2). By Corollary 2.1 and [14, Lemma 3.3], $M$ is a strongly separative right $R$-module. It follows from [14, Lemma 3.1] that $\text{End}_R(M)$ is a strongly separative ring.

(2)⇒(3). For any $N \cong M$, we have an idempotent $e \in \text{End}_R(M)$ such that $N \cong eM$. So $\text{End}_R(N) \cong e\text{End}_R(M) e$ is strongly separative. It follows by [14, Lemma 3.1] that $N$ is a strongly separative right $R$-module. Using [14, Lemma 3.3] and Corollary 2.2, we prove that $N$ has self-cancellation.

(3)⇒(1) is trivial.

Theorem 2.4 and Lemma 3.1 show that the concepts of self-cancellation of modules and strongly separative right modules coincide with each other. Let $M$ be a right $R$-module with the finite exchange property, and let $n \in \mathbb{N}$. As a result, we prove that $M$ has self-cancellation if and only if so has $nM$. 


§ 3. Strongly Separative Exchange Ideals

Following Ara et al. [3], we say that an ideal of a ring $R$ is strongly separative in case for all $A, B \in FP(I)$, $A \oplus A \cong A \oplus B \Rightarrow A \cong B$. In this section, we investigate strongly separative exchange ideals of a ring.

**Lemma 3.1.** Let $I$ be an exchange ideal of a ring $R$. Then $eRe$ is an exchange ring for all idempotents $e \in I$.

**Proof.** Let $e \in I$ be an idempotent. Given any $x \in eRe$, we have $x \in I$. Since $I$ is an exchange ideal of $R$, we have an idempotent $f \in I$ such that $f \in Rx$ and $1 - f \in R(1 - x)$. Hence $fe = f$, and then $(efe)^2 = efe \in (eRe)x$. In addition, we have $e - efe = e(1 - f)e \in (eRe)(e - x)$. Consequently, $eRe$ is an exchange ring.

**Lemma 3.2.** Let $I$ be an exchange ideal of a ring $R$. If $A \in FP(I)$, then $A \cong e_1 R \oplus \cdots \oplus e_n R$ for some idempotents $e_1, \ldots, e_n \in I$.

**Proof.** Suppose that $A$ is a finitely generated projective right $R$-module such that $A = AI$. Then we have a right $R$-module $B$ such that $A \oplus B \cong nR$ for some $n \in \mathbb{N}$. Let $e : nR \to A$ be the projection onto $A$. Then $A \cong e(nR)$, whence $\text{End}_R(A) \cong eM_n(R)e$. It follows from $A = AI$ that $e(nR) = e(nR)I \subseteq nI$. Set $e = (\alpha_1, \ldots, \alpha_n) \in M_n(R)$. Then $e(1, 0, \ldots, 0)^T \in nI$, and hence $\alpha_1 \in nI$. Likewise, $\alpha_2, \ldots, \alpha_n \in nI$. Therefore $e \in M_n(I)$. Since $I$ is an exchange ideal of $R$, $M_n(I)$ is an exchange ideal of $M_n(R)$ from [1, Theorem 14]. According to Lemma 3.1, $\text{End}_R(A)$ is an exchange ring. That is, $A$ has the finitely exchange property. Set $M = A \oplus B$. Then $M = A \oplus B = \bigoplus_{i=1}^n R_i$ with all $R_i \cong R$. By the finite exchange property of $A$, we have $B_i \subseteq R_i$ ($1 \leq i \leq n$) such that $M = A \oplus \bigoplus_{i=1}^n B_i$.

Assume that $B_i \oplus A_i = R_i$ ($1 \leq i \leq n$). Then $A \oplus \bigoplus_{i=1}^n B_i = \bigoplus_{i=1}^n A_i \oplus \bigoplus_{i=1}^n B_i$. Hence $A \cong A_1 \oplus \cdots \oplus A_n$. Clearly, we have idempotents $e_i$ such that $A_i \cong e_i R$ ($1 \leq i \leq n$). It follows from $A = AI$ that $A \bigotimes_R(R/I) = 0$; hence, $A_i \bigotimes_R(R/I) = 0$ ($1 \leq i \leq n$). That is, each $(e_i R) \bigotimes_R(R/I) = 0$; hence, $e_i \in e_i R = e_i R I \subseteq I$. Therefore $A \cong e_1 R \oplus \cdots \oplus e_n R$ with all $e_i \in I$.

**Lemma 3.3.** Let $I$ be an exchange ideal of a ring $R$. Then the following statements are equivalent:

1. $I$ is strongly separative.
2. For any idempotent $e \in I$, $eR$ has self-cancellation.

**Proof.** Let $e \in I$ be an idempotent. By Lemma 3.1, $\text{End}_R(eR) \cong eRe$ is an exchange ring. Thus $eR$ has the finite exchange property.

Suppose that $I$ is strongly separative. Given $A \oplus A \cong A \oplus B$ as right $eRe$-modules, then $A \bigotimes eRe \oplus A \bigotimes eRe \cong A \bigotimes eRe \oplus B \bigotimes eRe$ as right $R$-modules. Clearly, $A \bigotimes eRe, B \bigotimes eRe \in FP(I)$. Hence $A \bigotimes eRe \cong B \bigotimes eRe$, and then $A \bigotimes eRe \bigotimes eRe \cong B \bigotimes eRe \bigotimes eRe$. Since $eRe \bigotimes eRe \cong eRe$ as right $eRe$-modules, we deduce that $A \cong B$. Thus $eRe$ is a strongly separative ring by [3, Lemma 5.1]. That is, $\text{End}_R(eR)$ is a strongly separative ring. Using Theorem 2.4, $eR$ has self-cancellation.
Conversely, assume now that $A \oplus A \cong A \oplus B$ with $A, B \in FP(I)$. By Lemma 3.2, we have idempotents $e_1, \ldots, e_n \in I$ such that $A \cong e_1 R \oplus \cdots \oplus e_n R$. Then we have
\[ e_1 R \oplus (e_2 R \oplus e_n R \oplus A) \cong e_1 R \oplus (e_2 R \oplus e_n R \oplus B). \]
Since $e_1 R$ has self-cancellation, it follows by Theorem 2.2 that
\[ e_2 R \oplus (e_3 R \oplus \cdots \oplus e_n R \oplus A) \cong e_2 R \oplus (e_3 R \oplus \cdots \oplus e_n R \oplus B). \]
Likewise, we have $e_3 R \oplus \cdots \oplus e_n R \oplus A \cong e_3 R \oplus \cdots \oplus e_n R \oplus B$. Furthermore, we deduce that $A \cong B$. Therefore $I$ is strongly separative.

**Theorem 3.1.** Let $I$ be an exchange ideal of a ring $R$. Then the following statements are equivalent:

1. $I$ is strongly separative.
2. Every $A \in FP(I)$ has self-cancellation.

**Proof.** (1)⇒(2). Let $A \in FP(I)$. According to Lemma 3.2, we have idempotents $e_1, \ldots, e_n \in I$ such that $A \cong e_1 R \oplus \cdots \oplus e_n R$. It follows by Lemma 3.3, Theorem 2.4 and [14, Lemma 3.1 and Theorem 3.2] that $\text{End}_R(A)$ is a strongly separative ring.

(2)⇒(1). For any idempotent $e \in I$, $eR$ has self-cancellation. Therefore we complete the proof by Lemma 3.3.

Let $I$ be a strongly separative exchange ideal of a ring $R$, and let $n \in \mathbb{N}$. As a consequence of Theorem 3.1, one can prove that $M_n(I)$ is a strongly separative exchange ideal of the ring $M_n(R)$.

**References**