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REGULARITY RESULTS FOR SOME QUASI-LINEAR ELLIPTIC SYSTEMS INVOLVING CRITICAL EXPONENTS***

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Abstract

The authors show the regularity of weak solutions for some typical quasi-linear elliptic systems governed by two p-Laplacian operators. The weak solutions of the following problem with lack of compactness are proved to be regular when $a(x)$ and α, β, p, q satisfy some conditions:

$$\begin{cases} -\Delta_p u + a(x)|u|^{\alpha-1}|v|^{\beta+1}u = |u|^{p^*-2}u, & x \in \Omega, \\ -\Delta_q v + a(x)|u|^{\alpha+1}|v|^{\beta-1}v = |v|^{q^*-2}v, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset R^N$ $(N \geq 3)$ is a smooth bounded domain.

Keywords Elliptic equation system, p-Laplacian operator, Critical Sobolev exponent, Regularity
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§ 1 . Introduction

In this paper, we consider the regularity of the weak solutions of the following quasi-linear elliptic systems

$$\begin{cases} -\Delta_p u + a(x)|u|^{\alpha-1}|v|^{\beta+1}u = |u|^{p^*-2}u, & x \in \Omega, \\ -\Delta_q v + a(x)|u|^{\alpha+1}|v|^{\beta-1}v = |v|^{q^*-2}v, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N$ $(N \geq 3)$ is a smooth bounded domain, Δ_p is the p-Laplacian operator, namely $\Delta_p u \doteq \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. In addition, we assume that $1 \leq p < N$, $1 \leq q < N$, $\alpha + 1 > 0$, $\beta + 1 > 0$ and $a(x) \in L^\infty(\Omega)$. For the positive constants α, β, p, q and N, the following inequality is valid:

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} \leq 1, \tag{1.2}$$

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where p^* and q^* are the critical Sobolev exponents, i.e., $p^* = \frac{Np}{N-p}$ and $q^* = \frac{Nq}{N-q}$.

Analogous results have been obtained for single elliptic equation (see [1]). As for the existence and regularity results for some quasi-linear elliptic equations or systems on R^N, we refer to [2, 3] and the references therein. The condition (1.2) can guarantee that the C^1 functional corresponding with the problem (1.1) defined as follows exists.

$$I(u,v) = \frac{\alpha+1}{p}\int_\Omega |\nabla u|^p dx + \frac{\beta+1}{q}\int_\Omega |\nabla v|^q dx - \frac{\alpha+1}{p^*}\int_\Omega |u|^{q^*} dx \\ - \frac{\beta+1}{q^*}\int_\Omega |v|^{q^*} dx + \int_\Omega a(x)|u|^{\alpha+1}|v|^{\beta+1} dx.$$

By variational methods, the critical point theory and Lions' concentration compactness principle, we can prove the existence of nontrivial weak solutions to the problem (1.1) similar to the arguments of [4]. Thus, it is necessary to show the regularity of the weak solutions to problem (1.1). In this paper, we obtain the regularity result by applying Morse's iterative scheme to this elliptic systems.

We shall denote by $W_0^{1,p}(\Omega)$ the Sobolev space obtained as the closure of $C_0^\infty(\Omega)$ in the norm $||u||^p = \int_\Omega |\nabla u|^p dx$. For simplicity, we denote $\int_\Omega \cdot\, dx$ by $\int \cdot\, dx$ and denote different positive constants by C which may change from line to line. The natural space setting for our problem is the space $E \doteq W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. We prove first any weak solution $(u,v) \in E$ belongs to $L^\infty(\Omega) \times L^\infty(\Omega)$, then (u,v) belongs to $C^{1,\mu}(\Omega) \times C^{1,\nu}(\Omega)$, $\mu > 0$, $\nu > 0$.

Now, We state our main results in this paper.

Theorem 1.1. *Let $(u,v) \in E$ be the weak solution of problem* (1.1)*, and suppose that* (1.2) *holds and the following condition holds*

$$\frac{\beta+1}{p^* - (\alpha+1)} \le \frac{q^*}{p^*} \le \frac{q^* - (\beta+1)}{\alpha+1}. \tag{1.3}$$

Then $(u,v) \in L^\infty(\Omega) \times L^\infty(\Omega)$.

Theorem 1.2. *Under the conditions of Theorem* 1.1*, it holds that $(|\nabla u|, |\nabla v|) \in L^\infty(\Omega) \times L^\infty(\Omega)$, moreover there exist two positive constants $\mu > 0$, $\nu > 0$ such that $(u,v) \in C^{1,\mu}(\Omega) \times C^{1,\nu}(\Omega)$.*

Example 1.1. Take $N = 4$, $p = q = 3$, and let $\alpha + 1 = \beta + 1 = 4$. One can see that the conditions (1.2) and (1.3) hold.

Example 1.2. When $N = 3$, $p = q = 2$, if $\alpha + 1 = \beta + 1 = 3$, then (1.2) and (1.3) are satisfied. Hence the classical solutions of the problem (1.1) are obtained.

§ 2 . Proofs of Main Results

Proof of Theorem 1.1. First we suppose $u \geq 0, v \geq 0$ and $s \geq 1, \omega = u^s \in L^p(\Omega)$, where s is to be determined. In a way similar to [1], let $\omega_L = \eta \cdot u \cdot u_L^{s-1}$, where $L \in R^+$, and

$$u_L(x) = \begin{cases} u(x), & \text{if } \ u \leq L, \\ L, & \text{if } \ u > L. \end{cases}$$

Take $\eta \in C_0^\infty(\Omega)$, $\eta \equiv 1$ if $x \in B_R(x_0)$, $\forall x_0 \in \overline{\Omega}$ fixed; $\eta \equiv 0$ if $x \notin B_{R+r}(x_0)$, and $0 < r < R$ such that $|\nabla \eta| \leq \frac{1}{r}$. Then

$$\nabla \omega_L = u \cdot u_L^{s-1} \cdot \nabla \eta + \eta \cdot u_L^{s-1}(\nabla u + (s-1)\nabla u_L),$$

$$\begin{aligned} \int |\nabla \omega_L|^p dx \leq 2^{p-1} \int |\nabla \eta|^p u^p \cdot u_L^{p(s-1)} dx \\ + 4^{p-1} \int \eta^p \cdot u_L^{p(s-1)} (|\nabla u|^p + (s-1)^p |\nabla u_L|^p) dx. \end{aligned} \tag{2.1}$$

Since $(u, v) \in E$ is a weak solution of the problem (1.1), for any $\varphi \in W_0^{1,p}(\Omega)$, it follows that

$$\int |\nabla u|^{p-2} \nabla u \nabla \varphi dx + \int a(x)|u|^{\alpha-1}|v|^{\beta+1} u \varphi dx = \int u^{p^*-1} \varphi dx. \tag{2.2}$$

Let $\varphi = \eta^p u \cdot u_L^{p(s-1)}$, then $\varphi \in W_0^{1,p}(\Omega)$. By (2.2) we get

$$\begin{aligned} &\sum_{i=1}^N \int \Big[|\nabla u|^p \cdot \eta^p \cdot u_L^{p(s-1)} + p(s-1)|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial u_L}{\partial x_i} \eta^p u \cdot u_L^{p(s-1)-1} \\ &+ p|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_i} \eta^{p-1} u \cdot u_L^{p(s-1)} \Big] dx + \int a(x) \eta^p u^{\alpha+1} |v|^{\beta+1} u_L^{p(s-1)} dx \\ = &\int u^{p^*} \eta^p \cdot u_L^{p(s-1)} dx. \end{aligned} \tag{2.3}$$

Then for any $\varepsilon > 0$, we deduce that

$$\begin{aligned} &\int |\nabla u|^p \eta^p u_L^{p(s-1)} dx + p(s-1) \int |\nabla u_L|^p \eta^p u_L^{p(s-1)} dx \\ \leq &-p \int \sum_{i=1}^N |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_i} \eta^{p-1} \cdot u \cdot u_L^{p(s-1)} dx \\ &- \int a(x) \eta^p u^{\alpha+1} |v|^{\beta+1} u_L^{p(s-1)} dx + \int u^{p^*} \eta^p \cdot u_L^{p(s-1)} dx \\ \leq &\int p |\nabla u|^{p-2} \eta^{p-2} \cdot u_L^{p(s-1)} (\varepsilon |\nabla u|^2 \eta^2 + C_\varepsilon |\nabla \eta|^2 \cdot u^2) dx \\ &+ \int |a(x)| \eta^p u^{\alpha+1} |v|^{\beta+1} u_L^{p(s-1)} dx + \int u^{p^*} \eta^p \cdot u_L^{p(s-1)} dx \\ = &\ \varepsilon \int |\nabla u|^p \eta^p u_L^{p(s-1)} dx + C_\varepsilon \int |\nabla u|^{p-2} \eta^{p-2} |\nabla \eta|^2 \cdot u^2 \cdot u_L^{p(s-1)} dx \\ &+ \int |a(x)| \eta^p u^{\alpha+1} |v|^{\beta+1} u_L^{p(s-1)} dx + \int u^{p^*} \eta^p \cdot u_L^{p(s-1)} dx. \end{aligned}$$

Choose $\varepsilon = \frac{1}{2}$, then

$$\begin{aligned}
&\int |\nabla u|^p \eta^p u_L^{p(s-1)} dx + 2p(s-1) \int |\nabla u_L|^p \eta^p u_L^{p(s-1)} dx \\
\leq{}& C \int |\nabla u|^{p-2} \eta^{p-2} |\nabla \eta|^2 \cdot u_L^{p(s-1)} dx \\
&+ \int |a(x)| \eta^p u^{\alpha+1} |v|^{\beta+1} u_L^{p(s-1)} dx + \int u^{p^*} \eta^p \cdot u_L^{p(s-1)} dx.
\end{aligned}$$

Notice that for any $\varepsilon > 0$, the following inequality holds

$$\int |\nabla u|^{p-2} |\nabla \eta|^2 \cdot \eta^{p-2} u^2 \cdot u_L^{p(s-1)} dx \leq \varepsilon \int |\nabla u|^p \eta^p u_L^{p(s-1)} dx + C_\varepsilon \int |\nabla \eta|^p u^p \cdot u_L^{p(s-1)} dx.$$

We may choose ε such that

$$\begin{aligned}
&\int |\nabla u|^p \eta^p u_L^{p(s-1)} dx + 4Cp(s-1) \int |\nabla u_L|^p \eta^p u_L^{p(s-1)} dx \\
\leq{}& C \int |\nabla \eta|^p u^p u_L^{p(s-1)} dx + \int |a(x)| \eta^p u^{\alpha+1} |v|^{\beta+1} u_L^{p(s-1)} dx + \int u^{p^*} \eta^p \cdot u_L^{p(s-1)} dx. \quad (2.4)
\end{aligned}$$

By (2.1) and (2.4), we obtain

$$\begin{aligned}
&\int_\Omega |\nabla \omega_L|^p dx = \int_\Omega |\nabla (\eta u u_L^{s-1})|^p dx \\
\leq{}& C \cdot s^{p-1} \int_\Omega (|\nabla \eta|^p u^p \cdot u_L^{p(s-1)} + \eta^p a(x) |v|^{\beta+1} u^{\alpha+1} \cdot u_L^{p(s-1)} + \eta^p u^{p^*} \cdot u_L^{p(s-1)}) dx. \quad (2.5)
\end{aligned}$$

Let $s = \frac{p^*}{p}$, then $\omega = u^{\frac{p^*}{p}} \in L^p(\Omega)$. By (2.5) and Sobolev inequality, we have

$$\begin{aligned}
&\Big[\int_\Omega \Big(\eta u \cdot u_L^{\frac{p^*-p}{p}} \Big)^{p^*} dx \Big]^{\frac{p}{p^*}} \leq C \int_\Omega \Big| \nabla \Big(\eta u u_L^{\frac{p^*-p}{p}} \Big) \Big|^p dx \\
\leq{}& C \Big(\frac{p^*}{p} \Big) \int_\Omega (|\nabla \eta|^p u^p u_L^{p^*-p} + |a(x)| \eta^p u^{\alpha+1} |v|^{\beta+1} u_L^{p^*-p} + \eta^p u^{p^*} \cdot u_L^{p^*-p}) dx. \quad (2.6)
\end{aligned}$$

Notice that

$$\int_\Omega \eta^p u^{p^*} \cdot u_L^{p^*-p} dx = \int_{B_{R+r}} (\eta^p u^p u_L^{p^*-p}) \cdot u^{p^*-p} dx \leq \Big[\int_\Omega (\eta u u_L^{\frac{p^*-p}{p}})^{p^*} dx \Big]^{\frac{p}{p^*}} \Big[\int_{B_{R+r}} u^{p^*} dx \Big]^{\frac{p^*-p}{p^*}}.$$

Similarly, for $\alpha + 1 \geq p$, we have

$$\begin{aligned}
&\int_\Omega \eta^p u^{\alpha+1} |v|^{\beta+1} \cdot u_L^{p^*-p} dx = \int_{B_{R+r}} (\eta^p u^p u_L^{p^*-p}) \cdot u^{\alpha+1-p} |v|^{\beta+1} dx \\
\leq{}& \Big[\int_\Omega (\eta u u_L^{\frac{p^*-p}{p}})^{p^*} dx \Big]^{\frac{p}{p^*}} \Big[\int_{B_{R+r}} (u^{\alpha+1-p} |v|^{\beta+1})^{\frac{p^*}{p^*-p}} dx \Big]^{\frac{p^*-p}{p^*}}.
\end{aligned}$$

Clearly

$$\int_\Omega u^{\frac{p^*(\alpha+1-p)}{p^*-p}} |v|^{\frac{p^*(\beta+1)}{p^*-p}} dx \leq \Big(\int_\Omega |u|^{p^*} dx \Big)^{\frac{\alpha+1-p}{p^*-p}} \Big(\int_\Omega |v|^{\frac{p^*(\beta+1)}{p^*-\alpha+1}} dx \Big)^{\frac{p^*-(\alpha+1)}{p^*-p}} \leq C. \quad (2.7)$$

Here we have used (1.3) and $(u,v)\in E$ in the last inequality. As for the case $0<\alpha+1<p$, the same result holds after the arguments similar to (2.11). By the continuity of integral, for any $\varepsilon_0>0$ we may choose R,r such that

$$\int_{B_{R+r}} u^{p^*}dx<\varepsilon_0,\qquad \int_{B_{R+r}} u^{\frac{p^*(\alpha+1-p)}{p^*-p}}|v|^{\frac{p^*(\beta+1)}{p^*-p}}dx<\varepsilon_0.$$

Since $u\in W_0^{1,p}(\Omega)$, it follows that $u\in L^{p^*}(\Omega)$. Take ε_0 such that

$$C\cdot\Big(\frac{p^*}{p}\Big)^{p-1}\cdot\varepsilon_0^{\frac{p^*-p}{p}}=\frac{1}{4},$$

then

$$\int_\Omega \eta^p u^{\alpha+1}|v|^{\beta+1}\cdot u_L^{p^*-p}dx+\int_\Omega \eta^p u^{p^*}\cdot u_L^{p^*-p}dx\le\frac{1}{2}\Big[\int_\Omega(\eta u\cdot u_L^{\frac{p^*-p}{p}})^{p^*}dx\Big]^{\frac{p}{p^*}}.$$

Substituting it into (2.6), we get

$$\Big[\int_\Omega(\eta u\cdot u_L^{\frac{p^*-p}{p}})^{p^*}dx\Big]^{\frac{p}{p^*}}\le C\int_\Omega|\nabla\eta|^p u^p\cdot u_L^{p^*-p}dx\le C.$$

Letting $L\to\infty$, by Fatou's lemma, we have

$$\Big(\int_{B_R} u^{\frac{p^{*2}}{p}}dx\Big)^{\frac{p}{p^*}}\le C. \tag{2.8}$$

By (2.5) and Sobolev inequality, we obtain

$$\begin{aligned}&\Big[\int_\Omega(\eta u\cdot u_L^{s-1})^{p^*}dx\Big]^{\frac{p}{p^*}}\\ &\le C\cdot s^{p-1}\int_\Omega(|\nabla\eta|^p u^p\cdot u_L^{p(s-1)}+\eta^p|v|^{\beta+1}u^{\alpha+1}\cdot u_L^{p(s-1)}+\eta^p u^{p^*}\cdot u_L^{p(s-1)})dx.\end{aligned}\tag{2.9}$$

If $p<\alpha+1<p^*$, then

$$\begin{aligned}\int\eta^p u^{\alpha+1}|v|^{\beta+1}u_L^{p(s-1)}dx&=\int(\eta u u_L^{s-1})^p\cdot u^{\alpha+1-p}|v|^{\beta+1}dx\\ &\le\Big(\int(\eta u u_L^{s-1})^{p^*}dx\Big)^{\frac{p}{p^*}}\Big(\int(u^{\alpha+1-p}|v|^{\beta+1})^{\frac{p^*}{p^*-p}}dx\Big)^{\frac{p^*-p}{p^*}}.\end{aligned}$$

In view of (2.7), it follows that

$$\Big[\int(\eta u\cdot u_L^{s-1})^{p^*}dx\Big]^{\frac{p}{p^*}}\le C\cdot s^{p-1}\int(|\nabla\eta|^p u^p\cdot u_L^{p(s-1)}+\eta^p u^{p^*}\cdot u_L^{p(s-1)})dx. \tag{2.10}$$

If $0<\alpha+1<p$, we have

$$\begin{aligned}&\int\eta u^{\alpha+1}|v|^{\beta+1}u_L^{p(s-1)}dx\le\int(\eta u u_L^{s-1})^{\alpha+1}\cdot u_L^{[p-(\alpha+1)](s-1)}|v|^{\beta+1}dx\\ &\le\Big(\int(\eta u u_L^{s-1})^{p^*}dx\Big)^{\frac{\alpha+1}{p^*}}\Big(\int_{R+r}(u^{[p-(\alpha+1)](s-1)}|v|^{\beta+1})^{\frac{p^*}{p^*-(\alpha+1)}}dx\Big)^{\frac{p^*-(\alpha+1)}{p^*}}.\end{aligned}$$

For a fixed $A>1$, we may choose R, r small suitably such that

$$\Big(\int(\eta u u_L^{s-1})^{p^*}dx\Big)^{\frac{\alpha+1}{p^*}}\le A\Big(\int(\eta u u_L^{s-1})^{p^*}dx\Big)^{\frac{p}{p^*}}.$$

Note that

$$\begin{aligned}&\int_{R+r} u^{\frac{[p-(\alpha+1)](s-1)p^*}{p^*-(\alpha+1)}}|v|^{\frac{p^*(\beta+1)}{p^*-(\alpha+1)}}dx\\ \le&\Big(\int|u|^{p^*}dx\Big)^{\frac{[p-(\alpha+1)](s-1)}{p^*-(\alpha+1)}}\Big(\int|v|^{\frac{p^*(\beta+1)}{p^*-(\alpha+1)-[p-(\alpha+1)](s-1)}}dx\Big)^{\frac{p^*-(\alpha+1)-[p-(\alpha+1)](s-1)}{p^*-(\alpha+1)}}.\end{aligned}$$

In view of (1.3), we may choose $s\ge1$ suitably such that

$$\int|v|^{\frac{p^*(\beta+1)}{p^*-(\alpha+1)-[p-(\alpha+1)](s-1)}}dx$$

exists. Then we get

$$\begin{aligned}&\int\eta^p u^{\alpha+1}|v|^{\beta+1}u_L^{p(s-1)}dx\\ \le&\int(\eta u u_L^{s-1})^{\alpha+1}\cdot u_L^{[p-(\alpha+1)](s-1)}|v|^{\beta+1}dx\\ \le& A\Big(\int(\eta u u_L^{s-1})^{p^*}dx\Big)^{\frac{p}{p^*}}\Big(\int_{2R}(u^{[p-(\alpha+1)](s-1)}|v|^{\beta+1})^{\frac{p^*}{p^*-(\alpha+1)}}dx\Big)^{\frac{p^*-(\alpha+1)}{p^*}}.\end{aligned}\tag{2.11}$$

According to the continuity of integral, we also obtain (2.10).

Now, let $t=\frac{p^{*2}}{p(p^*-p)}$. Using Holder inequality, we have

$$\int\eta^p u^{ps}dx\le C\Big(\int_{R+r}(\eta^p u^{ps})^{\frac{t}{t-1}}\Big)^{1-\frac1t}$$

and

$$\begin{aligned}&\int_{R+r}\eta^p u^{p^*+ps-p}dx=\int_{R+r}(\eta^p u^{ps})\cdot u^{p^*-p}dx\\ \le&\Big(\int_{R+r}(\eta^p u^{ps})^{\frac{t}{t-1}}dx\Big)^{1-\frac1t}\Big(\int_{R+r}u^{(p^*-p)t}dx\Big)^{\frac1t}\\ =&\Big(\int_{R+r}(\eta^p u^{ps})^{\frac{t}{t-1}}dx\Big)^{1-\frac1t}\Big(\int_{R+r}u^{\frac{p^{*2}}{p}}dx\Big)^{\frac1t}.\end{aligned}$$

By (2.8), there results

$$\int_{R+r}\eta^p u^{p^*+ps-p}dx\le C\Big(\int_{R+r}(\eta^p u^{ps})^{\frac{t}{t-1}}dx\Big)^{1-\frac1t}.\tag{2.12}$$

Combining with (2.10), we obtain

$$\begin{aligned}\Big[\int(\eta u u_L^{s-1})^{p^*}dx\Big]^{\frac{p}{p^*}}\le& C\Big(1+\frac{1}{r^p}\Big)s^{p-1}\Big(\int_{R+r}u^{s\cdot\frac{pt}{t-1}}dx\Big)^{1-\frac1t}\\ \le& Cr^{-p}s^{p-1}\Big(\int_{R+r}u^{s\cdot\frac{pt}{t-1}}dx\Big)^{1-\frac1t}.\end{aligned}$$

Thus

$$\Big[\int(\eta u u_L^{s-1})^{p^*}dx\Big]^{\frac{1}{s}} \le C^{\frac{p^*}{ps}}\cdot r^{\frac{-p^*}{s}}\cdot s^{\frac{p^*(p-1)}{ps}}\cdot\Big(\int_{R+r}u^{s\frac{pt}{t-1}}dx\Big)^{\frac{p^*}{ps}\frac{t-1}{t}}. \tag{2.13}$$

By the same methods of Theorem 1.1 (see [1]), setting $\tau = \frac{p^*(t-1)}{pt} > 1$, letting $s_i = \tau^i$, $r_i = 2^{-i}r$, $B_i = B_{R+2^{-i}r}(x_0)$ and letting $L \to \infty$ in (2.13) for $i = 1, 2, \cdots$, we have

$$\Big[\int_{B_R}(u^s)p^*dx\Big]^{\frac{1}{s}} \le C^{\frac{p^*}{ps}}\cdot r^{\frac{-p^*}{s}}\cdot s^{\frac{p^*(p-1)}{ps}}\cdot\Big(\int_{R+r}u^{s\frac{pt}{t-1}}dx\Big)^{\frac{p^*}{ps}\frac{t-1}{t}}. \tag{2.14}$$

Set $W = u^{\frac{pt}{t-1}}$, $T_i = (\int_{B_i}W^{s_i}dx)^{\frac{1}{s_i}}$, and apply (2.14) iteratively,

$$\begin{aligned}
T_{i+1} &= \Big(\int_{B_{i+1}}u^{\frac{pt}{t-1}\tau^{i+1}}dx\Big)^{\frac{1}{\tau^{i+1}}} = \Big(\int_{B_{i+1}}u^{\tau^i\cdot p^*}dx\Big)^{\frac{1}{\tau^{i+1}}}\\
&\le C^{\frac{1}{\tau^i}}\cdot r^{\frac{-p^*}{\tau^i}}\cdot 2^{\frac{p^*i}{\tau^i}}(\tau^i)^{\frac{p^*(p-1)}{p\tau^i}}\cdot\Big(\int_{B_i}u^{\tau^i\frac{p^*}{t-1}}dx\Big)^{\frac{1}{\tau^i}}\\
&= C^{\frac{1}{\tau^i}}\cdot r^{\frac{-p^*}{\tau^i}}\cdot 2^{\frac{p^*i}{\tau^i}}(\tau^i)^{\frac{p^*(p-1)}{p\tau^i}}\cdot T_i\\
&\le C^{\sum\limits_{k=0}^{i}(\frac{1}{\tau})^k}\cdot(r^{-p^*})^{\sum\limits_{k=0}^{i}(\frac{1}{\tau})^k}\cdot(2^{p^*})^{\sum\limits_{k=0}^{i}\frac{k}{\tau^k}}\cdot e^{\frac{p^*(p-1)}{p}\sum\limits_{k=0}^{i}\frac{k\ln\tau}{\tau^k}}\cdot T_0\\
&\le C^{\sum\limits_{k=0}^{i}(\frac{1}{\tau})^k}\cdot(r^{-p^*})^{\sum\limits_{k=0}^{i}(\frac{1}{\tau})^k}\cdot(2^{p^*})^{\sum\limits_{k=0}^{i}\frac{k}{\tau^k}}\cdot e^{\frac{p^*(p-1)}{p}\sum\limits_{k=0}^{i}\frac{k\ln\tau}{\tau^k}}\cdot\Big(\int_{R+r}u^{\frac{pt}{t-1}}dx\Big).
\end{aligned}$$

Note that

$$\sum_{k=0}^{\infty}\Big(\frac{1}{\tau}\Big)^k = \frac{\tau}{\tau-1} < +\infty, \qquad \sum_{k=0}^{\infty}\frac{k}{\tau^k} < +\infty.$$

Since $\frac{pt}{t-1} < p^*$ implies

$$\int_{B_{R+r}}u^{\frac{pt}{t-1}}dx < +\infty,$$

it follows that $T_{i+1} \le C$. Letting $i \to \infty$, we can conclude that $||u||^{\frac{pt}{t-1}}_{L^\infty(B_R)} \le C$, i.e., $||u||_{L^\infty(B_R)} \le C$. Since x_0 is an arbitrary point in Ω and $\overline{\Omega}$ is compact, $u \in L^\infty(\Omega)$ follows by the finitely covered for $u \ge 0$.

If the sign of u is changeable, write $u = u^+ - u^-$, where $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$. Let $\omega_L = \eta u^+ u_L^{+(s-1)}$, and take $\varphi = \eta^p u^+\cdot u_L^{+p(s-1)}$ in (2.2). We may deduce in the same way that $u^+ \in L^\infty(\Omega)$.

Because of the oddness of the first equation in the elliptic systems (1.1), $(-u, -v)$ is also a weak solution of the elliptic systems, thus $u^- \in L^\infty(\Omega)$. Noting that $|u| = u^+ + u^-$, we obtain $u \in L^\infty(\Omega)$.

Taking the place of u, p by v, q, corresponding with (1.3), we can also conclude in the same way as above that $v \in L^\infty(\Omega)$. This completes the proof of Theorem 1.1.

Proof of the Theorem 1.2. By Theorem 1.1, the fact that (u, v) is a solution of (1.1) implies that $(u, v) \in L^\infty(\Omega)\times L^\infty(\Omega)$. In a way similar to the arguments of Corollary 5.4

in [3], we can conclude that there exist $\mu > 0$, $\nu > 0$ such that $(u, v) \in C^{1,\mu}(\Omega) \times C^{1,\nu}(\Omega)$, thus the proof is completed by the results of Tolksdorf [5] and Serrin [6].

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