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α-TRANSIENCE AND α-RECURRENCE FOR RANDOM WALKS AND LÉVY PROCESSES****

ZHANG Huizeng* ZHAO Minzhi** YING Jiangang***

Abstract

The authors investigate the α-transience and α-recurrence for random walks and Lévy processes by means of the associated moment generating function, give a dichotomy theorem for not one-sided processes and prove that the process X is quasi-symmetric if and only if X is not α-recurrent for all $\alpha < 0$ which gives a probabilistic explanation of quasi-symmetry, a concept originated from C. J. Stone.

Keywords α-Transience, α-Recurrence, Quasi-symmetry, Lévy process
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§ 1. Introduction

Random walks and Lévy processes on $\mathbb{R}^n$ are very interesting and important (see [1, 3, 4, 6–8]). The state space of any Lévy process is a locally compact Abelian group G. If G is compactly generated, then G is the direct sum of a compact group D, a vector group V, and a lattice group L (see [2, p.90]). There are two integers a and b such that V is isomorphic to $\mathbb{R}^a$ and L is isomorphic to $\mathbb{Z}^b$. Thus the properties of Lévy processes on $\mathbb{R}^n$ may be generalized to Lévy processes on a compactly generated locally compact Abelian group with little effort. Let X be a Lévy process on $\mathbb{R}^n$ with convolution semigroup $\{\pi_t\}$ or a random walk with transition probability μ. We know that Lévy processes (or random walks) can be divided into two classes: recurrence and transience. If the genuine dimension of X is greater than 2, then X is transient. Thus the question that how to classify the transient Lévy process or random walks more finely is very interesting.

In the present paper, we shall discuss α-transience and α-recurrence with $\alpha \leq 0$ of Lévy processes (resp. random walk). We shall show that for any $\alpha < 0$, X is α-recurrent if and only if $E(e^{-\alpha L_N}) = \infty$ for all open neighborhoods of 0; while X is α-transient if and only if $E(e^{-\alpha L_N}) < \infty$ for all bounded open neighborhoods of 0. We call a probability measure ν quasi-symmetric if $\limsup\limits_{i\to\infty} \nu^i(K)^{\frac{1}{i}} = 1$ for some compact set K. The property of quasi-symmetric measure has been studied in [9]. We call X quasi-symmetric if π_1 (or if μ) is quasi-symmetric. We shall show that X is quasi-symmetric if and only if X is α-recurrent for all $\alpha < 0$, that is to say, the speed at which X escapes from any bounded open set can not be any exponential. This gives a probabilistic explanation of quasi-symmetry. If X is

Manuscript received May 21, 2003.
*Department of Mathematics, Zhejiang University, Hangzhou 310027, China.
School of Computer and Computing Science, Zhejiang University City College, Hangzhou 310015, China. **E-mail:** huizengzhang@sina.com
Institute of Mathematics, Fudan University, Shanghai 200433, China. **E-mail: greenzmz@sohu.com
***Institute of Mathematics, Fudan University, Shanghai 200433, China. **E-mail:** jgying@fudan.edu.cn
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not one-sided, then a dichotomy theorem holds. We may also use the moment generating function to characterize the α-transience and α-recurrence. On our another paper [?], to classify the quasi-symmetric Lévy processes more finely, we study the polynomial recurrence and polynomial transience.

Throughout this paper, for any finite positive measure μ on $\mathbb{R}^n$, define the characteristic function of μ on $\mathbb{R}^n$ as $\hat{\mu}(x) := \int_{\mathbb{R}^n} e^{i(x,y)}\mu(dy)$ and define the moment generating function of μ as $\pounds\mu(x) := \int_{\mathbb{R}^n} e^{(x,y)}\mu(dy)$. If i is a positive integer, we use μ^i to denote the i-fold convolution of μ. Let $\mu^0 := \delta_0$. For any Borel set A, let L_A denote the last exit time from A, let $G(A)$ denote the closed group generated by A and let $\overline{A}$ denote the closure of A. For any $x \in \mathbb{R}^n$, let $x^\perp$ denote the space $\{u : (u,x) = 0\}$ and δ_x denote the Dirac measure at the point x. Let $\aleph$ denote the collection of all sets $N \subseteq \mathbb{R}^n$ such that N is relative compact open and $0 \in N$. For any $h \geq 0$, let $I_h := \{(x^1, x^2, \cdots, x^n) \in \mathbb{R}^n : |x^i| \leq \frac{h}{2} \text{ for } 1 \leq i \leq n\}$.

§ 2. α-Transience and α-Recurrence of Random Walks

Let $X = \{X_i; P^x\}$ be a random walk on $\mathbb{R}^n$ with transition probability measure μ, where μ is genuinely n-dimensional, that is, there is no proper linear subspace of $\mathbb{R}^n$ contains $\operatorname{supp}\mu$. We use P to denote P^0 for convenience. A point $x \in \mathbb{R}^n$ is called possible if for each neighborhood N of 0 there is an integer $i \geq 0$ such that $P(X_i \in x + N) > 0$. We denote the set of all possible points by Σ. Then $\Sigma = \overline{\bigcup_{i=0}^{\infty} \operatorname{supp}\mu^i}$ which is the smallest closed semigroup that contains 0 and $\operatorname{supp}\mu$. The closed group generated by $\operatorname{supp}\mu^i - \operatorname{supp}\mu^i$ is independent of the positive integer i and we denote it by G_1. Let G be the group generated by $\operatorname{supp}\mu$. Then $G_1 \subseteq G$. Let m_G be the Haar measure on G and let $\wp$ denote the collection of all bounded Borel sets A on G such that $m_G(A) > 0$ and $m_G(\partial_G(A)) = 0$. Here $\partial_G(A)$ is the boundary of A relative to G.

We say μ is normalized (see [5, pp.64–75], or see [7]) if there is an integer n_1, $0 \leq n_1 \leq n$, and there are real numbers $\alpha_1, \cdots, \alpha_{n_1}$ such that

$$\hat{\mu}(2\pi n_1, \cdots, 2\pi n_{n_1}, 0, \cdots, 0) = \exp(2\pi i(n_1\alpha_1 + \cdots + n_{n_1}\alpha_{n_1}))$$

for all integers $n_1, \cdots, n_{n_1}$, and $|\hat{\mu}(\theta)| < 1$ for all other values of θ.

For any $\alpha \in \mathbb{R}$ and any Borel subset A on $\mathbb{R}^n$, define $V^\alpha(A) = \sum_{i=0}^{\infty} e^{-\alpha i}\mu^i(A)$. Then V^α is a measure with support Σ. In this section, we assume that $\alpha \leq 0$ which is the only interesting case.

Definition 2.1. *Given any x, y, we say that y can be reached from x, and write $x \curvearrowright y$, if for any $N \in \aleph$, $P^x(X_i \in y + N) > 0$ for some integer $i \geq 0$. We say that x and y communicate, and write $x \leftrightarrow y$, if x and y can be reached from each other.*

Lemma 2.1. (1) *Suppose that A_1, A_2, A are Borel sets and $A_1 + A_2 \subseteq A$. Then $\mu^{i+j}(A) \geq \mu^i(A_1)\mu^j(A_2)$ for any integers i, j.*

(2) *The relation $\leftrightarrow$ is an equivalent relation on $\mathbb{R}^n$.*

Proof. (1) Since $A_1 + A_2 \subseteq A$, $A_1 \subseteq A - z$ for all $z \in A_2$. For any integers i, j,

$$\mu^{i+j}(A) = \int_{\mathbb{R}^n} \mu^i(A - z)\mu^j(dz) \geq \int_{A_2} \mu^i(A - z)\mu^j(dz) \geq \mu^i(A_1)\mu^j(A_2).$$

(2) We need only to show that $x \leftrightarrow y$ and $y \leftrightarrow z$ imply that $x \leftrightarrow z$. For any $N \in \aleph$, there is $N_1 \in \aleph$ such that $N_1 + N_1 \subseteq N$. Since $x \curvearrowright y$ and $y \curvearrowright z$, $\mu^i(y + N_1 - x) =$

$P^x(X_i \in y+N_1) > 0$ and $\mu^j(z+N_1-y) = P^y(X_j \in z+N_1) > 0$ for some i,j. Then by (1), $P^x(X_{i+j} \in z+N) = \mu^{i+j}(z+N-x) \geq \mu^i(y+N_1-x)\mu^j(z+N_1-y) > 0$. Hence $x \curvearrowright z$. Similarly, $z \curvearrowright x$. Therefore $x \leftrightarrow y$.

Then $\mathbb{R}^n$ is divided into disjoint equivalent classes called communicating classes. Clearly, the set that can be reached from x is $x+\Sigma$ and the set that can communicate with x is $x+\Sigma\cap(-\Sigma)$.

A vector u is said to be a sided vector for μ if $u \neq 0$ and $\mu\{x : (u,x) > 0\} = 0$, strictly sided vector for μ if $\mu\{x : (u,x) \geq 0\} = 0$. The set of all sided vectors for μ, including 0, is a convex cone containing 0 which we denote by $\operatorname{side}\mu$. Similarly, the set of all strictly sided vectors for μ which we denote by $\operatorname{sside}\mu$ is a convex cone that does not contain 0. We say μ is not one-sided if $\operatorname{side}\mu = \{0\}$. Otherwise μ is one-sided. We say μ is strictly one-sided if $\operatorname{sside}\mu \neq \emptyset$. Let $g = \pounds\mu$, the moment generating function of μ. It can attain its infimum if and only if μ is not one-sided. If μ is not one-sided, then there exists a unique point u_0 such that $g(u_0) = \inf g$ (see [**?**, **?**]). The probability μ is quasi-symmetric if and only if $g(x) > 1$ for all $x \neq 0$ (see [9]), that is to say, $g(0) = \inf g$.

Theorem 2.1. *The measure μ is quasi-symmetric if and only if* $\limsup_{i\to\infty} \mu^i(x+N)^{\frac{1}{i}} = 1$ *for all $x \in G$ and all $N \in \aleph$. Particularly, if $G = G_1$, then* $\limsup$ *can be replaced by* $\lim$.

Proof. We need only to prove the only if part. Suppose that μ is quasi-symmetric. By [9], there is a compact set K such that $\lim_{i\to\infty} \mu^i(K)^{\frac{1}{i}} = 1$. We may assume that $K \subseteq G$ and $0 = m_G(\partial_G(K)) < m_G(K)$ without loss of generality. Firstly, if $G = G_1$, by [6, Corollary 1], for any $A \in \wp$, $\lim_{i\to\infty} \frac{\mu^i(A)}{\mu^i(K)} = \frac{m_G(A)}{m_G(K)}$. Thus $\lim_{i\to\infty} \mu^i(A)^{\frac{1}{i}} = 1$. For any $x \in G$ and $N \in \aleph$, there is $A \in \wp$ such that $A \subset N$. Therefore $\lim_{i\to\infty} \mu^i(x+N)^{\frac{1}{i}} = 1$.

Next, let $\rho = \frac{1}{2}\mu + \frac{1}{2}\delta_0$. Then $0 \in \operatorname{supp}\rho$ and hence $G(\operatorname{supp}\rho - \operatorname{supp}\rho) = G(\operatorname{supp}\rho) = G$. Since $\pounds\rho(x) = \frac{1}{2}\pounds\mu(x) + \frac{1}{2} > 1$ for all $x \neq 0$, the measure ρ is also a quasi-symmetric probability. So for any $x \in G$ and $N \in \aleph$, $\lim_{i\to\infty} \rho^i(x+N)^{\frac{1}{i}} = 1$. If for some $x \in G$ and $N \in \aleph$, $\limsup_{i\to\infty} \mu^i(x+N)^{\frac{1}{i}} < 1$, then for some $c < 0$, there is i_0 such that $\mu^i(x+N) < e^{ci}$ for all $i > i_0$. Hence

$$\rho^k(x+N) = \frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}\mu^i(x+N) \leq \frac{1}{2^k}\Big[\sum_{i=0}^{k}\binom{k}{i}e^{ci} + \sum_{i=0}^{i_0}\binom{k}{i}\Big] = \frac{1}{2^k}\Big[(1+e^c)^k + \sum_{i=0}^{i_0}\binom{k}{i}\Big].$$

Since $\lim_{k\to\infty} \binom{k}{i}^{\frac{1}{k}} = 1 < 1+e^c$, there exists an integer $k_0 > i_0$ such that when $k > k_0$, $\binom{k}{i} \leq (1+e^c)^k$ for all $i \leq i_0$. It follows that $\rho^k(x+N) \leq \frac{i_0+2}{2^k}(1+e^c)^k$ for all $k > k_0$, and then $\limsup_{k\to\infty} \rho^k(x+N)^{\frac{1}{k}} \leq \frac{1+e^c}{2} < 1$. This is a contradiction which shows that $\limsup_{i\to\infty} \mu^i(x+N)^{\frac{1}{i}} = 1$ for all $x \in G$ and all $N \in \aleph$.

Let $D = \{x : g(x) < \infty\}$, the effective domain of D. For any $x \in D$, define

$$\mu^x(dy) := \frac{e^{(x,y)}}{g(x)}\mu(dy).$$

Then μ^x is also a probability measure and has the same support with μ. This is similar to Doob's h-transform of μ. Let g^x be the moment generating function of μ^x. Then $g^x(y) = \frac{g(x+y)}{g(x)}$. Thus μ^x is quasi-symmetric for some $x \in D$ if and only if μ is not one-sided. In this case $g(x) = \inf g$.

Theorem 2.2. *The group G is a communicating class if and only if μ is not one-sided.*

Proof. Suppose G is a communicating class. For any $u \in \mathrm{side}\,\mu$, $\mathrm{supp}\,\mu \subseteq \{x : (u, x) \leq 0\}$ which is a closed semigroup. Then $\Sigma \subseteq \{x : (u, x) \leq 0\}$. Hence $G = \Sigma \cap (-\Sigma) \subseteq u^{\perp}$. Since μ is genuinely n-dimensional, $u = 0$. Therefore μ is not one-sided.

If μ is not one-sided, then μ^x is quasi-symmetric for some $x \in D$. By Theorem **??**, $G \subseteq \Sigma$ and hence $\Sigma = G$ which is a group. It follows that G is a communicating class.

Corollary 2.1. *Any closed semigroup S of $\mathbb{R}^n$ is a group if and only if either both $S \cap \{x : (u, x) > 0\}$ and $S \cap \{x : (u, x) < 0\}$ are empty or both of them are non-empty for any $u \neq 0$.*

Proof. It suffices to show the sufficiency. If $S = \{0\}$ then S is a group. Now we assume that the linear space generated by S is just $\mathbb{R}^n$ without loss of generality. Then for any $u \neq 0$, $S \cap \{x : (u, x) > 0\}$ is non-empty. Since S is separable, there are countable vectors $\{x_i\}$ in S such that $\overline{\{x_1, x_2, \cdots\}} = S$. Let $\nu := \sum\limits_{i=1}^{\infty} \frac{1}{2^i} \delta_{x_i}$. Then ν is a probability measure on $\mathbb{R}^n$ with $\mathrm{supp}\,\nu = S$. Thus ν is not one-sided. By Theorem **??**, $S \cup \{0\}$ is a group. There is $x \neq 0$ such that $x \in S$. Then $-x \in S$. Since S is a semigroup, $0 = x + (-x) \in S$. Therefore S is a group.

Definition 2.2. *A state $x \in \mathbb{R}^n$ is α-recurrent if $V^{\alpha}(x + N) = \infty$ for all $N \in \aleph$, and α-transient if $V^{\alpha}(x + N) < \infty$ for some $N \in \aleph$. The random walk $\{X_i\}$ is said to be α-transient if V^{α} is a Radon measure; α-recurrent if $V^{\alpha}(x + N) = \infty$ for all $x \in \Sigma$ and $N \in \aleph$.*

Proposition 2.1. *Suppose that $x \curvearrowright y$. If x is α-recurrent, then y is α-recurrent.*

Proof. For any $N \in \aleph$, there is $N_1 \in \aleph$ such that $N_1 + N_1 \subseteq N$. Since $x \curvearrowright y$, $\mu^i(y + N_1 - x) = P^x(X_i \in y + N_1) > 0$ for some i. By Lemma **??**, for any integer j, $\mu^{i+j}(y + N) \geq \mu^i(y + N_1 - x)\mu^j(x + N_1)$. Our result follows as desired.

This proposition tells us that the α-transience and α-recurrence are class properties. The process X is α-recurrent if and only if the state 0 is α-recurrent. If μ is not one-sided, then X is α-transient if and only if 0 is α-transient.

Proposition 2.2. (1) *If X is α-transient, then $\limsup\limits_{i\to\infty} \mu^i(N)^{\frac{1}{i}} \leq e^{\alpha}$ for any $N \in \aleph$.*

(2) *If $\limsup\limits_{i\to\infty} \mu^i(N)^{\frac{1}{i}} < e^{\alpha}$ for any $N \in \aleph$, then X is α-transient.*

(3) *If X is α-recurrent, then $\limsup\limits_{i\to\infty} \mu^i(x + N)^{\frac{1}{i}} \geq e^{\alpha}$ for any $x \in \Sigma$ and $N \in \aleph$.*

(4) *If $\limsup\limits_{i\to\infty} \mu^i(N)^{\frac{1}{i}} > e^{\alpha}$ for any $N \in \aleph$, then X is α-recurrent.*

Proof. Evidently, (2) and (3) hold. If $\limsup\limits_{i\to\infty} \mu^i(N)^{\frac{1}{i}} > e^{\alpha}$, then $\limsup\limits_{i\to\infty} \mu^i(N)^{\frac{1}{i}} > e^{\beta} > e^{\alpha}$ for some β. Thus there is a sequence $i_1 < i_2 < \cdots$, such that $\mu^{i_j}(N) > e^{\beta i_j}$. Hence $V^{\alpha}(N) \geq \sum\limits_{j=1}^{\infty} e^{-\alpha i_j}\mu^{i_j}(N) > \sum\limits_{j=1}^{\infty} e^{(\beta-\alpha)i_j} = \infty$. Therefore (1) and (4) hold.

We call X quasi-symmetric if the transition probability μ is quasi-symmetric. By this proposition and by Theorem **??**, we get the following corollary. It gives a probabilistic explanation of quasi-symmetric random walks.

Corollary 2.2. *The random walk X is quasi-symmetric if and only if X is α-recurrent for all $\alpha < 0$. Particularly, if X is recurrent, then X is quasi-symmetric.*

Since any symmetric probability measure is quasi-symmetric (see [9]), any symmetric random walk is α-recurrent for all $\alpha < 0$.

Lemma 2.2. *For any Borel set A, $\limsup P(L_A \geq i)^{\frac{1}{i}} = \limsup \mu^i(A)^{\frac{1}{i}}$. If $V^0(A) < \infty$, then $\limsup \left[\sum_{j=i}^{\infty} \mu^j(A)\right]^{\frac{1}{i}} = \limsup \mu^i(A)^{\frac{1}{i}}$.*

Proof. Clearly, $\sum_{j=i}^{\infty} \mu^j(A) \geq P(L_A \geq i) \geq \mu^i(A)$. If $\limsup \mu^i(A)^{\frac{1}{i}} = 1$, then our result holds. Now suppose that $\limsup \mu^i(A)^{\frac{1}{i}} < 1$, then $V^0(A) < \infty$. Hence $c_1 := \limsup \left[\sum_{j=i}^{\infty} \mu^j(A)\right]^{\frac{1}{i}} \leq 1$. If $\limsup \mu^i(A)^{\frac{1}{i}} < c < c_1$, then $c < 1$ and there exists an integer n_0, such that for any $j > n_0$, $\mu^j(A) < c^j$. For any $i > n_0$, $\sum_{j=i}^{\infty} \mu^j(A) < \sum_{j=i}^{\infty} c^j = \frac{c^i}{1-c}$. It follows that $\limsup \left[\sum_{j=i}^{\infty} \mu^j(A)\right]^{\frac{1}{i}} \leq c < c_1$. It is a contradiction which completes our proof.

Lemma 2.3. *If μ is normalized and quasi-symmetric, then for all $x \in G$ and all $h > 1$, $\lim_{k\to\infty} \mu^k(x + I_h)^{\frac{1}{k}} = 1$.*

Proof. By the proof of Lemma 5 in [7], $\lim_{k\to\infty} \mu^k(I_1)^{\frac{1}{k}} = 1$. For any $h > 1$, $I_1 + I_{h-1} \subseteq I_h$. For any fixed $x \in G$, since μ is quasi-symmetric, there is an integer i such that $\mu^i(x + I_{h-1}) > 0$. By Lemma **??**, $\mu^{i+j}(x + I_h) \geq \mu^i(x + I_{h-1})\mu^j(I_1)$ for any integer j. Consequently, $\lim_{k\to\infty} \mu^k(x + I_h)^{\frac{1}{k}} = 1$.

Lemma 2.4. *Suppose that $x \in D$. Then for any bounded Borel set A, there exists two positive constants k_1 and k_2 such that for all integer k,*

$$k_1 g(x)^k (\mu^x)^k(A) \leq \mu^k(A) \leq k_2 g(x)^k (\mu^x)^k(A).$$

Proof. Since $\mu^k(dy) = g(x)^k e^{-(x,y)}(\mu^x)^k(dy)$, $\mu^k(A) = g(x)^k \int_A e^{-(x,y)}(\mu^x)^k(dy)$. Thus our lemma holds with $k_1 = \inf_{y\in\overline{A}} e^{-(x,y)} > 0$ and $k_2 = \sup_{y\in\overline{A}} e^{-(x,y)} < \infty$ that are independent of k.

If μ is not one-sided, then we have the following dichotomy theorem.

Theorem 2.3. *Suppose that μ is not one-sided and $g(u) = \inf g = e^{\alpha_0}$. Let $x \in G$ and $N \in \aleph$. The following properties hold.*

(1) *There is a constant $M < \infty$ such that $\mu^i(x + N) \leq M e^{\alpha_0 i}$ for all integer i.*

(2) $\limsup_{k\to\infty} P(L_{x+N} \geq k)^{\frac{1}{k}} = \limsup_{k\to\infty} \mu^k(x + N)^{\frac{1}{k}} = e^{\alpha_0}$. ($\limsup$ *may be replaced by* $\lim$ *if $G = G_1$.) There exists $N_0 \in \aleph$, such that $\lim_{k\to\infty} P(L_{y+N_0} \geq k)^{\frac{1}{k}} = \lim_{k\to\infty} \mu^k(y + N_0)^{\frac{1}{k}} = e^{\alpha_0}$ for all $y \in G$.*

(3) *The random walk X is α-transient provided $\alpha > \alpha_0$, and α-recurrent provided $\alpha < \alpha_0$. It is either α_0-recurrent or α_0-transient. It is α_0-recurrent (resp. α_0-transient) if and only if $\{(\mu^u)^i\}$ is recurrent (resp. transient). When $n \geq 3$, X is α_0-transient.*

Proof. Since μ^u is a probability, the random walk with transition probability μ^u is either recurrent or transient and is transient when $n \geq 3$. By Lemma **??** and Corollary **??**,

(1) and (3) hold. For any $x \in G$ and $N \in \aleph$, by Lemma **??**,

$$\limsup_{k\to\infty} \mu^k(x+N)^{\frac{1}{k}} = g(u) \limsup_{k\to\infty} (\mu^u)^k(x+N)^{\frac{1}{k}},$$
$$\liminf_{k\to\infty} \mu^k(x+N)^{\frac{1}{k}} = g(u) \liminf_{k\to\infty} (\mu^u)^k(x+N)^{\frac{1}{k}}.$$

Since μ^u is quasi-symmetric, by Theorem **??**, Lemma **??** and Lemma **??**, (2) holds.

If μ is one-sided, it is getting more complicated. In general, let $\alpha_1 = \inf\{\alpha : X$ is α-transient$\}$ and $\alpha_2 = \sup\{\alpha : X$ is α-recurrent$\}$. By Lemma **??** and by Proposition **??**, we get the following theorem.

Theorem 2.4. *Suppose that $x \in \Sigma$ and $N \in \aleph$.*

(1) *$\alpha_2 \le \alpha_1 \le \alpha_0$, where $\alpha_0 = \inf \ln g$.*

(2) *The random walk X is α-transient provided $\alpha > \alpha_1$ and α-recurrent provided $\alpha < \alpha_2$. It is neither α-recurrent nor α-transient if $\alpha_2 < \alpha < \alpha_1$.*

(3) $e^{\alpha_2} \le \limsup_{k\to\infty} P(L_{x+N} \ge k)^{\frac{1}{k}} = \limsup_{k\to\infty} \mu^k(x+N)^{\frac{1}{k}} \le e^{\alpha_1}$.

Thus (α_1, α_2) characterize the escape speed and we call (α_1, α_2) the decay parament for X. If X is quasi-symmetric, then $\alpha_1 = \alpha_2 = 0$ and the speed at which X escapes from any bounded open set N is slower than any exponential. If $\alpha_0 = -\infty$, then $\alpha_1 = \alpha_2 = -\infty$. Particularly, if X is strictly one-sided, then X is α-transient for all $\alpha \le 0$. The escaping speed is quicker than any exponential.

If X is not one-sided, then $\alpha_2 = \alpha_1 = \alpha_0$. It is difficult to determine α_1 and α_2 if X is one-sided. For any finite measure ν on $\mathbb{R}^n$, let $(\alpha_1^\nu, \alpha_2^\nu)$ be the decay parameter for $\{\nu^i\}$ and let $\alpha_0^\nu = \inf \ln g^\nu$ where g^ν is the moment generating function of ν. We say ν is not essentially one-sided if the restriction of ν to the linear space generated by $\operatorname{supp}\nu$ is not one-sided, that is, for any $u \ne 0$, either both $\nu\{x : (u,x) > 0\}$ and $\nu\{x : (u,x) < 0\}$ are zero or both are positive. We have the following comparison result.

Lemma 2.5. (Comparison) *If $\nu \le \mu$, then $\alpha_1^\nu \le \alpha_1$ and $\alpha_2^\nu \le \alpha_2$. In particular, if ν is not essential one-sided, then $\alpha_2 \ge \alpha_0^\nu$.*

Proof. Clearly, for any α, that $\{\mu^i\}$ is α-transient implies that $\{\nu^i\}$ is α-transient. Hence $\alpha_1^\nu \le \alpha_1$. For any α, if $\{\nu^i\}$ is α-recurrent, then 0 is an α-recurrent state for $\{\nu^i\}$. Since $\nu \le \mu$, 0 is also an α-recurrent state for $\{\mu^i\}$. Hence $\{\mu^i\}$ is α-recurrent. It follows that $\alpha_2^\nu \le \alpha_2$. Suppose that ν is not essential one-sided. Let ν_1 be the restriction of ν to the space generated by $\operatorname{supp}\nu$. Then ν_1 is not one-sided, $\inf g^{\nu_1} = \inf g^\nu$ and $\alpha_2^{\nu_1} = \alpha_2^\nu$. It follows that $\alpha_2^\nu = \alpha_0^\nu$ and hence $\alpha_2 \ge \alpha_0^\nu$.

Theorem 2.5. *If there is $u \in \operatorname{side}\mu$ such that $u \ne 0$ and $g(x) < \infty$ for all x in $u^\perp$, then $\alpha_2 = \alpha_1 = \alpha_0$.*

Proof. We may assume $\alpha_0 > -\infty$. Let ν_1 be the restriction of μ to $u^\perp$. Then $g^{\nu_1} \le g$ and $\inf g^{\nu_1} \le \inf g$. We shall show that $\inf g^{\nu_1} = \inf g$. Otherwise $\inf g > \inf g^{\nu_1} + 2\varepsilon$ for some $\varepsilon > 0$. Since $\operatorname{supp}\nu_1 \subset u^\perp$, there is a point $x \in u^\perp$ such that $g^{\nu_1}(x) < \inf g^{\nu_1} + \varepsilon$. Then $\inf g > g^{\nu_1}(x) + \varepsilon$. By our condition, $g(x) < \infty$. Now for any $t > 0$, by Lebesgue's dominated convergence theorem,

$$\lim_{t\to+\infty} g(x+tu) = \lim_{t\to+\infty} \int_{(u,y)\le 0} e^{(x,y)} e^{t(u,y)} \mu(dy) = \int_{u^\perp} e^{(x,y)} \mu(dy) = g^{\nu_1}(x).$$

It follows that $\inf g \le g^{\nu_1}(x)$. It is a contradiction which shows that $\inf g^{\nu_1} = \inf g > 0$ and $\nu_1 \ne 0$.

If ν_1 is not one-sided, then we finish. Otherwise for any $u_1 \neq 0$ with $u_1 \in \operatorname{side}\nu_1$, let ν_2 be the restriction of ν_1 to $u_1^\perp \cap u^\perp$. Since $g^{\nu_1}(x) \leq g(x) < \infty$ for all $x \in u^\perp$, $\inf g^{\nu_2} = \inf g^{\nu_1} = \inf g > 0$. Continue this program until ν_i is not one-sided or $i = n$. This i is denoted by i_0. Since ν_n is the restriction of ν to $\{0\}$, ν_n is not one-sided. Therefore ν_{i_0} is not one-sided and $\inf g^{\nu_{i_0}} = \inf g = e^{\alpha_0}$. By Lemma **??**, $\alpha_2 \geq \alpha_0$ and hence $\alpha_2 = \alpha_1 = \alpha_0$.

For any Borel set A, let $T_A = \inf\{j \geq 0 : X_j \in A\}$. Then $T_A \circ \theta_i < \infty$ if and only if $L_A \geq i$.

Lemma 2.6. *Suppose that $x \in G$, $N, N_1 \in \aleph$ with $N_1 + N_1 \subseteq N$. Let $T_i = i + T_{x+N_1} \circ \theta_i$ and $S_i = i + T_{x+N} \circ \theta_i$. Then for any $\alpha \in \mathbb{R}$ and integer $i \geq 0$,*

$$V^\alpha(N_1)E(e^{-\alpha T_i}; T_i < \infty) \leq E\sum_{h=i}^{\infty} e^{-\alpha h} 1_{x+N}(X_h) \leq V^\alpha(N-N)E(e^{-\alpha S_i}; S_i < \infty).$$

In Particular,

$$V^0(N_1)P(L_{x+N_1} \geq i) \leq \sum_{h=i}^{\infty} \mu^h(x+N) \leq V^0(N-N)P(L_{x+N} \geq i).$$

Proof. Let $Y = \sum_{j=0}^{\infty} e^{-\alpha j} 1_{x+N}(X_j)$. Then Y is $\mathcal{F}$-measurable. For any stopping time T, $Y \circ \theta_T = \sum_{j=0}^{\infty} e^{-\alpha j} 1_{x+N}(X_j \circ \theta_T) = e^{\alpha T} \sum_{j=T}^{\infty} e^{-\alpha j} 1_{x+N}(X_j)$.

Firstly, since $T_i \geq i$, by the strong Markov property, one has

$$\begin{aligned} E\sum_{j=i}^{\infty} e^{-\alpha j} 1_{x+N}(X_j) &\geq E\sum_{j=T_i}^{\infty} e^{-\alpha j} 1_{x+N}(X_j) \\ &= E(e^{-\alpha T_i} Y \circ \theta_{T_i}) = E[e^{-\alpha T_i} E^{X_{T_i}}(Y); T_i < \infty]. \end{aligned}$$

We know that $X_{T_i} \in x + N_1$ if $T_i < \infty$. By that $N_1 + N_1 \subseteq N$, for any $y \in N_1$,

$$E^{x+y}(Y) = E\sum_{k=0}^{\infty} e^{-\alpha k} 1_{N-y}(X_k) \geq E\sum_{k=0}^{\infty} e^{-\alpha k} 1_{N_1}(X_k) = V^\alpha(N_1).$$

Hence $E\sum_{j=i}^{\infty} e^{-\alpha j} 1_{x+N}(X_j) \geq V^\alpha(N_1)E(e^{-\alpha T_i}; T_i < \infty)$.

Secondly, by the strong Markov property, we have

$$\begin{aligned} E\sum_{j=i}^{\infty} e^{-\alpha j} 1_{x+N}(X_j) &= E\sum_{j=S_i}^{\infty} e^{-\alpha j} 1_{x+N}(X_j) \\ &= E(e^{-\alpha S_i} Y \circ \theta_{S_i}) = E[e^{-\alpha S_i} E^{X_{S_i}}(Y); S_i < \infty]. \end{aligned}$$

We know that $X_{S_i} \in x + N$ if $S_i < \infty$. For any $y \in N$,

$$E^{x+y}(Y) = E\sum_{k=0}^{\infty} e^{-\alpha k} 1_{N-y}(X_k) \leq E\sum_{k=0}^{\infty} e^{-\alpha k} 1_{N-N}(X_k) = V^\alpha(N-N).$$

Hence $E\sum_{j=i}^{\infty} e^{-\alpha j} 1_{x+N}(X_j) \leq V^\alpha(N-N)E(e^{-\alpha S_i}; S_i < \infty)$.

Theorem 2.6. *Suppose that* $\alpha < 0$. *Then* X *is* α*-recurrent if and only if* $E(e^{-\alpha L_{x+N}}) = \infty$ *for all* $x \in \Sigma$ *and* $N \in \aleph$; X *is* α*-transient if and only if* $E(e^{-\alpha L_N}) < \infty$ *for all* $N \in \aleph$.

Proof. If X is recurrent, then $P(L_{x+N} = \infty) = 1$ for all $x \in \Sigma$ and $N \in \aleph$. Thus our theorem holds. Now we shall assume that X is transient. For any $x \in \Sigma$ and $N \in \aleph$,

$$E[e^{-\alpha L_{x+N}} - 1] = E\Big[\sum_{i=0}^{\infty}(e^{-\alpha} - 1)e^{-\alpha i}1_{\{L_{x+N}-1 \geq i\}}\Big] = \sum_{i=0}^{\infty}(e^{-\alpha} - 1)e^{-\alpha i}P(L_{x+N} \geq i+1).$$

Let $c_1 = \frac{1}{V^0(N-N)}$ and $c_2 = \frac{1}{V^0(N)}$. Then $c_1, c_2 > 0$. Now by the lemma above,

$$\begin{aligned} &c_1 \sum_{i=0}^{\infty}(e^{-\alpha} - 1)e^{-\alpha i} \sum_{j=i+1}^{\infty} \mu^j(x+N) \\ &\leq E[e^{-\alpha L_{x+N}} - 1] \leq c_2 \sum_{i=0}^{\infty}(e^{-\alpha} - 1)e^{-\alpha i} \sum_{j=i+1}^{\infty} \mu^j(x+N+N). \end{aligned}$$

Thus our theorem holds since for any Borel set A,

$$\sum_{i=0}^{\infty}(e^{-\alpha} - 1)e^{-\alpha i} \sum_{j=i+1}^{\infty} \mu^j(A) = \sum_{j=1}^{\infty} \mu^j(A) \sum_{i=0}^{j-1}(e^{-\alpha} - 1)e^{-\alpha i} = V^{\alpha}(A) - V^0(A).$$

§ 3. α-Transience and α-Recurrence of Lévy Processes

Let $X = (X_t; P^x)$ be a genuinely n-dimensional Lévy process on $\mathbb{R}^n$ with convolution semigroup $\pi = \{\pi_t; t > 0\}$. Let $P = P^0$ and $E = E^0$ for convenience. A point $x \in \mathbb{R}^n$ is called possible if for each neighborhood N of 0 there is $t > 0$ such that $\pi_t(x+N) = P(X_t \in N + x) > 0$. We denote the set of all possible points by Σ. Then $\Sigma = \overline{\bigcup_{t>0} \operatorname{supp} \pi_t}$ which is a closed sub-semigroup of $\mathbb{R}^n$. Let G be the smallest closed group including Σ. The closed group generated by $\operatorname{supp} \pi_t - \operatorname{supp} \pi_t$ is independent of t (see [3, Proposition 5.1]) and we denote it by G_1. Then $G_1 \subseteq G$. The following dichotomy theorem is well known (see [1, 3, 4]).

Theorem 3.1. (1) *The Lévy process* X *is either recurrent or transient.*
(2) X *is recurrent if and only if* $P(L_N = \infty) = 1$ *for all* $N \in \aleph$.
(3) X *is transient if and only if* $P(L_N = \infty) = 0$ *for all* $N \in \aleph$.
(4) *If* $n \geq 3$, *then* X *is transient.*

Thus Lévy processes are divided into two classes: recurrent and transient. However, from the examples shown below, we may see that there is still big difference among transient Lévy processes. Though the uniform translation and the Poisson process are both transient, their escaping speed from a compact set is quite different. The former is much quicker than the latter. In this section, we aim to distinguish those transient Lévy processes more precisely.

Example 3.1. The uniform translation X with $X_t = t$ is transient. It is easy to see that for any compact set K, $\lim_{t \to \infty} e^{-\alpha t} P(L_K > t) = 0$ for all $\alpha \in \mathbb{R}$.

Example 3.2. The Poisson process X with parameter $\lambda > 0$ is also transient. For any fixed nonnegative integer d, we have $P(L_{\{d\}} > t) = P(X_t \le d) = e^{-\lambda t} \sum\limits_0^d \frac{(\lambda t)^i}{i!}$. Then

$$\lim_{t\to\infty} e^{-\alpha t} P(L_{\{d\}} > t) = \lim_{t\to\infty} e^{-(\alpha+\lambda)t} \sum_0^d \frac{(\lambda t)^i}{i!} = \begin{cases} 0, & \alpha > -\lambda, \\ 1, & \alpha = -\lambda \text{ and } d = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

For any $\alpha \in \mathbb{R}$ and any Borel set A, define $V^\alpha(A) = \int_0^\infty e^{-\alpha t}\pi_t(A)\,dt$. We consider $\alpha \le 0$ which is the only interesting case. Similarly as random walk, we have

Definition 3.1. *Given any x, y, we say that y can be reached from x, and write $x \curvearrowright y$, if for any $N \in \aleph$, $P^x(X_t \in y + N) > 0$ for some $t > 0$. We say that x and y communicate, and write $x \leftrightarrow y$, if x and y can be reached from each other. A state x is α-recurrent if $V^\alpha(x + N) = \infty$ for all $N \in \aleph$, and α-transient if $V^\alpha(x + N) < \infty$ for some $N \in \aleph$.*

Proposition 3.1. (1) *Suppose that A_1, A_2, A are Borel sets and $A_1 + A_2 \subseteq A$. Then $\pi_{t+s}(A) \ge \pi_t(A_1)\pi_s(A_2)$ for any $t, s > 0$.*
(2) *The relation $\leftrightarrow$ is an equivalent relation on $\mathbb{R}^n$.*
(3) *Suppose $x \curvearrowright y$. If x is α-recurrent, then y is α-recurrent.*

Thus $\mathbb{R}^n$ is divided into disjoint equivalent classes called communicating classes. The group G is a communicating class if and only if Σ is a group. The α-transience (or α-recurrence) is a class property. We call X (strictly) one-sided if π_1 is (strictly) one-sided. Similarly, X is not one-sided if π_1 is not one-sided.

Theorem 3.2. *The group G is a communicating class if and only if X is not one-sided.*

Proof. Suppose G is a communicating class. For any $u \in \operatorname{side}\pi_1$, we have $u \in \operatorname{side}\pi_t$ for all $t > 0$ and hence $\Sigma \subseteq \{x : (u, x) \le 0\}$. Then $G = \Sigma \cap (-\Sigma) \subseteq u^\perp$. Since X is genuinely n-dimensional, $u = 0$. It follows that X is not one-sided. Conversely, if X is not one-sided, then for any $u \ne 0$, $\Sigma \cap \{x : (u, x) > 0\} \supseteq \operatorname{supp}\pi_1 \cap \{x : (u, x) > 0\} \ne \emptyset$. By Corollary **??**, Σ is a group and hence G is a communicating class.

Definition 3.2. *The Lévy process $\{X_t\}$ is said to be α-transient if V^α is a Radon measure. It is said to be α-recurrent if $V^\alpha(N + x) = \infty$ for all $x \in \Sigma$ and $N \in \aleph$.*

It is easy to see that 0-recurrence or 0-transience is just the usual recurrence and transience defined in [1, 3, 4], and as α decreases, the set of α-transient Lévy processes decreases, while the set of α-recurrent ones increases. The Lévy process X is α-recurrent (resp. α-transient) if and only if all states $x \in \Sigma$ are α-recurrent (resp. α-transient). By Proposition **??**, X is α-recurrent if and only if 0 is α-recurrent. If X is not one-sided, then X is α-transient if and only if 0 is α-transient. For any $h > 0$, $\{X_{hi}\}$ is a random walk on $\mathbb{R}^n$ with transition probability π_h.

Lemma 3.1. *Suppose that $N \in \aleph$ and N_1, N_2 are Borel sets with $N + N_1 \subset N_2$.*
(1) *There is $h > 0$, such that $\inf\limits_{0\le t\le h} \pi_t(N) > 0$.*
(2) *For such h,*

$$\limsup_{i\to\infty} \pi_{hi}(N_2)^{\frac{1}{ih}} \ge \limsup_{t\to\infty} \pi_t(N_1)^{\frac{1}{t}} \ge \limsup_{i\to\infty} \pi_{hi}(N_1)^{\frac{1}{ih}},$$
$$\liminf_{i\to\infty} \pi_{hi}(N_2)^{\frac{1}{ih}} \ge \liminf_{t\to\infty} \pi_t(N_2)^{\frac{1}{t}} \ge \liminf_{i\to\infty} \pi_{hi}(N_1)^{\frac{1}{ih}}.$$

Proof. We need to prove (2). Let $\varepsilon = \inf_{0\le t\le h} \pi_t(N)$. Then $\varepsilon > 0$. Firstly, for any $0 \le t \le h$, $\pi_{(i+1)h}(N_2) \ge \pi_{ih+t}(N_1)\pi_{h-t}(N) \ge \varepsilon\pi_{ih+t}(N_1)$. For any β, if $\limsup_{t\to\infty} \pi_t(N_1)^{\frac{1}{t}} > e^\beta$, then $\beta < 0$ and there is $i_1 < i_2 < \cdots$ and $\{t_j\}$ with $i_j \in \mathbb{Z}^+$ and $0 \le t_j \le h$ such that $\pi_{i_jh+t_j}(N_1) > e^{\beta(i_jh+t_j)}$. Thus $\pi_{(i_j+1)h}(N_2) > \varepsilon e^{\beta(i_jh+t_j)} \ge \varepsilon e^{\beta(i_j+1)h}$. So $\limsup_{i\to\infty} \pi_{hi}(N_2)^{\frac{1}{ih}} \ge e^\beta$. This shows that $\limsup_{i\to\infty} \pi_{hi}(N_2)^{\frac{1}{ih}} \ge \limsup_{t\to\infty} \pi_t(N_1)^{\frac{1}{t}}$.

Secondly, for any $0 \le t \le h$, $\pi_{ih+t}(N_2) \ge \pi_{ih}(N_1)\pi_t(N) \ge \varepsilon\pi_{ih}(N_1)$. For any β, if $\liminf_{i\to\infty} \pi_{hi}(N_1)^{\frac{1}{ih}} > e^\beta$, then $\beta < 0$ and there is some integer i_0, such that $\pi_{ih}(N_1) > e^{\beta ih}$ provided the integer $i > i_0$. Then $\pi_{ih+t}(N_2) \ge \varepsilon e^{\beta ih} \ge \varepsilon e^{\beta(ih+t)}$ for all integer $i > i_0$ and $0 \le t \le h$. This shows that $\liminf_{t\to\infty} \pi_t(N_2)^{\frac{1}{t}} \ge \liminf_{i\to\infty} \pi_{hi}(N_1)^{\frac{1}{ih}}$. We complete the proof of (2) since the other inequalities are obvious.

We say $\{\pi_t\}$ (or X) is quasi-symmetric if it satisfies

Condition 3.1. *There exists a compact subset $K \subset \mathbb{R}^n$ such that* $\limsup_{t\to\infty} \pi_t(K)^{\frac{1}{t}} = 1$.

Condition 3.1 was introduced by S. C. Port and C. J. Stone [3] to give a ratio limit theorem. The process X is quasi-symmetric if and only if the probability measure π_1 is quasi-symmetric (see [**?**, **?**]).

Theorem 3.3. (1) *The $\{\pi_t\}$ is quasi-symmetric if and only if* $\limsup_{t\to\infty} \pi_t(y+B)^{\frac{1}{t}} = 1$ *for all $y \in G$ and all $B \in \aleph$. Furthermore, if $G = G_1$, then* lim sup *can be replaced by* lim.

(2) *Suppose X is quasi-symmetric and π_1 is normalized, then for any $r > 1$,* $\lim_{t\to\infty} \pi_t(y+I_r)^{\frac{1}{t}} = 1$ *for all $y \in G$.*

Proof. The if part of (1) is got by the definition. Now suppose that $\{\pi_t\}$ is quasi-symmetric. For any $y \in G$ and any $B \in \aleph$, since $G = \Sigma = \overline{\bigcup_{t>0} \operatorname{supp} \pi_t}$, there is $x \in \operatorname{supp} \pi_{t_0}$ with some $t_0 > 0$ and $N \in \aleph$, such that $x + N \subset y + B$. For this N, there is $N_1 \in \aleph$ such that $N_1 + N_1 \subset N$. By Lemma **??**, there is $h > 0$ such that $\inf_{0\le t\le h} \pi_t(N_1) > 0$ and $t_0 = jh$ for some integer j. Then $x \in \bigcup_{i=0}^{\infty} \operatorname{supp} \pi_{hi}$ and π_h is quasi-symmetric. If $G = G_1$, then $G(\operatorname{supp} \pi_h) = G(\operatorname{supp} \pi_h - \operatorname{supp} \pi_h)$. Therefore the only if part of (1) holds by Lemma **??** and by Theorem **??**. Suppose $r > 1$. Then there is $r_0 > 1$ and $B \in \aleph$ such that $I_{r_0} + B \subset I_r$. We have shown that for any $y \in G$, there is $x \in G$, $h > 0$ and $N, N_1 \in \aleph$, such that $x + N \subset y + B$, $N_1 + N_1 \subset N$, $\inf_{0\le t\le h} \pi_t(N_1) > 0$ and $x \in \bigcup_{i=0}^{\infty} \operatorname{supp} \pi_{hi}$. The measure π_h is quasi-symmetric and normalized since $\hat{\pi}_h = (\hat{\pi}_1)^h$. Now Lemma **??** and Lemma **??** yield the statement (2).

Lemma 3.2. *Suppose N, N_1, N_2 as in Lemma* **??** *and $h > 0$.*

(1) *If* $\sum_{i=0}^{\infty} e^{-\alpha hi}\pi_{hi}(N_1) = \infty$, *then* $\int_0^\infty e^{-\alpha t}\pi_t(N_2)\,dt = \infty$.

(2) *If* $\inf_{0\le t\le h} \pi_t(N) > 0$ *and* $\int_0^\infty e^{-\alpha t}\pi_t(N_1)\,dt = \infty$, *then* $\sum_{i=0}^{\infty} e^{-\alpha hi}\pi_{hi}(N_2) = \infty$.

Proof. Since $\alpha \le 0$, we have

$$\int_0^\infty e^{-\alpha t}\pi_t(N_2)\,dt = \sum_{i=0}^\infty \int_{ih}^{(i+1)h} e^{-\alpha t}\pi_t(N_2)\,dt$$
$$\geq \sum_{i=0}^\infty e^{-\alpha hi}\int_0^h \pi_{hi}(N_1)\pi_s(N)\,ds = \int_0^h \pi_s(N)\,ds\Big[\sum_{i=0}^\infty e^{-\alpha hi}\pi_{hi}(N_1)\Big].$$

It follows that (1) holds since $\int_0^h \pi_s(N)\,ds > 0$.

Let $c = \inf_{0\leq t\leq h} \pi_t(N)$. Then $c > 0$. For any $0 \leq t \leq h$, $\pi_{(i+1)h}(N_2) \geq c\pi_{ih+t}(N_1)$. Thus $c\int_0^h \pi_{ih+t}(N_1)\,dt \leq h\pi_{(i+1)h}(N_2)$. It follows that

$$c\int_0^\infty e^{-\alpha s}\pi_s(N_1)\,ds = c\sum_{i=0}^\infty \int_{ih}^{(i+1)h} e^{-\alpha s}\pi_s(N_1)\,ds$$
$$\leq c\sum_{i=0}^\infty e^{-\alpha h(i+1)}\int_0^h \pi_{ih+t}(N_1)\,dt \leq h\sum_{i=0}^\infty e^{-\alpha h(i+1)}\pi_{(i+1)h}(N_2).$$

Therefore if $\int_0^\infty e^{-\alpha t}\pi_t(N_1)\,dt = \infty$, then $\sum\limits_{i=0}^\infty e^{-\alpha hi}\pi_{hi}(N_2) = \infty$.

Proposition 3.2. (1) *If for some $h > 0$, $\{X_{hi}\}$ is αh-recurrent, then X is α-recurrent.*

(2) *For any $h > 0$, X is α-transient if and only if $\{X_{hi}\}$ is αh-transient.*

(3) *For any $h > 0$, $\inf_{0\leq t\leq h} \pi_t(N) > 0$ for all $N \in \aleph$ if and only if $\inf_{0\leq t\leq 1} \pi_t(N) > 0$ for all $N \in \aleph$. In this case, for any $h > 0$, X is α-recurrent if and only if the random walk $\{X_{hi}\}$ is αh-recurrent.*

Proof. By Lemma **??**, (1) and the necessity of (2) hold. If X is not α-transient, then there is $N \in \aleph$ such that $\int_0^\infty e^{-\alpha t}\pi_t(N)\,dt = \infty$. For any $a > 0$, let $B_a = \{x : \|x\| < a\}$. There is h_0 such that $\inf_{0\leq t\leq h_0} \pi_t(B_1) > 0$. Since $\pi_{jt}(B_j) \geq \pi_{(j-1)t}(B_{j-1})\pi_t(B_1) \geq \cdots \geq \pi_t(B_1)^j$, $\inf_{0\leq t\leq jh_0} \pi_t(B_j) > 0$ for all positive integer j. Hence for any $h > 0$, there is $N_1 \in \aleph$ such that $\inf_{0\leq t\leq h} \pi_t(N_1) > 0$. By Lemma **??**, $\sum\limits_{i=0}^\infty e^{-\alpha hi}\pi_{hi}(N + N_1) = \infty$. Thus $\{X_{hi}\}$ is not αh-transient. Hence the sufficiency of (2) holds. Suppose that $\inf_{0\leq t\leq h} \pi_t(N) > 0$ for all $N \in \aleph$. Fix any integer $j > 1$. For any $N \in \aleph$, there is $N_1 \in \aleph$ such that $\underbrace{N_1 + \cdots + N_1}_{j} \subseteq N$. Let $c := \inf_{0\leq t\leq h} \pi_t(N_1)$. Then $c > 0$. Since $\pi_{jt}(N) \geq \pi_t(N_1)^j$, $\inf_{0\leq t\leq jh} \pi_s(N) \geq c^j > 0$. Thus (3) holds.

Proposition 3.3. (1) *If X is α-transient, then $\limsup_{t\to\infty} \pi_t(N)^{\frac{1}{t}} \leq e^\alpha$ for any $N \in \aleph$.*

(2) *If $\limsup_{t\to\infty} \pi_t(N)^{\frac{1}{t}} < e^\alpha$ for any $N \in \aleph$, then X is α-transient.*

(3) *If X is α-recurrent, then $\limsup_{t\to\infty} \pi_t(x + N)^{\frac{1}{t}} \geq e^\alpha$ for any $x \in \Sigma$ and $N \in \aleph$.*

(4) *If $\limsup_{t\to\infty} \pi_t(N)^{\frac{1}{t}} > e^\alpha$ for any $N \in \aleph$, then X is α-recurrent.*

Proof. The statements (2) and (3) are obvious. For any $N \in \aleph$ and $\varepsilon > 0$, there is $h > 0$, such that $\pi_t(N) \geq \varepsilon$ for all $0 \leq t \leq h$. If X is α-transient, then $\{X_{hi}\}$ is αh-transient and hence $\limsup_{i\to\infty} \pi_{hi}(N+N)^{\frac{1}{ih}} \le e^{\alpha}$. By Lemma **??**, one has

$$\limsup_{t\to\infty} \pi_t(N)^{\frac{1}{t}} \le \limsup_{i\to\infty} \pi_{hi}(N+N)^{\frac{1}{ih}} \le e^{\alpha}.$$

Therefore (1) holds. If $\limsup_{t\to\infty} \pi_t(N)^{\frac{1}{t}} > e^{\alpha}$, then $\limsup_{i\to\infty} \pi_{hi}(N+N)^{\frac{1}{ih}} > e^{\alpha}$. Consequently, $\sum_{i=0}^{\infty} e^{-\alpha hi}\pi_{hi}(N+N) = \infty$. By Lemma **??** , $\int_0^{\infty} e^{-\alpha t}\pi_t(N+N+N)\,dt = \infty$. Therefore (4) holds.

Corollary 3.1. *The Lévy process X is quasi-symmetric if and only if X is α-recurrent for all $\alpha < 0$. In particular, if X is symmetric, then X is α-recurrent for all $\alpha < 0$. If X is recurrent, then X is quasi-symmetric.*

This corollary gives a probabilistic explanation of quasi-symmetric Lévy processes. Let $g = \pounds\pi_1$. Then $\pounds\pi_t = g^t$. For any $x \in D := \{g < \infty\}$, define $\pi_t^x(dy) := \frac{e^{(x,y)}}{g(x)^t}\pi_t(dy)$. Then $\{\pi_t^x\}$ is also a convolution semigroup. The following lemma is obvious by the compactness of $\overline{A}$.

Lemma 3.3. *Suppose $x \in D$. Then for any bounded Borel set A, there exist two positive constants k_1 and k_2 such that for all $t > 0$,*

$$k_1 g(x)^t \pi_t^x(A) \le \pi_t(A) \le k_2 g(x)^t \pi_t^x(A).$$

By Lemma **??**, Proposition **??**, Theorem **??** and by Theorem **??**, we get the following dichotomy theorem.

Theorem 3.4. *Suppose that X is not one-sided and $g(u) = \inf g = e^{\alpha_0}$. Let $x \in G$ and $N \in \aleph$.*

(1) *There is a constant M such that $\pi_t(x+N) \le Me^{\alpha_0 t}$.*

(2) $\limsup_{t\to\infty} \pi_t(x+N)^{\frac{1}{t}} = e^{\alpha_0}$. *(Here* lim sup *can be replaced by* lim *if $G = G_1$.) There exists a $N_0 \in \aleph$, such that $\lim_{t\to\infty} \pi_t(y+N_0)^{\frac{1}{t}} = e^{\alpha_0}$ for all $y \in G$.*

(3) *The Lévy process X is α-transient provided $\alpha > \alpha_0$ and α-recurrent provided $\alpha < \alpha_0$. It is either α_0-recurrent or α_0-transient. It is α_0-recurrent (resp. α_0-transient) if and only if the Lévy process $\{\pi_t^u\}$ is recurrent (resp. transient). When $n \ge 3$, X is α_0-transient.*

Let $\alpha_1 = \inf\{\alpha : X \text{ is } \alpha\text{-transient}\}$ and $\alpha_2 = \sup\{\alpha : X \text{ is } \alpha\text{-recurrent}\}$. By Lemma **??** and Proposition **??**, we get the following theorem.

Theorem 3.5. (1) *$\alpha_0 \ge \alpha_1 \ge \alpha_2$, where $\alpha_0 = \inf \ln g$.*

(2) *For any $x \in \Sigma$ and $N \in \aleph$, $e^{\alpha_2} \le \limsup_{t\to\infty} \pi_t(x+N)^{\frac{1}{t}} \le e^{\alpha_1}$.*

(3) *The Lévy process X is α-transient provided $\alpha > \alpha_1$ and α-recurrent provided $\alpha < \alpha_2$. It is neither α-recurrent nor α-transient if $\alpha_2 < \alpha < \alpha_1$.*

We call (α_1, α_2) the decay parament for X. If $\alpha_0 = -\infty$, then $\alpha_1 = \alpha_2 = -\infty$.

Corollary 3.2. *If X is strictly one-sided, then X is α-transient for all $\alpha \le 0$.*

For any convolution $\{\nu_t\}$ with $\nu_1(\mathbb{R}^n) \le 1$, let $(\alpha_1^{\nu}, \alpha_2^{\nu})$ be the decay parameter for $\{\nu_t\}$ and let $\alpha_0^{\nu} = \inf \ln g^{\nu}$ where g^{ν} is the moment generating functions of ν_1. Similarly as that in random walk, we have

Proposition 3.4. (1) *If $\nu_1 \le \pi_1$, then $\alpha_1^\nu \le \alpha_1$ and $\alpha_2^\nu \le \alpha_2$. Particularly, if ν_1 is not essential one-sided, then $\alpha_2 \ge \alpha_0^\nu$.*

(2) *If there is $u \in \operatorname{side}\pi_1$ such that $u \ne 0$ and $g(x) < \infty$ for all x in $u^\perp$, then $\alpha_2 = \alpha_1 = \alpha_0$.*

For any Borel set A, let $T_A = \inf\{t > 0 : X_t \in A\}$, the first hitting time of A.

Lemma 3.4. *Suppose that $x \in G$, $N, N_1 \in \aleph$ with $\overline{N}_1 + N_1 \subseteq N$. Let $T_t = t + T_{x+N_1} \circ \theta_t$ and $S_t = t + T_{x+N} \circ \theta_t$. Then for any $\alpha \in \mathbb{R}$ and $t \ge 0$, we have*

$$V^\alpha(N_1)E(e^{-\alpha T_t}; T_t < \infty) \le E\int_t^\infty e^{-\alpha h} 1_{x+N}(X_h)\,dh \le V^\alpha(N - \overline{N})E(e^{-\alpha S_t}; S_t < \infty).$$

Proof. Let $Y = \int_0^\infty e^{-\alpha s} 1_{x+N}(X_s)\,ds$. Then Y is $\mathcal{F}$-measurable. Firstly, since $T_t \ge t$,

$$\begin{aligned} E\int_t^\infty e^{-\alpha h} 1_{x+N}(X_h)\,dh &\ge E\int_{T_t}^\infty e^{-\alpha h} 1_{x+N}(X_h)\,dh \\ &= E(e^{-\alpha T_t} Y \circ \theta_{T_t}) = E[e^{-\alpha T_t} E^{X_{T_t}}(Y); T_t < \infty]. \end{aligned}$$

We know that $X_{T_t} \in x + \overline{N}_1$ provided $T_t < \infty$. By that $\overline{N}_1 + N_1 \subseteq N$, for any $y \in \overline{N}_1$,

$$E^{x+y}(Y) = E\int_0^\infty e^{-\alpha s} 1_{N-y}(X_s)\,ds \ge E\int_0^\infty e^{-\alpha s} 1_{N_1}(X_s)\,ds = V^\alpha(N_1).$$

Hence $E\int_t^\infty e^{-\alpha h} 1_{x+N}(X_h)\,dh \ge V^\alpha(N_1)E(e^{-\alpha T_t}; T_t < \infty)$.

Secondly, by the strong Markov property

$$\begin{aligned} E\int_t^\infty e^{-\alpha h} 1_{x+N}(X_h)\,dh &= E\int_{S_t}^\infty e^{-\alpha h} 1_{x+N}(X_h)\,dh \\ &= E(e^{-\alpha S_t} Y \circ \theta_{S_t}) = E[e^{-\alpha S_t} E^{X_{S_t}}(Y); S_t < \infty]. \end{aligned}$$

We know that $X_{S_t} \in x + \overline{N}$ provided $S_t < \infty$. For any $y \in \overline{N}$,

$$E^{x+y}(Y) = E\int_0^\infty e^{-\alpha s} 1_{N-y}(X_s)\,ds \le E\int_0^\infty e^{-\alpha s} 1_{N-\overline{N}}(X_s)\,ds = V^\alpha(N - \overline{N}).$$

Hence $E\int_t^\infty e^{-\alpha h} 1_{x+N}(X_h)\,dh \le V^\alpha(N - \overline{N})E(e^{-\alpha S_t}; S_t < \infty)$.

Corollary 3.3. *Suppose that $x \in G$, $N, N_1 \in \aleph$ with $\overline{N}_1 + N_1 \subseteq N$. Then for any $t > 0$, we have*

$$V^0(N_1)P(L_{x+N_1} > t) \le \int_t^\infty \pi_h(x + N)\,dh \le V^0(N - \overline{N})P(L_{x+N} > t).$$

Theorem 3.6. *Suppose that $\alpha < 0$. Then X is α-recurrent if and only if $E(e^{-\alpha L_{x+N}}) = \infty$ for all $x \in \Sigma$ and $N \in \aleph$; is α-transient if and only if $E(e^{-\alpha L_N}) < \infty$ for all $N \in \aleph$.*

Proof. If X is recurrent, then $P(L_{x+N} = \infty) = 1$ for all $x \in \Sigma$ and $N \in \aleph$. Thus our theorem holds. Now we shall assume that X is transient. For any $x \in \Sigma$ and $N \in \aleph$, there holds

$$E[e^{-\alpha L_{x+N}} - 1] = E\Big[\int_0^\infty -\alpha e^{-\alpha t} 1_{\{L_{x+N} > t\}}\,dt\Big] = \int_0^\infty -\alpha e^{-\alpha t} P(L_{x+N} > t)\,dt.$$

Let $c_1 = \frac{1}{V^0(N-\overline{N})}$ and $c_2 = \frac{1}{V^0(N)}$. Then $c_1, c_2 > 0$. Now by Corollary **??**, we have

$$c_1 \int_0^\infty -\alpha e^{-\alpha t} \int_t^\infty \pi_h(x+N)\, dh\, dt$$
$$\leq E[e^{-\alpha L_{x+N}} - 1] \leq c_2 \int_0^\infty -\alpha e^{-\alpha t} \int_t^\infty \pi_h(x+N+\overline{N})\, dh\, dt.$$

Thus our theorem holds since for any Borel set A,

$$\int_0^\infty -\alpha e^{-\alpha t} \int_t^\infty \pi_h(A)\, dh\, dt = \int_0^\infty \pi_h(A) \int_0^h -\alpha e^{-\alpha t}\, dt\, dh = V^\alpha(A) - V^0(A).$$

Thus the classification of α-transient and α-recurrent is determined by the exponential moments of the last exit times. This classification is more precise. For any probability measure μ and any $\lambda > 0$, define $\mu_t^\lambda := e^{-\lambda t} \sum\limits_{i=0}^{\infty} \frac{(\lambda t)^i \mu^i}{i!}$. Then $\{\mu_t^\lambda\}$ is a convolution semigroup. Let $\Sigma = \overline{\bigcup\limits_{i=0}^{\infty} \operatorname{supp} \mu^i}$. Then $\Sigma = \operatorname{supp} \mu_t^\lambda$ for any $t > 0$. Let g and g_λ be the moment generating function of μ and of μ_1^λ respectively. Then $g_\lambda = e^{\lambda(g-1)}$. Thus $\inf \ln g_\lambda = \lambda(e^{\inf \ln g} - 1)$. For any Borel set A and any $\beta \in \mathbb{R}$, let $V_\lambda^\beta(A) = \int_0^\infty e^{-\beta t} \mu_t^\lambda(A)\, dt$ and let $V^\beta(A) = \sum\limits_{i=0}^{\infty} e^{-\beta i} \mu^i(A)$.

Proposition 3.5. *The following properties hold:*
(1) *The measure μ is quasi-symmetric if and only if μ_1^λ is quasi-symmetric.*
(2) *The set* $\operatorname{side} \mu = \operatorname{side} \mu_1^\lambda$. *Thus, μ is one-sided if and only if μ_1^λ is one-sided.*
(3) *The measure μ_1^λ is not strictly one-sided.*
(4) *The Lévy process $\{\mu_t^\lambda\}$ is $-\lambda$-recurrent.*
(5) *Suppose $\beta = \lambda(e^\alpha - 1)$. Then for any Borel set A, $V_\lambda^\beta(A) = \dfrac{1}{\lambda e^\alpha} V^\alpha(A)$.*

Proof. By the equality $g_\lambda = e^{\lambda(g-1)}$, we get (1). The statement (2) is trivial. Since $\mu_1^\lambda(\{0\}) \geq e^{-\lambda} > 0$, μ_1^λ is not strictly one-sided. Since

$$V_\lambda^{-\lambda}(\{0\}) = \int_0^\infty e^{\lambda t} \mu_t^\lambda(\{0\})\, dt \geq \int_0^\infty dt = \infty,$$

$\{\mu_t^\lambda\}$ is $-\lambda$-recurrent. For any Borel set A,

$$V_\lambda^\beta(A) = \int_0^\infty e^{-(\beta+\lambda)t} \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} \mu^i(A)\, dt = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \mu^i(A) \int_0^\infty e^{-\lambda e^\alpha t} t^i\, dt.$$

It follows that $V_\lambda^\beta(A) = \sum\limits_{i=0}^{\infty} \frac{\lambda^i}{i!} \mu^i(A) \frac{i!}{(\lambda e^\alpha)^{i+1}} = \frac{1}{\lambda e^\alpha} V^\alpha(A)$.

Suppose the decay parameter for $\{\mu^i\}$ is (α_1, α_2), and the decay parameter for $\{\mu_t^\lambda\}$ is (β_1, β_2). Then by Proposition **??**(4), $\beta_2 \geq -\lambda$.

Corollary 3.4. (1) *The random walk $\{\mu^i\}$ is α-recurrent (resp. α-transient) if and only if the Lévy process $\{\mu_t^\lambda\}$ is $\lambda(e^\alpha - 1)$-recurrent (resp. $\lambda(e^\alpha - 1)$-transient).*
(2) $\beta_1 = \lambda(e^{\alpha_1} - 1)$ *and* $\beta_2 = \lambda(e^{\alpha_2} - 1)$.

Thus the α-transient or α-recurrent property of a random walk can be converted into the $\lambda(e^\alpha - 1)$-transient or $\lambda(e^\alpha - 1)$-recurrent property of some Lévy process.

§ 4. Examples

By Theorem **??**, if $\limsup_{t\to\infty} e^{-\alpha t}P(L_N > t) > 0$ for all $N \in \aleph$, then X is α-recurrent. But the converse is not always true.

Example 4.1. Let $\{b_t\}$ be the Gaussian convolution semigroup on $\mathbb{R}$ and $\pi_t = b_t * \delta_{-t}$. Let X be the Lévy process with convolution semigroup $\{\pi_t\}$. Since $\pounds\pi_1(1) = \inf \pounds\pi_1(x) = \inf e^{\frac{x^2}{2}-x} = e^{-\frac{1}{2}}$ and $\pi_t^1 = b_t$, X is α-transient provided $\alpha > -\frac{1}{2}$ and is α-recurrent provided $\alpha \le -\frac{1}{2}$. In fact for any compact set K with $m(K) > 0$, $\pi_t(K) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{t}{2}} \int_K e^{-\frac{x^2}{2t}} e^{-x}\, dx$. Hence $\pi_t(K) \sim \frac{1}{\sqrt{t}} e^{-\frac{t}{2}}$. It follows that $\lim_{t\to\infty} \pi_t(K)^{\frac{1}{t}} = e^{-\frac{1}{2}}$ and

$$V^\alpha(K) = \int_0^\infty e^{-\alpha t}\pi_t(K)\, dt \begin{cases} < \infty, & \alpha > -\dfrac{1}{2}, \\ = \infty, & \alpha \le -\dfrac{1}{2}. \end{cases}$$

For any $N \in \aleph$, let $c = \sup_{x\in N} |x|$. Then $c < \infty$ and

$$\int_t^\infty \pi_s(N)\, ds \le \frac{m(N)e^c}{\sqrt{2\pi t}} e^{-\frac{t}{2}} \int_t^\infty e^{-\frac{s-t}{2}}\, ds = \frac{2m(N)e^c}{\sqrt{2\pi t}} e^{-\frac{t}{2}}.$$

Therefore $\lim_{t\to\infty} e^{\frac{t}{2}} \int_t^\infty \pi_s(N)\, ds = 0$. By Corollary **??**, $\lim_{t\to\infty} e^{\frac{t}{2}} P(L_N > t) = 0$.

Example 4.2. The Brownian motion on $\mathbb{R}^n$ is transient whenever $n \ge 3$. But it is symmetric and hence is α-recurrent for all $\alpha < 0$. In fact for any compact set K with $m(K) > 0$, $P(L_K \ge t) \ge \pi_t(K) \sim t^{-\frac{n}{2}}$ and hence $V^\alpha(K) = \infty$ for all $\alpha < 0$.

The two examples above are of not one-sided Lévy processes. Finally we shall give two other examples. The first is a Lévy process whose decay parameter is (α_0, α_0). But it is neither α_0-recurrent nor α_0-transient. The second is a random walk whose decay parameter is not (α_0, α_0).

Example 4.3. Let $X^{(1)}$ be the Poisson process with $\lambda > 0$ on $\mathbb{R}$ and $X^{(2)}$ be the Brownian motion on $\mathbb{R}^n$. Suppose that $X^{(1)}$ and $X^{(2)}$ are independent. Let $X = (X^{(1)}, X^{(2)})$. Suppose its corresponding convolution is $\{\pi_t\}$. Then for any $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}^n$, $\pounds\pi_1(x_1, x_2) = e^{\lambda(e^{x_1}-1)+\frac{\|x_2\|^2}{2}}$. Thus $\inf \ln \pounds\pi_1 = -\lambda$.

For any nonnegative integer i and any compact subset K of $\mathbb{R}^n$ with $m(K) > 0$, we have

$$\pi_t(\{i\}\times K) = e^{-\lambda t}\frac{(\lambda t)^i}{i!}\Big[\int_K \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{\|x\|^2}{2t}}\, dx\Big] \sim e^{-\lambda t} t^{i-\frac{n}{2}}.$$

Thus

$$V^\alpha(\{i\}\times K) = \int_0^\infty e^{-\alpha t}\pi_t(\{i\}\times K)\, dt \begin{cases} < \infty, & \alpha > -\lambda, \\ = \infty, & \alpha < -\lambda, \\ < \infty, & \alpha = -\lambda \text{ and } i < \dfrac{n}{2} - 1, \\ = \infty, & \alpha = -\lambda \text{ and } i \ge \dfrac{n}{2} - 1. \end{cases}$$

Therefore X is α-transient if $\alpha > -\lambda$, α-recurrent if $\alpha < -\lambda$. Thus $\alpha_1 = \alpha_2 = \alpha_0 = -\lambda$. If $n \le 2$, then X is $-\lambda$-recurrent. If $n \ge 3$, then X is neither $-\lambda$-recurrent nor $-\lambda$-transient.

Example 4.4. Let $\mu = \frac{1}{2}\delta_{(1,0)} + \frac{1}{2}\pi_1 \times \delta_1$, where $\{\pi_t\}$ is the Cauchy convolution semigroup on $\mathbb{R}$. Then μ is a probability measure on $\mathbb{R}^2$. Since $\pounds\pi_1(x) = \infty$ for any $x \neq 0$,

$$\pounds\mu(x_1, x_2) = \frac{1}{2}\pounds\delta_{(1,0)}(x_1, x_2) + \frac{1}{2}\pounds\pi_1(x_1)\pounds\delta_1(x_2) = \begin{cases} \infty, & x_1 \neq 0, \\ \frac{1}{2} + \frac{1}{2}e^{x_2}, & x_1 = 0. \end{cases}$$

Thus $\inf \ln \pounds\mu = -\ln 2$.

For any integer i, using the binomial formula, we have

$$\mu^i = \frac{1}{2^i}\sum_{j=0}^{i}\binom{i}{j}\delta_{(1,0)}^j * (\pi_1 \times \delta_1)^{i-j} = \frac{1}{2^i}\sum_{j=0}^{i}\binom{i}{j}(\pi_{i-j} * \delta_j) \times \delta_{i-j}.$$

For any compact set K on $\mathbb{R}$ with $m(K) > 0$ and any nonnegative integer d, if $i > d$, then $\mu^i(K \times \{d\}) = \frac{1}{2^i}\binom{i}{i-d}(\pi_d * \delta_{i-d})(K)$. If $d = 0$, then $\mu^i(K \times \{d\}) = \frac{1}{2^i}\delta_i(K) = 0$ for sufficiently large i. Now we suppose that $d > 0$. The measure $\pi_d * \delta_{i-d}$ has density $\frac{d}{\pi[d^2+(x+d-i)^2]} \sim i^{-2}$. Thus $\mu^i(K \times \{d\}) \sim \frac{i^{d-2}}{2^i}$. Consequently

$$V^\alpha(K \times \{d\}) = \sum_{0}^{\infty} e^{-\alpha i}\mu^i(K \times \{d\}) \begin{cases} < \infty, & \alpha > -\ln 2, \\ = \infty, & \alpha \leq -\ln 2. \end{cases}$$

Therefore X is α-transient provided $\alpha > -\ln 2$, and X is neither α-transient nor α-recurrent provided $\alpha \leq -\ln 2$. Thus $\alpha_1 = \alpha_0 = -\ln 2$ and $\alpha_2 = -\infty$.

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