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TOEPLITZ ALGEBRAS ON DISCRETE GROUPS AND THEIR NATURAL MORPHISMS***

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Abstract

Let G be a discrete group, E_1 and E_2 be two subsets of G with $E_1 \subseteq E_2$, and $e \in E_2$. Denote by $\mathcal{T}^{E_1}$ and $\mathcal{T}^{E_2}$ the associated Toeplitz algebras. In this paper, it is proved that the natural morphism γ^{E_2,E_1} from $\mathcal{T}^{E_1}$ to $\mathcal{T}^{E_2}$ exists as a C^*-morphism if and only if E_2 is finitely covariant-lifted by E_1. Based on this necessary and sufficient condition, some applications are made.

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§ 1. Introduction

The object of the present paper is to study the natural morphisms between Toeplitz algebras. If G is a discrete group and $E \subseteq G$, one may form the associated Toeplitz algebra $\mathcal{T}^E$. Given two subsets E_1 and E_2 with $E_1 \subseteq E_2$, there is a natural morphism $\gamma^{E_2,E_1} : \mathcal{T}^{E_1} \to \mathcal{T}^{E_2}$. In some cases, this morphism fails to be a C^*-morphism, or even fails to be well defined. Our main task is to put forth a necessary and sufficient condition under which γ^{E_2,E_1} exists as a C^*-morphism.

Toward this end, a technique initiated by E. Park in [2] and generalized in [6] yielded the finite decomposition condition. However, as shown in [4], while this condition is sufficient for the existence of γ^{E_2,E_1}, it is not necessary. In this paper we put forth a more natural condition, called the finite covariant-lift condition. We will show that this latter condition is not only sufficient, but also necessary (see Theorem 2.1). Based on this new condition, some applications also have been made.

The paper is organized in the following way. In Section 2, we give the precise definition of the finite covariant-lift condition, and show that it is both necessary and sufficient for the existence of γ^{E_2,E_1}. In Section 3, we show that Toeplitz algebras associated to quasi-ordered groups have a certain universal property (see Corollary 3.1). In Section 4, we extend certain results from [4] to the nonabelian case. These results concern the natural morphism between Toeplitz algebras corresponding to a quasi-ordered group and its induced partially-ordered group. Using the finite covariant-lift condition, our arguments become much simpler than those in [4]. Finally, in Section 5 we turn to study Toeplitz algebras on discrete abelian groups. In [1], G. Murphy proved that if (G, G_+) is a discrete abelian ordered group, then

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the Toeplitz algebra $\mathcal{T}^{G_+}$ has a universal property for isometric representations of G_+. As a generalization, it was proved in [5] that the same property holds for Toeplitz algebras associated to discrete abelian quasi-ordered groups. In this section, we will show that the converse is also true (see Theorem 5.1).

§ 2. The Natural Morphisms Between Toeplitz Algebras on Discrete Groups

Let G be a discrete group and $\{\delta_g \,|\, g \in G\}$ be the usual orthonormal basis for $\ell^2(G)$, where

$$\delta_g(h) = \begin{cases} 1, & \text{if } \; g = h, \\ 0, & \text{otherwise} \end{cases}$$

for $g, h \in G$. For any $g \in G$, we define a unitary operator u_g on $\ell^2(G)$ by $u_g(\delta_h) = \delta_{gh}$ for $h \in G$. For any subset E of G, let $\ell^2(E)$ be the closed subspace of $\ell^2(G)$ generated by $\{\delta_g \,|\, g \in E\}$, and let p^E denote the projection from $\ell^2(G)$ onto $\ell^2(E)$.

Definition 2.1. *The C^*-algebra generated by $\{T_g^E := p^E u_g p^E \mid g \in G\}$ is denoted by $\mathcal{T}^E$ and is called the Toeplitz algebra with respect to E.*

Definition 2.2. *Let E_1 and E_2 be two subsets of G with $E_1 \subseteq E_2$. E_2 is said to be finitely covariant-lifted by E_1 if for any finite subset F of G, there exists $g_* \in G$ such that for any $g \in F$, $g \in E_2$ if and only if $g \cdot g_* \in E_1$.*

Remark 2.1. (i) Let E_1 and E_2 be two subsets of G with $E_1 \subseteq E_2$. Then E_2 is finitely covariant-lifted by E_1 if and only if the following three conditions are satisfied:

(1) For any two non-empty subsets $F_1 \subseteq E_2$ and $F_2 \subseteq G \setminus E_2$, there exists $g_* \in G$ such that $F_1 \cdot g_* \subseteq E_1$ and $F_2 \cdot g_* \subseteq G \setminus E_1$. A diagram illustrating such a condition is as follows:

$$\begin{array}{ccc} F_1 \subseteq E_2 & \qquad & F_2 \subseteq G \setminus E_2 \\ \downarrow & & \downarrow \\ F_1 \cdot g_* \subseteq E_1 & & F_2 \cdot g_* \subseteq G \setminus E_1; \end{array}$$

(2) For any finite non-empty subset $F_1 \subseteq E_2$, there exists $g_1 \in G$ such that $F_1 \cdot g_1 \subseteq E_1$;

(3) For any finite non-empty subset $F_2 \subseteq G \setminus E_2$, there exists $g_2 \in G$ such that $F_2 \cdot g_2 \subseteq G \setminus E_1$.

(ii) If $e \in E_2$ and E_2 is finitely covariant-lifted by E_1, then upon replacing the finite subset F by $F \cup \{e\}$, we see that g_* may be chosen in E_1. Furthermore, if $e \in E_1$, then we may choose $g_2 = e$, so the condition (3) above is satisfied automatically.

Theorem 2.1. *Let G be a discrete group and $E_1, E_2 \subseteq G$ with $E_1 \subseteq E_2$ and $e \in E_2$. Then the natural morphism $\gamma^{E_2,E_1} : \mathcal{T}^{E_1} \to \mathcal{T}^{E_2}$, which satisfies $\gamma^{E_2,E_1}(T_g^{E_1}) = T_g^{E_2}$ for any $g \in G$, exists as a C^*-morphism if and only if E_2 is finitely covariant-lifted by E_1.*

Proof. Suppose γ^{E_2,E_1} exists as a C^*-algebra morphism. For a contradiction, suppose that E_2 is not finitely covariant-lifted by E_1. Then Remark 2.1 implies that one of the following three cases must occur:

Case 1. There exist two finite non-empty subsets $F_1 \subseteq E_2$ and $F_2 \subseteq G \setminus E_2$ such that for any $g_* \in E_1$,

$$\text{if } \; F_1 \cdot g_* \subseteq E_1, \quad \text{then } \; (F_2 \cdot g_*) \cap E_1 \neq \emptyset. \tag{2.1}$$

Let $F_1 = \{g_1, g_2, \cdots, g_n\}$ and $F_2 = \{h_1, h_2, \cdots, h_m\}$. Set

$$T = \Big(\prod_{j=1}^{m}(1 - T^{E_1}_{h_j^{-1}} T^{E_1}_{h_j})\Big) \cdot \Big(\prod_{i=1}^{n} T^{E_1}_{g_i^{-1}} T^{E_1}_{g_i}\Big).$$

Then

$$\gamma^{E_2,E_1}(T) = \Big(\prod_{j=1}^{m}(1 - T^{E_2}_{h_j^{-1}} T^{E_2}_{h_j})\Big) \cdot \Big(\prod_{i=1}^{n} T^{E_2}_{g_i^{-1}} T^{E_2}_{g_i}\Big).$$

By (2.1) we know that $T = 0$, so $\gamma^{E_2,E_1}(T) = 0$. But clearly $\gamma^{E_2,E_1}(T)\,\delta_e = \delta_e \neq 0$, yielding a contradiction.

Case 2. There exists a finite subset $F = \{g_1, g_2, \ldots, g_n\} \subseteq E_2$, such that for any $g_* \in E_1$, there exists $g_{i_0} \in F$, such that $g_{i_0} \cdot g_* \notin E_1$. Let

$$T = \prod_{i=1}^{n} T^{E_1}_{g_i^{-1}} T^{E_1}_{g_i}.$$

Then $T = 0$, but $\gamma^{E_2,E_1}(T)\,\delta_e = \delta_e \neq 0$, which is a contradiction.

Case 3. There exists a finite subset $F = \{h_1, h_2, \cdots, h_m\} \subseteq G \setminus E_2$, such that for any $g_* \in E_1$, there exists $h_{j_0} \in F$, such that $h_{j_0} \cdot g_* \in E_1$. Let

$$T = \prod_{j=1}^{m}(1 - T^{E_1}_{h_j^{-1}} T^{E_1}_{h_j}).$$

Then $T = 0$, but $\gamma^{E_2,E_1}(T)\,\delta_e = \delta_e \neq 0$, which is a contradiction.

Now, for the reverse direction, suppose that E_2 is finitely covariant-lifted by E_1. Let T be an operator in $\mathcal{T}^{E_1}$ of the form

$$T = \sum_{i=1}^{m} \xi_i \prod_{j=1}^{n_i} T^{E_1}_{g_{ij}}.$$

Then

$$\gamma^{E_2,E_1}(T) = \sum_{i=1}^{m} \xi_i \prod_{j=1}^{n_i} T^{E_2}_{g_{ij}} \qquad \text{for} \;\; \xi_i \in C,\, g_{ij} \in G.$$

To show that γ^{E_2,E_1} is well defined and can be extended as a C^*-morphism, it suffices to show that $\|\gamma^{E_2,E_1}(T)\| \leq \|T\|$, as these operators are dense in $\mathcal{T}^{E_1}$.

Given $\varepsilon > 0$, there exists $\xi \in \ell^2(E_2)$ with finite support such that $\|\xi\| = 1$, and $\|\gamma^{E_2,E_1}(T)\| \leq \|\gamma^{E_2,E_1}(T)\xi\| + \varepsilon$. We may write $\xi = \sum\limits_{p=1}^{n} \eta_p \delta_{h_p}$ with $\eta_p \in C$ and $h_p \in E_2$ for $p = 1, 2, \cdots, n$. Let

$$F = \Big\{\Big(\prod_{j=l_i}^{n_i} g_{ij}\Big) \cdot h_p \,\Big|\, 1 \leq l_i \leq n_i,\ \forall i, p\Big\} \bigcup \{\, h_p \,|\, \forall p\,\}.$$

Then by the assumption there exists $g_* \in E_1$ such that for any $g \in F$, we have $g \in E_2$ if and only if $g \cdot g_* \in E_1$. So for any $\Big(\prod\limits_{j=l_i}^{n_i} g_{ij}\Big) \cdot h_p \in F$,

$$\Big(\prod_{j=l_i}^{n_i} g_{ij}\Big) \cdot h_p \in E_2 \quad \text{if and only if} \quad \Big(\prod_{j=l_i}^{n_i} g_{ij}\Big) \cdot h_p \cdot g_* \in E_1. \tag{2.2}$$

Let $\theta = \sum_{p=1}^{n} \eta_p \delta_{h_p g_*}$. Since $h_p \in E_2$, we know that $h_p \cdot g_* \in E_1$. Therefore $\theta \in \ell^2(E_1)$ with $\|\theta\| = \|\xi\| = 1$, and by (2.2) we know that $\|\gamma^{E_2,E_1}(T)\,\xi\| = \|T\,\theta\|$. Thus

$$\|\gamma^{E_2,E_1}(T)\| \le \|\gamma^{E_2,E_1}(T)\,\xi\| + \varepsilon = \|T\,\theta\| + \varepsilon \le \|T\| + \varepsilon.$$

Since ε is arbitrary, it follows that $\|\gamma^{E_2,E_1}\| \le \|T\|$, hence γ^{E_2,E_1} is bounded.

§ 3. The Weakly Universal Property of Toeplitz Algebras

Next, we discuss a certain universal property concerning the natural morphisms which will be used in the sequel. Throughout this section, G is a discrete group, G_+ is a sub-semigroup of G such that $e \in G_+$.

Let E be a subset of G with $e \in E$. We first clarify a necessary condition on E under which γ^{E,G_+} exists as a C^*-morphism.

If γ^{E,G_+} exists as a C^*-morphism, then for any $g \in G_+$,

$$1 - T^E_{g^{-1}} T^E_g = T^E_e - T^E_{g^{-1}} T^E_g = \gamma^{E,G_+}(1 - T^{G_+}_{g^{-1}} T^{G_+}_g) = 0.$$

Therefore $(1 - T^E_{g^{-1}} T^E_g)\,\delta_h = 0$ for any $g \in G_+$ and $h \in E$. It follows that $G_+ \cdot E \subseteq E$. Since $e \in E$, we conclude that $G_+ \subseteq E$.

Definition 3.1. *The Toeplitz algebra $\mathcal{T}^{G_+}$ is said to have a weakly universal property if for any subset E of G with $G_+ \subseteq E$ and $G_+ \cdot E \subseteq E$, γ^{E,G_+} exists as a C^*-morphism.*

For any $x \in G \setminus G_+$, let $H[x] = G_+ \cup (G_+ \cdot x)$. Clearly, $G_+ \cdot H[x] \subseteq H[x]$.

Theorem 3.1. *$\mathcal{T}^{G_+}$ has the weakly universal property if and only if $\gamma^{H[x],G_+}$ exists as a C^*-morphism for any $x \in G \setminus G_+$.*

Proof. Clearly the forward implication holds, for $G_+ \subseteq H[x]$ and $G_+ \cdot H[x] \subseteq H[x]$. It remains to verify the reverse implication.

Let E be any subset of G satisfying $G_+ \subseteq E$ and $G_+ \cdot E \subseteq E$. Since $e \in G_+$ and G_+ is a sub-semigroup, by Remark 2.1 and Theorem 2.1 it suffices to prove that the following two conditions are satisfied:

(1) For any two finite non-empty subsets $F_1 \subseteq E \setminus G_+$ and $F_2 \subseteq G \setminus E$, there exists $g_* \in G_+$ such that $F_1 \cdot g_* \subseteq G_+$ and $F_2 \cdot g_* \subseteq G \setminus G_+$.

(2) For any finite non-empty subset F of $E \setminus G_+$, there exists $g_* \in G_+$ such that $F \cdot g_* \subseteq G_+$.

First we consider the condition (1). Let $F_1 = \{x_1, x_2, \cdots, x_n\}$. Then $H[x_1] \subseteq E$, so $F_2 \subseteq G \setminus H[x_1]$. By the assumption $\gamma^{H[x_1],G_+}$ exists as a C^*-morphism, so by Theorem 2.1 we know that there exists $g_1 \in G_+$ such that $x_1 g_1 \in G_+$ and $F_2 \cdot g_1 \subseteq G \setminus G_+$. If $x_2 g_1 \notin G_+$, then since $G_+ \cdot x_2 \subseteq G_+ \cdot E \subseteq E$, we know that

$$(F_2 \cdot g_1) \cap H[x_2 g_1] = \emptyset.$$

By the assumption, $\gamma^{H[x_2 g_1],G_+}$ exists as a C^*-morphism, so there exists $g_2 \in G_+$ such that $(x_2 g_1) g_2 \in G_+$ and $(F_2 \cdot g_1) \cdot g_2 \subseteq G \setminus G_+$. Let $g = g_1 g_2$. Then $g \in G_+$ and $x_i \cdot g \in G_+$ for $i = 1, 2$; also $F_2 \cdot g \subseteq G \setminus G_+$. Pursuing this process, eventually we obtain $g_* \in G_+$ such that $F_1 \cdot g_* \subseteq G_+$ and $F_2 \cdot g_* \subseteq G \setminus G_+$.

The condition (2) is satisfied in a similar way.

Proposition 3.1. *$\gamma^{H[x],G_+}$ exists as a C^*-morphism for any $x \in G_+^{-1} \setminus G_+$, where $G_+^{-1} = \{g^{-1} \mid g \in G_+\}$.*

Proof. Since $x^{-1} \in G_+$ and G_+ is a sub-semigroup, we know that for any $g \in G$, $g \in H[x]$ if and only if $gx^{-1} \in G_+$. Now for any finite non-empty subset F of $H[x]$, let $g_* = x^{-1} \in G_+$. Then for any $g \in F$, $g \in H[x]$ if and only if $gg_* \in G_+$.

Remark 3.1. There exists (G, G_+) such that $\gamma^{H[x],G_+}$ fails to be a C^*-morphism for any given $x \notin G_+ \cup G_+^{-1}$. For example, consider $G = \mathbb{Z}^2$. Fix $k_0 \in \mathbb{Z}$, and let

$$G_+ = \{(0, m) \mid m \geq 0\} \cup \{(1, n) \mid n \geq k_0\} \cup \{(s, t) \mid s \geq 2\}.$$

Then G_+ is a sub-semigroup and given $g \in G_+ \setminus \{e\}$, either $(0, -1) + g \in G_+$ or $(-1, -k_0) + g \in G_+$. Also, if $x \notin G_+ \cup (-G_+)$, then $(0, -1) \notin H[x]$ (otherwise $-x \in (0, 1) + G_+ \subseteq G_+$, a contradiction). Similarly, $(-1, -k_0) \notin H[x]$.

Now suppose that $\gamma^{H[x],G_+}$ does exist as a C^*-morphism. By Theorem 2.1 there exists $g_* \in G_+$ such that

$$\begin{aligned} &x + g_* \in G_+, \\ &(0, -1) + g_* \notin G_+ \quad \text{and} \quad (-1, -k_0) + g_* \notin G_+. \end{aligned} \tag{3.1}$$

Since $x \notin G_+$, we know $g_* \in G_+ \setminus \{e\}$, so either $(0, -1) + g_* \in G_+$ or $(-1, -k_0) + g_* \in G_+$, a contradiction to (3.1).

Definition 3.2. *A pair (G, G_+) is said to be a quasi-ordered group if $e \in G_+, G_+ \cdot G_+ \subseteq G_+$ and $G = G_+ \cup G_+^{-1}$, where $G_+^{-1} = \{\, g^{-1} \,\big|\, g \in G_+ \,\}$. (G, G_+) is referred to as an ordered group if furthermore $G_+^0 = G_+ \cap G_+^{-1} = \{e\}$.*

Let us take a look at $\mathbb{Z}^2$, where $\mathbb{Z}$ is the integer group. By definition, $(\mathbb{Z}^2, \mathbb{Z}_+ \times \mathbb{Z})$ is a quasi-ordered group, while the lexico-ordered group $(\mathbb{Z}^2, \mathbb{Z}^2_{\ell ex})$ is an ordered group. It is easy to construct non-abelian quasi-ordered groups by choosing certain upper triangular invertible matrices over the real numbers.

By Theorem 3.1 and Proposition 3.1, we have the following corollary.

Corollary 3.1. *If (G, G_+) is a quasi-ordered group, then $\mathcal{T}^{G_+}$ has the weakly universal property.*

§ 4. Quasi-ordered Groups and Their Induced Partially Ordered Groups

This section and the next contain applications of the results discussed in Sections 2 and 3. In this section we investigate the natural morphism between Toeplitz algebras corresponding to a quasi-ordered group and its induced partially-ordered group. Results from [4] are extended nontrivially to the nonabelian case, while at the same time the technique of proof is greatly simplified.

Definition 4.1. *Let G be a discrete group and G_+ be a sub-semigroup of G containing the identity of G. A pair (V, M) is said to be an isometric representation of G_+ if M is a unital C^*-algebra and $V : G_+ \to M$ is a map satisfying*

(1) $V(e) = 1$;
(2) $V(g)V(h) = V(gh)$ *for any* $g, h \in G_+$;
(3) $V(g)^*V(g) = 1$ *for any* $g \in G_+$;
(4) $V(g)V(g)^* = 1$ *for any* $g \in G_+ \cap G_+^{-1}$.

Remark 4.1. If G is a discrete amenable group and (G, G_+) is a quasi-ordered group, then by [5, Theorem 3.5] $\mathcal{T}^{G_+}$ has a universal property for isometric representations of G_+. More precisely, for any isometric representation (V, M) of G_+, there exists a C^*-morphism $\pi_V : \mathcal{T}^{G_+} \to M$ such that $\pi_V(T_g^{G_+}) = V(g)$ for any $g \in G_+$.

Throughout the rest of this section, (G, G_+) denotes a quasi-ordered group such that $G_+^0 = G_+ \cap G_+^{-1}$ is non-trivial, that is, G_+^0 is neither equal to $\{e\}$ nor equal to G.

It is easy to show that

$$(G_+ \setminus G_+^0) \cdot G_+ = (G_+ \setminus G_+^0) = G_+ \cdot (G_+ \setminus G_+^0), \tag{4.1}$$

and

$$G = (G_+ \setminus G_+^0) \cup G_+^0 \cup (G_+ \setminus G_+^0)^{-1}.$$

Let $G_1 = (G_+ \setminus G_+^0) \cup \{e\}$. Then (G, G_1) is a partially ordered group in the sense that

$$e \in G_1, \quad G_1 \cdot G_1 \subseteq G_1, \quad G_1 \cap G_1^{-1} = \{e\} \quad \text{and} \quad G = G_1 \cdot G_1^{-1}. \tag{4.2}$$

Proposition 4.1. *Suppose that G_+^0 is infinite. Then*

(1) *γ^{G_+, G_1} exists as a C^*-morphism.*

(2) *For any M with $G_1 \subsetneq M \subsetneq G_+$ and $G_1 \cdot M \subseteq M$, γ^{M, G_1} fails to be a C^*-morphism.*

(3) *If G is amenable, then $\operatorname{Ker} \gamma^{G_+, G_1} = \mathbb{K}(\ell^2(G_1))$, where $\mathbb{K}(\ell^2(G_1))$ is the ideal of compact operators on $\ell^2(G_1)$.*

Proof. (1) We apply Theorem 2.1. Since $G = G_1 \cdot G_1^{-1}$, it is easy to show that for any finite subset F of G, there exists $g_1 \in G_1$ such that $F \cdot g_1 \subseteq G_1$. So it reduces to show that for finite non-empty subsets $F_1 \subseteq G_+$ and $F_2 \subseteq G \setminus G_+ = (G_+ \setminus G_+^0)^{-1}$, there exists $g_* \in G_1$ such that $F_1 \cdot g_* \subseteq G_1$ and $F_2 \cdot g_* \subseteq G \setminus G_1$.

Toward this end, we define a quasi-order on G by $x \ll y \Longleftrightarrow x^{-1} y \in G_+$ for $x, y \in G$. Suppose $F_2 = \{g_i^{-1} \,|\, g_i \in G_+ \setminus G_+^0, i = 1, 2, \cdots, n\}$. There exists $i_0 \in \{1, \cdots, n\}$ such that $g_{i_0} \ll g_i$ for all i. Let

$$E = \{g_i \,|\, g_i \ll g_{i_0}, \text{ and } g_{i_o} \ll g_i\} \quad \text{and} \quad H = \{g_{i_0}^{-1} g_i \,|\, g_i \in E\},$$

and observe $\{e\} \subseteq H \subseteq G_+^0$.

Since G_+^0 is infinite, we may choose $h_* \in G_+^0 \setminus H$. If we put $g_* = g_{i_0} \cdot h_*$, then $g_* \in (G_+ \setminus G_+^0) \cdot G_+ = G_+ \setminus G_+^0$. Therefore $F_1 \cdot g_* \subseteq G_+ \setminus G_+^0 \subseteq G_1$.

Finally, $F_2 \cdot g_* \subseteq G \setminus G_1$: Given any $g_i^{-1} \in F_2$, if $g_i \in E$, then $g_i^{-1} g_* \subseteq G_+^0 \setminus \{e\} \subseteq G \setminus G_1$. On the other hand, if $g_i \in F_2 \setminus E$, then $g_i^{-1} g_{i_0} \in G \setminus G_+$, so $g_i^{-1} \cdot g_* \subseteq (G \setminus G_+) \cdot E^0 = G \setminus G_+ \subseteq G \setminus G_1$.

(2) If $g_1 \in M \setminus G_1$ and $g_2 \in G_+ \setminus M$, then $g_1, g_2 \in G_+^0 \setminus \{e\}$. Let $F_1 = \{g_1\} \subseteq M$ and $F_2 = \{g_2\} \subseteq G \setminus M$. Note that for any $g_* \in G_1$, if $g_1 g_* \in G_1$, then g_* must be in $G_+ \setminus G_+^0$, which implies that $g_2 g_* \in G_+ \setminus G_+^0 \subseteq G_1$. By Theorem 2.1, we conclude that γ^{M, G_1} fails to be a C^*-morphism.

(3) First, we prove that $\mathcal{T}^{G_1}$ is irreducible. Let $T \in \mathbb{B}(\ell^2(G_1))$ be such that $TS = ST$ for any $S \in \mathcal{T}^{G_1}$. We prove that $T = \lambda$ for some $\lambda \in C$. Indeed, for any $t \in G_1 \setminus \{e\}$, $T_{t^{-1}}^{G_1} T \delta_e = T T_{t^{-1}}^{G_1} \delta_e = 0$, so $T \delta_e = \lambda \delta_e$ for some $\lambda \in C$. It follows that for any $g \in G_1$, $T \delta_g = T T_g^{G_1} \delta_e = T_g^{G_1} T \delta_e = \lambda \delta_g$, so $T = \lambda$.

Second, we show that $\mathbb{K}(\ell^2(G_1)) \subseteq \operatorname{Ker} \gamma^{G_+, G_1}$. If $x \in G_+^0 \setminus \{e\}$, then clearly $1 - T_x^{G_1} T_{x^{-1}}^{G_1}$ is a projection of rank one. Since $\mathcal{T}^{G_1}$ is irreducible, and $0 \neq 1 - T_x^{G_1} T_{x^{-1}}^{G_1} \in \operatorname{Ker} \gamma^{G_+, G_1} \cap$

$\mathbb{K}(\ell^2(G_1))$, by [3, Theorem 4.1.11] we know that $\mathbb{K}(\ell^2(G_1)) \subseteq \operatorname{Ker}\gamma^{G_+,G_1}$. Therefore, a C^*-morphism $\overline{\gamma^{G_+,G_1}} : \mathcal{T}^{G_1}/\mathbb{K}(\ell^2(G_1)) \to \mathcal{T}^{G_+}$ can be induced such that $\overline{\gamma^{G_+,G_1}}([T_g^{G_1}]) = T_g^{G_+}$ for any $g \in G$.

The proof will be finished if we show that $\overline{\gamma^{G_+,G_1}}$ is an isomorphism. Define $V : G_+ \to \mathcal{T}^{G_1}/\mathbb{K}(\ell^2(G_1))$ by $V(g) = [T_g^{G_1}]$ for any $g \in G_+$. Then V is an isometric representation of G_+. Since G is amenable, by [5, Theorem 3.5] we know that there is a C^*-morphism $\pi_V : \mathcal{T}^{G_+} \to \mathcal{T}^{G_1}/\mathbb{K}(\ell^2(G_1))$ such that $\pi_V(T_g^{G_+}) = [T_g^{G_1}]$ for any $g \in G_+$, and since $G = G_+ \cup G_+^{-1}$, we know that $\pi_V(T_g^{G_+}) = [T_g^{G_1}]$ for any $g \in G$. Therefore, $\overline{\gamma^{G_+,G_1}} = (\pi_V)^{-1}$.

Remark 4.2. In the special case when G is abelian and M is the positive part of an ordered group containing G_1, the preceding proposition was obtained in [4] (see [4, Theorem 3.2 and Corollary 3.3]). The proof of [4, Theorem 3.2] relied on a complicated technique (see [4, Theorem 2.4]). Our current proof is much simpler.

§ 5. Toeplitz Algebras Associated to Abelian Ordered Groups

Throughout this section, G is a discrete abelian group and G_+ is a sub-semigroup of G such that $0 \in G_+$ and $G_+ \cap G_+^{-1} = \{0\}$. It is well known that G admits a total order if and only if G is torsion-free. So in the following we always assume that G is torsion-free.

Proposition 5.1. *Suppose that G is abelian and torsion-free. Then the following two conditions are equivalent:*

(1) *(G, G_+) is an ordered group;*

(2) *If $x \in G$, then $2x \in G_+$ implies that $x \in G_+$, and $\mathcal{T}^{G_+}$ has the weakly universal property.*

Proof. That (1) implies (2) follows immediately from Corollary 3.1. So, we consider (2) implies (1).

Suppose that γ^{E,G_+} exists as a C^*-morphism for any E with $0 \in E$ and $G_+ + E \subseteq E$. If (G, G_+) is not an ordered group, then there exists some $x \in G$ such that $x \notin G_+ \cup (-G_+)$. Let

$$H = G_+ \cup (G_+ + x) \cup (G_+ + 3x).$$

Then $2x \notin G_+$ (otherwise, by the assumption $x \in G_+$), $2x \notin (G_+ + x) \cup (G_+ + 3x)$, therefore $2x \notin H$. Let $F_1 = \{x, 3x\} \subseteq H$ and $F_2 = \{2x\} \subseteq G \setminus H$. By hypothesis γ^{H,G_+} exists as a C^*-morphism, thus by Theorem 2.1 there exists $g_* \in G_+$ such that

$$x + g_* \in G_+, \quad 3x + g_* \in G_+; \tag{5.1}$$

$$2x + g_* \in G \setminus G_+. \tag{5.2}$$

Now (5.1) implies that $2(2x + g_*) = (x + g_*) + (3x + g_*) \in G_+$, which by hypothesis yields $2x + g_* \in G_+$, contradicting (5.2).

Example 5.1. Let $G = \mathbb{Z}^2$, α and β be two real numbers with $\alpha < \beta$. Let

$$G_+ = \{(m, n) \in \mathbb{Z}^2 \mid -\alpha m + n \geq 0 \text{ and } -\beta m + n \leq 0\}.$$

Toeplitz algebras associated to such a pair (G, G_+) were studied in [2]. By the preceding proposition, we know that there exists some E with $G_+ \subseteq E$ and $G_+ + E \subseteq E$, such that γ^{E,G_+} fails to be a C^*-morphism.

Lemma 5.1. *Suppose* $2x_0 \in G_+$ *but* $x_0 \notin G_+$. *If* $\mathcal{T}^{G_+}$ *has the weakly universal property, then* $\mathcal{T}^{H[G_+,x_0]}$ *also has the weakly universal property, where* $H[G_+,x_0] = G_+ \cup (G_+ + x_0)$.

Proof. Let G_1 be $H[G_+,x_0]$. Then G_1 is a sub-semigroup and $x_0 + G_1 \subseteq G_1$. To show that $\mathcal{T}^{G_1}$ has the weakly universal property, by Theorem 3.1 it suffices to prove that for any $y_0 \notin G_1$, $\gamma^{H[G_1,y_0],G_1}$ exists as a C^*-morphism, which in turn by Theorem 2.1 is equivalent to proving that for any finite collection $y_1, y_2, \cdots, y_n \notin H[G_1,y_0] = G_1 \cup (G_1 + y_0)$, there exists $g_* \in G_1$ such that

$$y_0 + g_* \in G_1, \quad \text{but} \quad y_i + g_* \notin G_1 \quad \text{for any} \quad i = 1, 2, \cdots, n.$$

Since $x_0 + G_1 \subseteq G_1$, we have $x_0 + H[G_1,y_0] \subseteq H[G_1,y_0]$, which implies $y_i - x_0 \notin H[G_1,y_0]$ for any $i = 1, 2, \cdots, n$. Clearly, $G_+ \subseteq H[G_1,y_0]$ and $G_+ + H[G_1,y_0] \subseteq H[G_1,y_0]$. By the assumption $\gamma^{H[G_1,y_0],G_+}$ exists as a C^*-morphism, hence by Theorem 2.1 (let $F = \{y_0, y_i, y_i - x_0,\ i = 1, 2, \cdots, n\}$), there exists $g_* \in G_+$ such that

$$y_0 + g_* \in G_+, \quad y_i + g_* \notin G_+, \quad y_i - x_0 + g_* \notin G_+.$$

So $y_0 + g_* \in G_1$ and $y_i + g_* \notin G_+ \cup (x_0 + G_+) = G_1$.

Lemma 5.2. *Suppose that* G *is torsion-free,* $2x_0 \in G_+$ *but* $x_0 \notin G_+$. *If such* x_0 *is unique, then* $\mathcal{T}^{G_+}$ *fails to have the weakly universal property.*

Proof. In order to obtain a contradiction, suppose that $\mathcal{T}^{G_+}$ has the weakly universal property.

First, note $2(x_0 + t) = 2x_0 + 2t \in G_+ + G_+ \subseteq G_+$ for any $t \in G_+ \setminus \{0\}$. Since x_0 is unique, we have $x_0 + t \in G_+$; hence

$$x_0 + G_+ \setminus \{0\} \subseteq G_+.$$

Second, $x_0 \in H[G_+,y] = G_+ \cup (G_+ + y)$ for $y \notin G_+$; otherwise, by the assumption $\gamma^{H[G_+,y],G_+}$ exists as a C^*-morphism, hence by Theorem 2.1 there exists $g_* \in G_+ \setminus \{0\}$ such that $y + g_* \in G_+$ but $x_0 + g_* \notin G_+$, yielding a contradiction. It follows that

$$G = G_+ \cup (x_0 - G_+). \tag{5.3}$$

Now let $G_1 = G_+ \cup (G_+ + x_0) = G_+ \cup \{x_0\}$. By Lemma 5.1 we know that $\mathcal{T}^{G_1}$ also has the weakly universal property. We claim (G, G_1) must be an ordered group. By Proposition 5.1 it suffices to verify that $2y_0 \in G_1$ implies $y_0 \in G_1$ for $y_0 \in G$. First, suppose that $2y_0 = x_0$. If $y_0 \notin G_+$, then by (5.3) we know that $y_0 = x_0 - t$ for some $t \in G_+$, so $x_0 = 2t \in G_+$, which is impossible. Next, suppose that $2y_0 \in G_+$. If $y_0 \notin G_+$, then by the uniqueness of x_0, $y_0 = x_0 \in G_1$. Thus (G, G_1) is an ordered group.

Define a total order $\leq_1$ on G by $x \leq_1 y \iff y - x \in G_1$. Since $x_0 \notin G_+$ and $G_1 = G_+ \cup \{x_0\}$, we know $x_0 - t \notin G_1$ for $t \in G_+ \setminus \{0\}$. Therefore $t - x_0 \in G_1$, which means that x_0 is the minimal positive element in G with respect to $\leq_1$.

Observe $-nx_0 \notin G_+$ (and hence is not in G_1) for $n \in N$. Otherwise $-2nx_0 \in G_+ \cap (-G_+) = \{0\}$, and since G is torsion free, this would imply $x_0 = 0$, which is impossible.

Finally, by the assumption γ^{G_1,G_+} exists as a C^*-morphism. Using Theorem 2.1 and the fact that $-nx_0 \notin G_1$ for $n \in N$, there exists $g_* \in G_+ \setminus \{0\}$ such that

$$x_0 + g_* \in G_+,$$
$$-x_0 + g_* \notin G_+, \tag{5.4}$$
$$-2x_0 + g_* \notin G_+. \tag{5.5}$$

Since x_0 is minimal, $g_* - x_0 \in G_1$. By (5.4) we know that $g_* - x_0 = x_0$, so $g_* = 2x_0$, a contradiction to (5.5).

Theorem 5.1. *Suppose that G is torsion-free, and for any sub-semigroup E containing G_+, $\mathcal{T}^E$ has the weakly universal property. Then (G, G_+) is an ordered group.*

Proof. By Proposition 5.1, it sufficies to verify that for any $x_0 \in G$, $2x_0 \in G_+$ implies $x_0 \in G_+$. Suppose that $x_0 \notin G_+$. We will show that x_0 is unique, and then use Lemma 5.2 to conclude that $\mathcal{T}^{G_+}$ does not have the weakly universal property, which is contrary to the assumption.

Let

$$G_1 = G_+ \cup (G_+ + x_0) \quad \text{and} \quad G_2 = G_+ \cup (G_+ \setminus \{0\} + x_0).$$

Then G_1 and G_2 are two sub-semigroups containing G_+. Since $G_2 \setminus \{0\} + x_0 \subseteq G_2$ and $\mathcal{T}^{G_2}$ has the weakly universal property, by Theorem 2.1 we have

$$G = G_2 \cup (x_0 - G_2) = (G_+ \cup (-G_+)) \cup (G_+ + x_0) \cup (x_0 - G_+). \tag{5.6}$$

So for any $y \notin G_+ \cup (-G_+)$, either $y - x_0 \in G_+$ or $x_0 - y \in G_+$.

First, we prove $3x_0 \in G_+$. In fact, by the assumption $\gamma^{H[G_+,-x_0],G_+}$ exists as a C^*-morphism, where $H[G_+, -x_0] = G_+ \cup (G_+ - x_0)$. Therefore there exists $g_* \in G_+ \setminus \{0\}$ such that

$$\begin{aligned} -x_0 + g_* &\in G_+, \\ -2x_0 + g_* &\notin G_+, \\ -3x_0 + g_* &\notin G_+ \end{aligned} \tag{5.7}$$

We assert $-2x_0 + g_* \notin -G_+$; indeed, if $-2x_0 + g_* = -t$ for some $t \in G_+$, then $x_0 - t = -x_0 + g_* \in G_+$, so $x_0 = (x_0 - t) + t \in G_+$, yielding a contradiction. Therefore, $-2x_0 + g_* \notin G_+ \cup (-G_+)$, hence either $(-2x_0 + g_*) - x_0 \in G_+$ or $x_0 - (-2x_0 + g_*) \in G_+$. By (5.7) we know $3x_0 - g_* \in G_+$, so $3x_0 = (3x_0 - g_*) + g_* \in G_+$.

Next, we prove for any $t \in G_+ \setminus \{0\}$, $t + x_0 \in G_+$, therefore $G_+ \setminus \{0\} + x_0 \subseteq G_+$.

Case 1. $t - 2x_0 \in G_+$. In this case, $t + x_0 = (t - 2x_0) + 3x_0 \in G_+$.

Case 2. $2x_0 - t \in G_+$. By the assumption γ^{G_1,G_+} exists as a C^*-morphism, therefore there exists $g_* \in G_+ \setminus \{0\}$ such that

$$x_0 + g_* \in G_+, \tag{5.8}$$

$$x_0 - t + g_* \notin G_+, \tag{5.9}$$

$$-t + g_* \notin G_+. \tag{5.10}$$

We assert $x_0 - t + g_* \notin -G_+$; otherwise $x_0 - g_* = (2x_0 - t) + (-x_0 + t - g_*) \in G_+ + G_+ \subseteq G_+$, which implies that $x_0 \in G_+$, a contradiction. By (5.6), (5.9) and (5.10) we have $t - g_* \in G_+$, and then $t + x_0 \in G_+$ follows from (5.8).

Case 3. $2x_0 - t \notin G_+ \cup (-G_+)$. Since $x_0 \notin G_+$, by (5.6) we have $t - x_0 \in G_+$, so $t + x_0 = 2x_0 + (t - x_0) \in G_+$.

Finally, since $G_+ \setminus \{0\} + x_0 \subseteq G_+$, we know from (5.6) that for any $y \notin G_+ \cup (-G_+)$, $x_0 - y \in G_+$. Now if $x_1 \in G$ such that $2x_1 \in G_+$ but $x_1 \notin G_+$, then $x_1 \notin G_+ \cup (-G_+)$, so $x_0 - x_1 \in G_+$. Exchanging x_0 with x_1, we have $x_1 - x_0 \in G_+$, so $x_1 = x_0$.

Remark 5.1. (1) Let G be a discrete abelian group, G_+ a sub-semigroup of G such that $0 \in G_+$ and $G = G_+ - G_+$ ($G_+ \cap (-G_+)$ is not necessarily $\{0\}$). Given any subset E with $G_+ \subseteq E$ and $G_+ + E \subseteq E$, a natural isometric representation $V : G_+ \to \mathcal{T}^E$ can be induced by $V(g) = T_g^E$ for any $g \in G_+$. So if $\mathcal{T}^{G_+}$ has the universal property for isometric representations of G_+, then there is a C^*-morphism $\pi_V : \mathcal{T}^{G_+} \to \mathcal{T}^E$ such that $\pi_V(T_g^{G_+}) = T_g^E$ for any $g \in G_+$. Moreover, since $G = G_+ - G_+$, we know for any $t \in G$, $t = g - h$ for some $g, h \in G_+$, so $\pi_V(T_t^{G_+}) = \pi_V((T_h^{G_+})^* T_g^{G_+}) = T_{-h}^E T_g^E = T_t^E$, i.e., $\pi_V = \gamma^{E,G_+}$, therefore in this case $\mathcal{T}^{G_+}$ has the weakly universal property. In view of this and [5, Theorem 3.5] we know that the reverse of Theorem 5.1 is also true.

(2) Many examples indicate that the following conjecture seems to be true:

Conjecture. *Let G be a discrete torsion-free abelian group, and (G, G_+) a partial-ordered group (for the definition, see (4.2)) which is not totally ordered . If there exists some $x_0 \notin G_+$ such that $2x_0 \in G_+$, then $\mathcal{T}^{G_+}$ fails to have the weakly universal property.*

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