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NONLINEAR EXPECTATIONS AND NONLINEAR MARKOV CHAINS**

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Abstract

This paper deals with nonlinear expectations. The author obtains a nonlinear generalization of the well-known Kolmogorov's consistent theorem and then use it to construct filtration-consistent nonlinear expectations via nonlinear Markov chains. Compared to the author's previous results, i.e., the theory of g-expectations introduced via BSDE on a probability space, the present framework is not based on a given probability measure. Many fully nonlinear and singular situations are covered. The induced topology is a natural generalization of L^p-norms and L^∞-norm in linear situations. The author also obtains the existence and uniqueness result of BSDE under this new framework and develops a nonlinear type of von Neumann-Morgenstern representation theorem to utilities and present dynamic risk measures.

Keywords Backward stochastic differential equations, Nonlinear expectation, Nonlinear expected utilities, Measure of risk, g-expectation, Nonlinear Markov chain, g-martingale, Nonlinear martingale, Nonlinear Kolmogorov's consistent theorem, Doob-Meyer decomposition
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§ 1 . Introduction

Let $(\Omega, \mathcal{F})$ be a measurable space and let $L_b(\mathcal{F})$ be the space of $\mathcal{F}$-measurable and bounded real functions. A nonlinear expectation is a continuous functional

$$\mathcal{E}[\,\cdot\,] : L_b(\mathcal{F}) \longrightarrow R$$

that is order preserving (i.e., $\mathcal{E}[X_1] \geq \mathcal{E}[X_2]$, if $X_1 \geq X_2$) and constant preserving (i.e., $\mathcal{E}[c] = c$).

If furthermore $\mathcal{E}[\,\cdot\,]$ is a linear functional, then it is a classical expectation under the (additive) probability measure P on $(\Omega, \mathcal{F})$ induced by

$$P(A) := \mathcal{E}[1_A], \qquad A \in \mathcal{F}. \tag{1.1}$$

In this case we have

$$\mathcal{E}[X] = \int_\Omega X(\omega) dP(\omega).$$

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It is well known that there is a 1-1 correspondence between linear expectations and additive probability measures. But this 1-1 correspondence fails in nonlinear situations. In general, given a nonlinear expectation $\mathcal{E}[\,\cdot\,]$, one can still derive a non additive probability measure P by (1.1). But there exist an infinite number of nonlinear expectations satisfying the same relation (see [8]). Thus in nonlinear situations the notion of expectation is more characteristic than that of non additive measures. We refer to [7] for a deeper investigation.

In the dynamic situation, a basic notion is the conditional expectation under a given filtration $\mathcal{F}_t$. This notion permits us to use the up-date information $\mathcal{F}_t$ to obtain the best estimate of a given random variable. The well-known martingale theory is fundamentally based on this notion (see [15]). As in linear situations, the conditional nonlinear expectation of a random variable X under $\mathcal{F}_t$ is an $\mathcal{F}_t$-measurable random variable $\mathcal{E}[X/\mathcal{F}_t]$ satisfying

$$\mathcal{E}[1_A\mathcal{E}[X/\mathcal{F}_t]] = \mathcal{E}[1_A X], \qquad \forall\, A \in \mathcal{F}_t.$$

A nonlinear expectation $\mathcal{E}[\,\cdot\,]$ is called $\mathcal{F}_t$-consistent if such $\mathcal{E}[X/\mathcal{F}_t]$ exists for all $t \geq 0$ and $X \in L_b(\mathcal{F})$. In nonlinear situations, there do exist non-consistence expectations. If $\mathcal{E}[\,\cdot\,]$ is $\mathcal{F}_t$-consistent, we then can develop the related nonlinear martingale theory in a way parallel to the classical one.

The following problems are theoretically interesting and practically important:

P1. Can we find a simple mechanism, which enables us to generate a large kind of filtration-consistent nonlinear expectation?

P2. For a given filtration consistent nonlinear expectation, is there a simple mechanism that determines the value of this expectation?

Problem P1 was investigated in [36] where a notion of g-expectation was introduced under the framework of the natural filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ generated by a d-dimensional Brownian motion $(B_t)_{0\leq t\leq T}$ in a probability space $(\Omega, \mathcal{F}, P)$. It is defined as follows. For each $\mathcal{F}_T$-measurable and L^2-integrable random variable X, we solve the following BSDE:

$$\begin{aligned} -dY_t^X &= g(t, Z_t^X)dt - Z_t^X B_t, \qquad t \in [0,T], \\ Y_T^X &= X. \end{aligned} \tag{1.2}$$

Here the mechanism is the function $g : (\omega, t, z) \in \Omega \times [0,T] \times R^d \longmapsto R$. It satisfies the usual conditions for a BSDE, i.e., Lipschitz and linear growth in z and $\mathcal{F}_t$-adapted. In addition we assume that $g(t,0) \equiv 0$. The g-expectation of X is defined by

$$\mathcal{E}_g[X] := Y_0^X.$$

We can check that it is an $\mathcal{F}_t$-consistent. In fact the corresponding conditional g-expectation of X given by $\mathcal{F}_t$ is nothing else but $\mathcal{E}_g[X|\mathcal{F}_t] = Y_t^X$. It is worth to point out that the expectation $E_Q[\,\cdot\,]$ under the probability Q defined by the well-known Girsanov transforation

$$\frac{dQ}{dP} = \exp\Big\{ \int_0^T b_s B_s - \frac{1}{2}\int_0^T |b_s|^2 ds\Big\}$$

is in fact the g-expectation for $g(t,z) = \langle b_t, z\rangle$, which is linear in z. When g is nonlinear in z, the notion of g-expectations can be considered as a nonlinear Girsanov transformation. Thus a large class of $\mathcal{F}_t$-consistent nonlinear expectations can be generated by a simple mechanism g. Once this function g is obtained, then the corresponding nonlinear expectation is uniquely determined by solving BSDE (1.2). We recall that in the last decade many numerical methods, algorithms and the related numerical analysis, i.e., convergence and converging rate, ect., have developed.

For an $\mathcal{F}_t$-consistent nonlinear expectation, one can introduce the notion of nonlinear martingales, submartingales and supermartingales. It is then natural to ask whether the abandunt results in the classical martingale theory have their counterparts under the framework of g-expectations. Many results have been obtained in this direction, among them the decomposition theorem of Doob-Meyer's type of g-supermartingales or submartingales has been proved for square-integrable situation by [37] and [10–12].

A natural question closely related to Problem P2 is: Is the notion of g-expectations general enough to include all regular $\mathcal{F}_t$-consistent nonlinear expectations? In the recent paper [8] we have the following result: if an $\mathcal{F}_t$-consistent nonlinear expectation $\mathcal{E}$ is $\mathcal{E}_{g^\mu}$ dominated with $g^\mu(z) := \mu|z|$ for some sufficiently large $\mu > 0$, then there exists a unique function g such that $\mathcal{E}[X] = \mathcal{E}_g[X]$ for all X (see Definition 2.2 for the notion of domination, it plays an important role in this paper). Nonlinear Doob-Meyer decomposition mentioned above plays a crucial role in the proof of this result (cf. [39] for a more systematic explanation and [38, 40] for more general results).

But on the other hand, we shall show in this paper that $\mathcal{E}_g[\,\cdot\,]$ is a quasi nonlinear expectation, i.e., the fully nonlinear situation can not be covered. Thus to solve Problem P2, we must find a new mechanism to generate a wider kind of nonlinear expectations.

In this paper we shall use a nonlinear Markov semigroup (or Markov chain) $(\mathcal{T}_t)_{t\geq 0}$ to generate a nonlinear expectation $\mathcal{E}[\,\cdot\,]$. In other words, the infinitesimal generator $\mathcal{A}$ of $(\mathcal{T}_t)_{t\geq 0}$ is the generator of the corresponding nonlinear expectations. In this situation, if $\mathcal{A}$ is quasilinear (resp. fully nonlinear) then $\mathcal{E}[\,\cdot\,]$ is also quasilinear (resp. fully nonlinear). Brievely, our procedure is as follows:

(1) We use a self-dominated nonlinear Markov semigroup $\mathcal{T}_t^*$ to generate a self-dominated and $\mathcal{F}_t$-consistent nonlinear expectation $\mathcal{E}^*$. In this step, we will obtain an extension of Kolmogorov consistent theorem for a family of finite dimensional nonlinear distributions induced by the the Markov semigroup $\mathcal{T}_t^*$. The condition of the self domination of $\mathcal{T}^*$ permits us to induce a norm under which $\mathcal{E}^*[\,\cdot\,]$ and $\mathcal{E}^*[\,\cdot\,|\mathcal{F}_t]$ are continuous.

(2) For an arbitrary $\mathcal{T}_t^*$-dominated Markov semigroup $\mathcal{T}_t$ we can use the same topology induced by $\mathcal{T}^*$ to generate the corresponding $\mathcal{F}_t$-consistent nonlinear expectation $\mathcal{E}[\,\cdot\,]$ which is $\mathcal{E}^*$-dominated. This $\mathcal{E}[\,\cdot\,]$ is therefore continuous under the given norm.

Let $g(z)$, $z \in R^d$ be a real Lipschitz function with Lipschitz constant $\mu > 0$. Then $\mathcal{E}_g$ is $\mathcal{E}_{g^\mu}$ dominated, so is the related nonlinear Markov chains. This implies that a large class of g-expectations can be also generated by the above approach. In this paper we shall also give some typical class of fully nonlinear Markov semigroups. They are either self dominated or dominated by some other self dominated fully nonlinear Markov semigroups. Thus the way to generate filtration consistent nonlinear expectations is largely extended. It is an important step towards to solve completely Problem P2.

On the other hand, since the classical linear Markov semigroups are self dominated, they are within our new framework. In fact in this special situation this method corresponds to the classical L^1 theory. We recall that the notion of g-expectations is essentially an L^2-theory.

Another advantage of this domination approach is that, unlike in BSDE theory, no prior probability space is required. In fact, the continuity and completeness of the generated nonlinear expectation is under the norm induced by the given self dominated Markov semigroup. This constitutes a new "probability space".

We shall also study the existence and uniqueness of BSDE under this new "probability space". This extends BSDE theory to fully nonlinear situations.

This paper is organized as follows: In Section 2, we shall introduce the notion of dominated and self-dominated nonlinear pre-expectations, introduce the norms and then take the completions. We thus have a generalized notion of "probability space". In Section 3 we introduce the notion of families of finite-dimensional distributions corresponding to a nonlinear expectation and prove the related nonlinear Kolmogorov consistent theorem. The notion and examples of nonlinear Markov chains (i.e., nonlinear Markov semigroups) will be given and studied in Section 4. In Section 5, we shall construct the filtration consistent nonlinear expectation corresponding to a nonlinear Markov semigroup. In Section 6 we shall prove an existence and uniqueness theorem of BSDE under this new probability space. In Section 7 we discuss the relation between nonlinear expectations and nonlinear expected utilities.

The systematic research on filtration-consistent nonlinear expectations begins from [36]. The formal definition is only given in 2002. Many interesing and largely open problems are still to be explored.

§ 2. Nonlinear Expectations

2.1. Examples

A financial market consists of a non-risky asset, called the bond, with price $P_0(t)$ satisfying

$$\frac{dP_0(t)}{dt} = r_t P_0(t), \qquad P_0(0) = 1,$$

and a risky asset, called the stock, with price $P^v(t)$ satisfying

$$dP(t) = P(t)[b_t dt + \sigma_t dB(t)], \qquad P(0) = p.$$

where B_t, $t \geq 0$ is a Brownian motion. We assume that b_t, σ_t and σ_t^{-1} are uniformly bounded and $\mathcal{F}_t^B$ adapted, where $\mathcal{F}_t^B$ is the filtration generated by the Brownian motion B. We assume that an investor invests $\pi_0(t) = n_0(t)P_0(t)$ in the bond and $\pi(t)n(t)P(t)$ in the stock. His total wealth at time t is $y_t = \pi_0(t) + \pi(t)$. Under the self-financing condition, his wealth evolves according to

$$dy_t = n_0(t)dP_0(t) + n(t)dP(t),$$

or

$$dy_t = [ry + (b_t - r)\pi(t)]dt + \sigma_t \pi(t) dB_t.$$

Without loss of generality, we may assume that $r_t \equiv 0$ (otherwise we can take the discount $\exp(-\int_0^t r_s ds)$). Thus

$$dy_t = \sigma_t \pi(t)[dB_t + \sigma_t^{-1} b_t dt]. \tag{2.1}$$

Let ξ be a bounded contingent claim at maturity $t = T$. It is an $\mathcal{F}_T^B$-measurable positive and bounded random variable. We can solve Equation (2.1) with the terminal condition $y_T = \xi$. This is a backward stochastic differential equation whose solution is a pair (y_t, π_t). The value y_0 is the cost to replicate ξ at the time $t = 0$. y_0 is the non-arbitrage price of the contingent claim ξ. It can be expressed as

$$y_0 = E_{X_T}[\xi] = E[X_T \xi],$$

where

$$X_T = \exp\Big[-\int_0^T \sigma_s^{-1} b_s dB_s - \frac{1}{2}\int_0^T |\sigma_s^{-1} b_s|^2 ds \Big].$$

Example 2.1. Consider a market where the short-selling is prohibited, i.e., $\pi(t) \geq 0$. Then the replication can be achieved by a penalty method

$$\begin{aligned} dy_t^\beta &= \sigma_t \pi^\beta(t)[dB_t + \sigma_t^{-1} b_t dt] - \beta[\sigma_t \pi^\beta(t)]^- dt, \\ y_T^\beta &= \xi. \end{aligned}$$

For each given $\beta \geq 0$, the solution $y_0^\beta = \mathcal{E}^\beta[\xi]$ is a g-expectation. The selling price of the contingent claim ξ under the prohibition of short selling is

$$\mathcal{E}^\infty[\xi] := \lim_{\beta \nearrow \infty} \mathcal{E}^\beta[\xi].$$

Both $\mathcal{E}^\beta[\xi]$ and $\mathcal{E}^\infty[\xi]$ are nonlinear expectations.

Example 2.2. Consider a large investor who influences the stocks price via controlling the volatility: $\sigma_t = \sigma(v_t)$ and the rate of expected return: $b_t = b(v_t)$, where v_t, $t \geq 0$ is his control policy and is $\mathcal{F}_t^B$ adapted with values in a control domain U. Here we assume that

$$b = b(v), \quad \sigma = \sigma(v), \quad \sigma^{-1}(v), \qquad v \in U,$$

are uniformly bounded functions. Thus we have

$$dy_t = \sigma(v_t)\pi_t[dB_t + \sigma^{-1}(v_t) b(v_t) dt].$$

In this situation the non-arbitrage price of a contingent claim at maturity $t = T$ is

$$y_0^v = E^v[\xi] = E[X_T^v \xi],$$

where $E^v[\,\cdot\,]$ is the expectation under P^v via the Girsonov transformation

$$X_T = \exp\Big[-\int_0^T \sigma^{-1}(v_s) b(v_s) dB_s - \frac{1}{2}\int_0^T |\sigma^{-1} b|^2 (v_s) ds \Big].$$

Thus, in order to replicate a contingent claim ξ, the minimum cost he need to pay at the time $t=0$ is

$$\mathcal{E}_*[\xi] := \inf_{v(\,\cdot\,)} E^v[\xi].$$

Example 2.3. On the other hand, a small investor who knows only that $b(v)$ and $\sigma(v)$ are ranged in $v \in U$ but who has no further inside information should consider the worst case. For him, the cost of the replication is

$$\mathcal{E}^*[\xi] := \sup_{v(\,\cdot\,)} E^v[\xi].$$

Both $\mathcal{E}^*$ and $\mathcal{E}_*$ are nonlinear expectations.

2.2. General framework

In the above examples our arguments are based on a given Wiener measure. We now introduce a self-closed framework. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $L_b(\mathcal{F})$ be the linear space of all $\mathcal{F}$-measurable real functions such that

$$\sup_{\omega\in\Omega} |X(\omega)| < \infty. \tag{2.2}$$

Let D be a linear subspace of $L_b(\mathcal{F})$ such that

(i) $1 \in D$;

(ii) if $X \in D$ then $|X| \in D$.

D constitutes a vector lattice. A typical example is

$$D = \Big\{ \sum_{i=1}^{N} a_i \mathbf{1}_{A_i},\ \{A_i\}_{i=1}^N \ \text{ is a partition of } \ (\Omega, \mathcal{F}),\ a_i \in R \Big\}.$$

We shall define pre-expectations on D and then take the completion under a norm induced from these pre-expectations.

Definition 2.1. *$\mathcal{E} : D \longrightarrow R$ is said to be a nonlinear pre-expectation defined on D if it satisfies*

(E1) *monotonicity:*

$$\textit{if } X_1(\omega) \geq X_2(\omega),\ \forall\, \omega \in \Omega,\ \textit{then } \ \mathcal{E}[X_1] \geq \mathcal{E}[X_2];$$

(E2) *constant-preserving:*

$$\mathcal{E}[c] = c \qquad \textit{for each constant } c.$$

If moreover D is a Fréchet space equipped with a quasi-norm $\|\cdot\|$ and $\mathcal{E}[\,\cdot\,]$ is continuous under this norm, then $\mathcal{E}$ is called a nonlinear expectation on $(D, \|\cdot\|)$.

Definition 2.2. *Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be two nonlinear pre-expectations on D. $\mathcal{E}^1[\,\cdot\,]$ is said to be dominated (resp. strongly dominated) by $\mathcal{E}^2[\,\cdot\,]$, or $\mathcal{E}^2$-dominated (resp. strongly dominated), if*

$$\mathcal{E}^1[X] - \mathcal{E}^1[Y] \leq \mathcal{E}^2[|X-Y|], \qquad \forall\, X, Y \in D$$

$$(\textit{resp. } \ \mathcal{E}^1[X] - \mathcal{E}^1[Y] \leq \mathcal{E}^2[Y-X], \qquad \forall\, X, Y \in D).$$

$\mathcal{E}$ is said to be self-dominated (resp. strongly self-dominated) if it is dominated (resp. strong -ly dominated) by itself.

Remark 2.1. If the constraint (E2) is reduced to (U2) $u(x) := \mathcal{E}[x]$, $x \in R$, is a deterministic, continuous and strictly increasing function such that $u(0) = 0$, then all conclusions derived in this section as well as in the next two sections still hold true. A functional satisfying (E1) and (U2) is called a pre-utility. It plays an important role in economics and finance. We shall discuss it in the last section.

Let $\mathcal{E}^*$ be a self-dominated nonlinear pre-expectation on D. We introduce a quasi-norm:

$$\|X\|_* := \mathcal{E}^*[|X|], \qquad X \in D.$$

Since

$$\mathcal{E}^*[X+Y] \le \mathcal{E}^*[X] + \mathcal{E}^*[|Y|], \qquad \forall\, X, Y \in D, \tag{2.3}$$

we have

$$\mathcal{E}^*[|X+Y|] \le \mathcal{E}^*[|X|+|Y|] \le \mathcal{E}^*[|X|] + \mathcal{E}^*[|Y|],$$

or

$$\|X+Y\|_* \le \|X\|_* + \|Y\|_*. \tag{2.4}$$

In particular, for each integer $n \ge 1$, we have

$$\|nX\|_* \le n\,\|X\|_*. \tag{2.5}$$

Lemma 2.1. *We have*

$$\lim_{\alpha_n \to 0} \|\alpha_n X\|_* = 0, \tag{2.6}$$

$$\lim_{\|X_n\|_* \to 0} \|\alpha X_n\|_* = 0. \tag{2.7}$$

Particularly

$$\|X\|_* = 0 \quad \textit{implies} \quad \|\alpha X\|_* = 0. \tag{2.8}$$

Proof. The first limit is due to the fact that

$$\|\alpha_n X\|_* \le \left\|\alpha_n \max_\omega |X(\omega)|\right\|_* = |\alpha_n| \max_\omega |X(\omega)| \to 0.$$

For the second limit, we fix an integer $i \ge 1$, such that $|\frac{\alpha}{i}| \le 1$. Since $|\frac{\alpha}{i} X_n| \le |X_n|$, we then have

$$\|\alpha X_n\|_* = \left\|i \frac{\alpha}{i} X_n\right\|_* \le i \left\|\frac{\alpha}{i} X_n\right\|_* \le i\,\|X_n\|_* \to 0, \qquad \text{as } \|X_n\|_* \to 0.$$

The set of null-elements under $\|\cdot\|_*$ is denoted by D_0^*:

$$D_0^* := \{X \in D;\ \|X\|_* = 0\}. \tag{2.9}$$

We have

Lemma 2.2. *D_0^* is a linear subspace of D.*

Proof. By (2.8), $X \in D_0^*$ and $\alpha \in R$ implies $\alpha X \in D_0^*$. Now let $X, Y \in D_0^*$ and $\alpha, \beta \in R$. We have

$$\|\alpha X + \beta Y\|_* \le \|\alpha X\|_* + \|\beta Y\|_* = 0.$$

This completes the proof.

Lemma 2.3. *We have*

$$\|X + Y\|_* = \|X\|_*, \qquad \forall\, X \in D,\ Y \in D_0^*.$$

Proof. By (2.4), we have

$$\|X + Y\|_* \le \|X\|_* + \|Y\|_* = \|X\|_*.$$

On the other hand

$$\|X + Y\|_* = \|X + Y\|_* + \|-Y\|_* \ge \|X\|_*.$$

From the above results, we can introduce an equivalent relation $\sim$ in D, i.e., $X \sim Y$ iff $X - Y \in D_0^*$. The quotient linear space under this equivalent relation is denoted by D/D_0^*. For each $\{X\} \in D/D_0^*$ with $X \in D$ a representative element of $\{X\}$, we denote by

$$\mathcal{E}[\{X\}] := \mathcal{E}[X], \qquad \|\{X\}\|_* := \|X\|_*. \tag{2.10}$$

Remark 2.2. In this space D/D_0^*, $\{X\} = \{Y\}$ (resp. $\{X\} \ge \{Y\}$) means there exists a null-element $Z \in D_0^*$ such that $X(\omega) + Z(\omega) = Y(\omega)$ (resp. $X(\omega) + Z(\omega) \ge Y(\omega)$), for all $\omega \in \Omega$.

It is clear that $(D/D_0^*, \|\cdot\|_*)$ constitutes a linear quasi-normed space. Its completion is denoted by $\|\cdot\|_*$. We thus have the following theorem.

Theorem 2.1. *Let $\mathcal{E}^*$ be a self-dominated nonlinear pre-expectation defined on D and let D_0^* be the linear subspace of $\|\cdot\|_*$-null elements with $\|\cdot\|_* = \mathcal{E}^*[|\cdot|]$. Then, with definition (2.10), $\mathcal{E}^*$ defined on the quotient space D/D_0^* is a self-dominated pre-expectation and $\|\cdot\|_*$ is a quasi-norm on D/D_0^*. Consequently, the completion of D/D_0^* under $\|\cdot\|_*$, denoted by $[D]_*$, is a Fréchet space (F-space in short).*

Since

$$|\mathcal{E}^*[X] - \mathcal{E}^*[Y]| \le \mathcal{E}^*[|X - Y|] = \|X - Y\|_*, \qquad \forall\, X, Y \in D,$$

we then have

Corollary 2.1. *Let $\mathcal{E}^*$ be a self-dominated (resp. strongly self-dominated) nonlinear pre-expectation defined on D. $\mathcal{E}^*$ can be continuously and uniquely extended to the F-space $[D]_*$. It is a self-dominated (resp. strongly self-dominated) expectation such that*

$$|\mathcal{E}^*[X] - \mathcal{E}^*[Y]| \le \|X - Y\|_*, \qquad \forall\, X, Y \in [D]_*,$$

and such that

$$\mathcal{E}^*[\{X\}] - \mathcal{E}^*[\{Y\}] = \mathcal{E}^*[X] - \mathcal{E}^*[Y], \qquad \forall\, X, Y \in D.$$

Corollary 2.2. *Let $\mathcal{E}$ be an $\mathcal{E}^*$-dominated (resp. strongly dominated) pre-expectation defined in D. Then $\mathcal{E}$ can be uniquely extended to the Fréchet space $[D]_*$ satisfying*

$$|\mathcal{E}[X]-\mathcal{E}[Y]| \le \|X-Y\|_*, \qquad \forall\, X,Y \in [D]_*.$$

Moreover, this extension $\mathcal{E}$ is an $\mathcal{E}^*$-dominated (resp. strongly dominated) nonlinear expectation.

2.3. Examples

We give some examples of nonlinear expectations.

Example 2.4. A linear pre-expectation $E[\,\cdot\,]$ is strongly self-dominated. The completion space $[L_b(\mathcal{F})]_*$ is $L^1(\Omega,\mathcal{F},P^*)$ with $P^*(A):=E^*[1_A]$.

Example 2.5. An extremely strong nonlinear pre-expectation and an extremely weak one on $L_b(\mathcal{F})$ in the sense of domination are respectively

$$\mathcal{E}^\infty[X]:=\sup_{\omega\in\Omega} X(\omega), \quad \text{and} \quad \mathcal{E}_\infty[X]:=\inf_{\omega\in\Omega} X(\omega).$$

$\mathcal{E}^\infty$ is strongly self-dominated. $\mathcal{E}_\infty$ is dominated by $\mathcal{E}^\infty$. It induces a Banach space $[D]_\infty$ under the norm: $\|X\|_\infty := \sup_{\omega\in\Omega}|X(\omega)|$.

Remark 2.3. If a nonlinear pre-expectation $\mathcal{E}^1$ defined on D is dominated (resp. strongly dominated) by some other one $\mathcal{E}^2$, then $\mathcal{E}^1$ is also dominated (resp. strongly dominated) by $\mathcal{E}^\infty$. Thus $\mathcal{E}^1$ can be continuously and uniquely extended to $[D]_\infty$.

Example 2.6. Let $f: R \longrightarrow R$ be continuous and strictly increasing function and let f^{-1} be its inverse. Given a nonlinear pre-expectation $\mathcal{E}$. We can construct another pre-expectation by

$$\mathcal{E}_f[X]:=f^{-1}([\mathcal{E}[f(X)]).$$

A typical example is $f_p(x) := (x^+)^p-(x^-)^p$ for some $\dot{p}\ge 1$. We have $\mathcal{E}_{f_p}[|X|] := ([\mathcal{E}[|X|^p)])^{1/p}$. If $\mathcal{E}$ is linear, then it is dominated by $\mathcal{E}_{f_p}$. The induced norm is L^p.

Example 2.7. A typical situation of the above $\mathcal{E}_f$ is in risk sensitive controls, where $f(x):=e^{\theta x}$ (see for example [30]).

Example 2.8. Let $\{E_i,\ i\in I\}$ be a family of linear pre-expectation defined on $L_b(\mathcal{F})$. We set

$$\mathcal{E}^*[X]:=\sup_{i\in I} E_i[X], \qquad \mathcal{E}_*[X]:=\inf_{i\in I} E_i[X].$$

We can check that $\mathcal{E}^*$ is a strongly self-dominated nonlinear pre-expectations. It follows from $\mathcal{E}_*=-\mathcal{E}^*[-X]$ and thus

$$\mathcal{E}_*[X]-\mathcal{E}_*[Y]=\mathcal{E}^*[-Y]-\mathcal{E}^*[-X]\le \mathcal{E}^*[X-Y]$$

that $\mathcal{E}_*$ is dominated by $\mathcal{E}^*$. We can check that $\|\cdot\|_*$ is a norm and $[L_b(\mathcal{F})]_*$ is a Banach space.

Example 2.9. Let $\{E_{ij},\ i\in I, j\in J\}$ be a family of linear pre-expectation defined on $L_b(\mathcal{F})$. We set

$$\mathcal{E}_i[X]:=\sup_{j\in J}E_{ij}[X],\qquad \mathcal{E}^*[X]:=\sup_{i\in I,\ j\in J}E_{ij}[X]=\sup_{i\in I}\mathcal{E}_i[X],$$
$$\mathcal{E}_*[X]:=\inf_{i\in I}\mathcal{E}_i[X]=\inf_{i\in I}\sup_{j\in J}E_{ij}[X],\qquad \mathcal{E}'_*[X]:=\sup_{i\in I}\inf_{j\in J}E_{ij}[X].$$

Since $\inf_{i\in I}\mathcal{E}_i[X]-\inf_{i\in I}\mathcal{E}_i[Y]\le\inf_{i\in I}\{\mathcal{E}_i[X]-\mathcal{E}_i[Y]\}$, we have

$$\begin{aligned}\mathcal{E}_*[X]-\mathcal{E}_*[Y]&\le\inf_{i\in I}\{\mathcal{E}_i[X]-\mathcal{E}_i[Y]\}\le\sup_{i\in I}|\mathcal{E}_i[X]-\mathcal{E}_i[Y]|\\&\le\sup_{i\in I}\mathcal{E}_i[|X-Y|]=\mathcal{E}^*[|X-Y|].\end{aligned}$$

Thus $\mathcal{E}_*$ is $\mathcal{E}^*$-dominated. Consequently, $\mathcal{E}'_*$ is also dominated by $\mathcal{E}^*$ since $\mathcal{E}'_*[X]=-\mathcal{E}_*[-X]$.

Example 2.10. Let $f(x,y):R^2\longrightarrow R$ be a continuous function such that $f(x,y)\ge f(x',y')$ if $x\ge x'$, $y\ge y'$ and such that $f(x,x)\equiv x$. Then with the notions of the precedent example, $\mathcal{E}[\,\cdot\,]:=f(\mathcal{E}^*[\,\cdot\,],\mathcal{E}_*[\,\cdot\,])$ is a nonlinear expectation on $[L_b(\mathcal{F})]_*$.

Example 2.11. (g-Expectation (see [36])) Let $(\mathcal{F}_t)_{t\ge0}$ be the filtration of a d-dimension -al Brownian $(B_t)_{t\ge0}$ in a probability space $(\Omega,\mathcal{F},P)$ and let D be the linear space of all bounded and $\left\{\bigcup_{T>0}\mathcal{F}_T\right\}$-measurable random variables. For each $X\in D$, there exists a $T>0$ such that X is $\mathcal{F}_T$-measurable. We solve the following 1-dimensional BSDE

$$-dY_t=g(Z_t)dt-Z_tdB_t,\qquad Y_T=X,$$

where the given generator $g(z):R^d\to R$ satisfies Lipschitz condition in z and $g(0)\equiv0$. We define

$$\mathcal{E}_g[X]:=Y_t|_{t=0}.$$

$\mathcal{E}_g[X]$ is called g-expectation. This g-expectation has $\mathcal{F}_t$-conditional expectation $\mathcal{E}_g[X/\mathcal{F}_t]$ $:=Y_t$. It is the only $\mathcal{F}_t$-measurable element satisfying

$$\mathcal{E}_g[1_A\mathcal{E}_g[X/\mathcal{F}_t]]=\mathcal{E}_g[1_AX],\qquad\forall\,A\in\mathcal{F}_t.$$

We thus call $\mathcal{E}_g$ a filtration-consistent nonlinear expectation. A particularly interesting case is $g_\mu(z):=\mu|z|$, where μ is a constant (see [7]) for an interesting application of $\mathcal{E}_{g_\mu}$ to economics and finance). When $\mu\ge0$, $\mathcal{E}_{g_\mu}$ is a strongly self-dominated nonlinear pre-expectation in D. If μ is bigger than the Lipschitz constant c of a generator g, then $\mathcal{E}_g$ is dominated by $\mathcal{E}_{g_\mu}$ (see [8]). In particular, when $\mu=0$, the related completion of D is $L^1(\Omega,\mathcal{F}_\infty,P)$.

Remark 2.4. A notion of $\mathcal{E}_{g_\mu}$-dominated and $\mathcal{F}_t$-consistent nonlinear expectation was introduced in [8]. We have proved that if the nonlinearity of an $\mathcal{E}_{g_\mu}$-dominated expectation depends only on the risk, then it is a g-expectation.

Example 2.12. (A Singular Case) Let $\alpha\in R^d$ be given. We set $g^\mu=\mu|\alpha\cdot z|,\forall\,z\in R^d$. In this case we have $\mathcal{E}_{g^\mu}[X]\ge\mathcal{E}_{g^\nu}[X]$, for $\mu\ge\nu$. $\mathcal{E}_{g^\nu}$ is strongly self-dominated

$$\mathcal{E}_{g^\mu}[X]-\mathcal{E}_{g^\mu}[Y]\le\mathcal{E}_{g^\mu}[X-Y].$$

We then define $\mathcal{E}^\infty[X] := \lim_{\mu\to\infty} \mathcal{E}_{g^\mu}[X]$. It is easy to check that $\mathcal{E}^\infty$ is still a strongly self-dominated expectation. In finance, this expectation is used in the pricing of contingent claims in an incomplete market (see [13, 14, 18, 19]).

2.4. L^∞_*-Norms

Let $\mathcal{E}^*$ be a self-dominated nonlinear pre-expectation defined on $L_b(\mathcal{F})$ and let $L_*(\mathcal{F})$ be the completion of $L_b(\mathcal{F})$ under the norm $\|\cdot\|_*$ in the sense of Theorem 2.1. We assume furthermore that

$$X_1, X_2, \cdots \in L_*(\mathcal{F}) \text{ and } X_n(\omega) \searrow 0,\ \forall\, \omega \in \Omega \quad \text{imply} \quad \mathcal{E}^*[X_n] \searrow 0.$$

Lemma 2.4. *Under this assumption we also have*

$$X, X_1, X_2, \cdots \in L_*(\mathcal{F}) \text{ and } X_n(\omega) \searrow X(\omega),\ \forall\, \omega \in \Omega \quad \text{imply} \quad \mathcal{E}^*[X_n] \searrow \mathcal{E}^*[X],$$

$$X, X_1, X_2, \cdots \in L_*(\mathcal{F}) \text{ and } X_n(\omega) \nearrow X(\omega),\ \forall\, \omega \in \Omega \quad \text{imply} \quad \mathcal{E}^*[X_n] \nearrow \mathcal{E}^*[X].$$

Proof. In fact, since $|X - X_n|(\omega) \searrow 0$, $\forall\, \omega \in \Omega$. By the self-domination of $\mathcal{E}^*$, we have, for the first situation,

$$0 \le \mathcal{E}^*[X_n] - \mathcal{E}^*[X] \le \mathcal{E}^*[|X - X_n|] \searrow 0,$$

and for the second situation,

$$0 \le \mathcal{E}^*[X_n] - \mathcal{E}^*[X] \le \mathcal{E}^*[|X - X_n|] \searrow 0.$$

Lemma 2.5. *Let $X \in L_*(\mathcal{F})$. Then $\mathcal{E}^*[|X|] - \mathcal{E}^*[|X| \wedge n] \le \mathcal{E}^*[|X| - |X| \wedge n] \searrow 0$ as $n \nearrow \infty$.*

Lemma 2.6. *Let $X \in L_*(\mathcal{F})$. Then*

$$\mathcal{E}^*[|X|] = 0 \Longleftrightarrow \mathcal{E}^*[\mathbf{1}_{\{|X|>0\}}] = 0.$$

Proof. Necessity. Let $\mathcal{E}^*[|X|] = 0$. Suppose by contraction that $\mathcal{E}^*[\mathbf{1}_{\{|X|>0\}}] > 0$. Since for each $\omega \in \Omega$, $\mathbf{1}_{\{|X|>\epsilon\}}(\omega) \nearrow \mathbf{1}_{\{|X|>0\}}(\omega)$ as $\epsilon \searrow 0$, we then have

$$\mathcal{E}^*[\mathbf{1}_{\{|X|>\epsilon\}}] \nearrow \mathcal{E}^*[\mathbf{1}_{\{|X|>0\}}] > 0.$$

It follows that there exists $\epsilon > 0$ such that $\mathcal{E}^*[\mathbf{1}_{\{|X|>\epsilon\}}] > 0$. Thus $\mathcal{E}^*[\epsilon\mathbf{1}_{\{|X|>\epsilon\}}] > 0$ and then $\mathcal{E}^*[|X|] \ge \mathcal{E}^*[\epsilon\mathbf{1}_{\{|X|>\epsilon\}}] > 0$. This contradicts $\mathcal{E}^*[|X|] = 0$.

Sufficiency. We first prove $\mathcal{E}^*[|X| \wedge n] = 0$ for each fixed $n = 1, 2, \cdots$. Since

$$\begin{aligned} 0 &= \mathcal{E}^*[n\mathbf{1}_{\{|X|>0\}}(\omega)] = \mathcal{E}^*[n\mathbf{1}_{\{(|X|\wedge n)>0\}}] \\ &\ge \mathcal{E}^*[(|X| \wedge n)\mathbf{1}_{\{(|X|\wedge n)>0\}}] = \mathcal{E}^*[|X| \wedge n], \end{aligned}$$

we have $\mathcal{E}^*[|X| \wedge n] = 0$. $|X| \wedge n] \nearrow |X|$ as $n \nearrow \infty$, $0 = \mathcal{E}^*[|X| \wedge n] \nearrow \mathcal{E}^*[|X|]$. Thus

$$\mathcal{E}^*[|X|] = 0.$$

For each $X \in L_*(\mathcal{F})$ we set

$$\text{ess}^* \sup_{\omega\in\Omega} X(\omega) : = \inf\{c \in R;\ c \geq X \text{ in } L_*(\mathcal{F})\},$$

$$\|X\|_{*\infty} : = \text{ess}^* \sup_{\omega\in\Omega} |X(\omega)|.$$

We then can define

$$L_*^\infty(\mathcal{F}) := \{X \in L_*(\mathcal{F});\ \|X\|_{*\infty} < +\infty\}.$$

By Lemma 2.6, $L_*^\infty(\mathcal{F})$ constitutes a Banach space under the norm $\|X\|_{*\infty}$.

Remark 2.5. If $\mathcal{E}^*$ is a linear expectation, then the space $L_*^\infty(\mathcal{F})$ becomes the classical $L^\infty(\mathcal{F})$.

Lemma 2.7. *Let $X,\ X_1, X_2, \cdots \in L_{*\infty}(\mathcal{F})$ be such that $\|X_n - X\|_{*\infty} \to 0$ and let $f : R \longrightarrow R$ be a continuous function. Then $f(X),\ f(X_1), f(X_2), \cdots \in L_*^\infty(\mathcal{F})$ and*

$$\|f(X_n) - f(X)\|_{*\infty} \to 0.$$

§ 3 . Distributions of Random Variables and Stochastic Processes

In this section we consider nonlinear distributions of R^d-valued random variables and R^d-valued stochastic processes. As in classic situations, the space R^d can be generalized to a Polish space $\mathcal{S}$. We shall give a nonlinear generalization of Kolmogorov's consistent theorem.

3.1. Distributions of random variables

Let $\mathcal{E}[\,\cdot\,]$ be a nonlinear pre-expectation on $L_b(\mathcal{F})$. We also denote by $L_b(\mathcal{B}(R^d))$, the space of $\mathcal{B}(R^d)$-measurable real functions defined on R^d such that $\sup\limits_{x\in R^d} |\phi(x)| < \infty$ holds for each $\phi \in L_b(\mathcal{B}(R^d))$. Let $X \in L_b(\mathcal{F})$ be given. The nonlinear distribution of X under $\mathcal{E}[\,\cdot\,]$ is defined by

$$\mathcal{T}[\phi] := \mathcal{E}[\phi(X)], \qquad \phi \in L_b(\mathcal{B}(R^d)).$$

This distribution $\mathcal{T}[\,\cdot\,]$ is again a nonlinear pre-expectation defined on $L_b(\mathcal{B}(R^d))$.

3.2. Family of finite-dimensional nonlinear distributions of a process

In the rest of this paper Ω will be a collection of R^d-valued processes defined on $R^+ = [0, \infty)$. A typical situation is $\Omega = C^d(R^+)$, the space of all R^d-valued continuous functions $(\omega_t)_{t\in R^+}$ equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^\infty 2^{-i}\Big[\Big(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|\Big) \wedge 1\Big]$$

with $\mathcal{F} = \mathcal{B}(C^d(R^+))$. Our formulation is also applied to some other canonical space such as $D(0, \infty)$, $(R^d)^{[0,\infty)}$. The space $L_b(\mathcal{F})$ is defined in (2.2). We also set

$$L_0(\mathcal{F}) := \{X(\omega) = \phi(\omega_{t_1}, \cdots, \omega_{t_m}), \quad \forall\, m \geq 1,\ t_1, \cdots, t_m \in R^+, \forall\, \phi \in L_b(\mathcal{B}[(R^d)^m])\}.$$

It is clear that $L_0(\mathcal{F})$ is a linear subspace of $L_b(\mathcal{F})$.

Given a nonlinear pre-expectation $\mathcal{E}[\,\cdot\,]$ defined on $L_b(\mathcal{F})$. The family of finite-dimension -al nonlinear distributions of the canonical process $(\omega_t)_{t\in R^+}$ under $\mathcal{E}[\,\cdot\,]$ is defined as follows. For each integer $m \geq 1$, $\phi \in L_b(\mathcal{B}[(R^d)^m])$ and $t_1, \cdots, t_m \in R^+$ (in this paper we always assume that $t_1, \cdots, t_m$ are different from each other), we set

$$\mathcal{T}_{t_1,\cdots,t_m}[\phi(\,\cdot\,)] := \mathcal{E}[\phi(\omega_{t_1}, \cdots, \omega_{t_m})], \qquad \phi \in L_b(\mathcal{B}[(R^d)^m]). \tag{3.1}$$

Since $(\omega_{t_1}, \cdots, \omega_{t_m})$ can be regarded as an $R^{m\times d}$-valued random variable, $\mathcal{T}_{t_1,\cdots,t_m}[\phi(\,\cdot\,)]$ is a nonlinear pre-expectation defined on $L_b(\mathcal{B}[(R^d)^m])$.

As in classical situations, this family of distributions is consistent in the following sense. Let Π_m be the set of all permutations of $(1, 2, \cdots, m)$, i.e., for each $\pi \in \Pi_m$,

$$\pi(1, 2, \cdots, m) = (\pi(1), \pi(2), \cdots, \pi(m)).$$

For each $\pi \in \Pi_m$ and $\phi \in L_b(\mathcal{B}[(R^d)^m])$, we also denote

$$\phi^\pi(x_1, \cdots, x_m) := \phi(x_{\pi(1)}, \cdots, x_{\pi(m)}).$$

We have the following obvious properties.

Lemma 3.1. *The family of finite-dimensional nonlinear distributions of the process* $(\omega_t)_{t\in[0,\infty)}$ *defined in* (3.1) *satisfies*

(k0) $\mathcal{T}_{t_1,\cdots,t_m}[\phi(\,\cdot\,)]$ *is a nonlinear pre-expectation on* $L_b(\mathcal{B}[(R^d)^m])$*;*

(k1) $\mathcal{T}_{t_1,\cdots,t_m}[\phi(\,\cdot\,)] = \mathcal{T}_{t_{\pi(1)},\cdots,t_{\pi(m)}}[\phi^\pi(\,\cdot\,)], \quad \forall\, \pi \in \Pi_m$*;*

(k2) *If* $\phi \in L_b(\mathcal{B}[(R^d)^m])$ *does not depend on* x_m*, i.e.,* $\phi = \phi(x_1, \cdots, x_{m-1})$*, then*

$$\mathcal{T}_{t_1,\cdots,t_m}[\phi(\,\cdot\,)] = \mathcal{T}_{t_1,\cdots,t_{m-1}}[\phi(\,\cdot\,)].$$

From (3.1) we immediately have

Lemma 3.2. *Let* $\mathcal{E}^1$ *and* $\mathcal{E}^2$ *be two pre-expectations defined on* $L_b(\mathcal{F})$ *and let* $\{\mathcal{T}^1_{t_1,\cdots,t_m}\}$ *and* $\{\mathcal{T}^2_{t_1,\cdots,t_m}\}$ *be the corresponding families of finite-dimensional distributions respectively induced by* $\mathcal{E}^1$ *and* $\mathcal{E}^2$ *in the sense of* (3.1)*. If* $\mathcal{E}^1$ *is* $\mathcal{E}^2$*-dominated* (*resp. strongly dominated*)*, then for each* $t_1, \cdots, t_m \in R^+$*,* $\mathcal{T}^1_{t_1,\cdots,t_m}$ *is also* $\mathcal{T}^2_{t_1,\cdots,t_m}$*-dominated* (*resp. strongly dominated*)*:*

$$\mathcal{T}^1_{t_1,\cdots,t_m}[\phi(\cdot)] - \mathcal{T}^1_{t_1,\cdot s,t_m}[\psi(\,\cdot\,)] \leq \mathcal{T}^2_{t_1,\cdots,t_m}[(\phi-\psi)(\,\cdot\,)], \qquad \forall\, \phi, \psi \in L_b(\mathcal{B}[(R^d)^m]).$$

Namely $\{\mathcal{T}^1_{t_1,\cdots,t_m}\}$ *is dominated* (*resp. strongly dominated*) *by* $\{\mathcal{T}^2_{t_1,\cdots,t_m}\}$*. In particular, if* $\mathcal{E}^*$ *is self-dominated* (*resp. strongly self-dominated*)*, then the family of finite-dimensional distributions* $\{\mathcal{T}^*_{t_1,\cdots,t_m}\}$ *is also self-dominated* (*resp. strongly self-dominated*)*.*

We shall give an extension of Kolmogorov's consistence theorem to nonlinear situations.

Theorem 3.1. (Nonlinear Kolmogorov Theorem)

(i) *Let*

$$\{\mathcal{T}_{t_1,\cdots,t_m}[\phi(\,\cdot\,)], \quad m \geq 1,\ t_1, \cdots, t_m \in R^+,\ \phi \in L_b(\mathcal{B}[(R^d)^m])\}$$

be a family of nonlinear pre-distributions satisfying (k0), (k1) *and* (k2). *Then there exists a unique nonlinear pre-expectation* $\mathcal{E}[\,\cdot\,]$ *defined on* $L_0(\mathcal{F})$ *such that*

$$\mathcal{E}[X]=\mathcal{T}_{t_1,\cdots,t_m}[\phi(\,\cdot\,)],\qquad \forall\, m\geq 1,\ t_1,\cdots,t_m\in R^+,\ \forall\, X\in L_0(\mathcal{F})$$
$$\text{with } X(\omega)=\phi(\omega_{t_1},\cdots,\omega_{t_m}),\ \phi\in L_b(\mathcal{B}[(R^d)^m]).$$

(ii) *If a family of nonlinear pre-distributions*

$$\{\mathcal{T}^*_{t_1,\cdots,t_m}[\phi(\,\cdot\,)],\ m\geq 1,\ t_1,\cdots,t_{m-1}\in R^+,\ \phi\in L_b(\mathcal{B}[(R^d)^m])\}$$

satisfying (k0), (k1) *and* (k2) *is self-dominated* (*resp. strongly self-dominated*), *then the corresponding nonlinear pre-expectation* $\mathcal{E}^*[\,\cdot\,]$ *is also self-dominated* (*resp. strongly self-dominated*). *Consequently, we can use Theorem* 2.1 *to extend* $\mathcal{E}^*$ *to the Fréchet space* $[L_0(\mathcal{F})]_*$ *under the quasi-norm* $\|X\|_*:=\mathcal{E}^*[|X|]$. *The extension* $\mathcal{E}^*$ *is a self-dominated* (*resp. strongly self-dominated*) *nonlinear expectation.*

(iii) *Furthermore, if a family of nonlinear pre-distributions* $\{\mathcal{T}_{t_1,\cdots,t_m}\}$ *satisfying conditions in* (i) *is* $\{\mathcal{T}^*_{t_1,\cdots,t_m}\}$*-dominated* (*resp. strongly dominated*), *then the corresponding nonlinear pre-expectation* $\mathcal{E}$ *is also* $\mathcal{E}^*$*-dominated* (*resp. strongly dominated*). *Consequently, we can use Theorem* 2.1 *to extend* $\mathcal{E}$ *to a* $\mathcal{E}^*$*-dominated* (*resp. strongly dominated*) *nonlinear expectation on* $[L_0(\mathcal{F})]_*$.

Proof. (i). From (k0), (k1) and (k2) we can consistently define a functional $\mathcal{E}[\,\cdot\,]:L_0(\mathcal{F})\to R$ such that

$$\mathcal{E}[\phi(\omega_{t_1},\cdots,\omega_{t_m})]:=\mathcal{T}_{t_1,\cdots,t_m}[\phi(\,\cdot\,)],\qquad \forall\,\phi\in L_b(\mathcal{B}[(R^d)^m]).$$

The uniqueness is clear. From the monotonicity and constant-preserving of $\mathcal{T}$, we have

$$\mathcal{E}[X]-\mathcal{E}[Y]\geq 0,\qquad \text{if } X\geq Y, \tag{3.2}$$

$$\mathcal{E}[c]=c. \tag{3.3}$$

Thus $\mathcal{E}^*$ is a nonlinear pre-expectation on $L_0(\mathcal{F})$.

(ii) and (iii). Thanks to the self-domination of $\mathcal{T}^*$, we have

$$\mathcal{T}^*_{t_1,\cdots,t_m}[\phi(\,\cdot\,)]-\mathcal{T}^*_{t_1,\cdots,t_m}[\psi(\,\cdot\,)]\leq \mathcal{T}^*_{t_1,\cdots,t_m}[|\phi-\psi|(\,\cdot\,)],\qquad \forall\phi,\psi\in L_b(\mathcal{B}[(R^d)^m]).$$

In other words,

$$\mathcal{E}^*[X]-\mathcal{E}^*[Y]\leq \mathcal{E}^*[|X-Y|],\qquad \forall\, X,Y\in L_0(\mathcal{F}). \tag{3.4}$$

The rest of the conclusions follows directly from Theorem 2.1 and its corollaries.

§ 4. Nonlinear Markov Chains

4.1. Nonlinear markov chains

For simplification, we only consider time-homogeneous nonlinear Markov chains. Nonhomogeneous situation can be treated similarly. We consider the following family of nonlinear pre-expectations, parametrized by $t\in R^+$,

$$\mathcal{T}_t[\phi]:L_b(\mathcal{B}(R^d))\longrightarrow L_b(\mathcal{B}(R^d)),\qquad t\geq 0. \tag{4.1}$$

In certain cases it is convenient to consider some (lattice) subspaces of $L_b(\mathcal{B}(R^d))$, such as $C_b(R^d)$ (uniformly continuous and bounded real functions on R^d), instead of $L_b(\mathcal{B}(R^d))$.

Definition 4.1. *A family of nonlinear pre-expectations* (4.1) *is called a Markov chain if it satisfies*

(m1) *For each fixed* $(t,x) \in R^+ \times R^d$, $\mathcal{T}_t[\phi](x)$ *is a nonlinear pre-expectation defined on* $L_b(\mathcal{B}(R^d))$.

(m2) $\mathcal{T}_0[\phi](x) = \phi(x)$.

(m3) $\mathcal{T}_t[\phi](x)$ *satisfies the following Chapman* (*semigroup*) *formula*

$$\mathcal{T}_t \circ \mathcal{T}_s[\phi] := \mathcal{T}_s[\mathcal{T}_s[\phi]] = \mathcal{T}_{t+s}[\phi]. \tag{4.2}$$

Definition 4.2. *Let* $\mathcal{T}^1_\cdot$, $\mathcal{T}^2_\cdot$ *be two Markov chains defined on* $L_b(\mathcal{B}(R^d))$. $\mathcal{T}^1_\cdot$ *is said to be dominated* (*resp. strongly dominated*) *by* $\mathcal{T}^2_\cdot$ *if for each* $t \in R^+$, *the pre-expectation* $\mathcal{T}^1_t[\,\cdot\,]$ *is dominated* (*resp. strongly dominated*) *by* $\mathcal{T}^2_t[\,\cdot\,]$. *A Markov chain* $\mathcal{T}_\cdot$ *defined on* $L_b(\mathcal{B}(R^d))$ *is said to be self-dominated* (*resp. strongly self-dominated*) *if it is dominated* (*resp. strongly dominated*) *by itself.*

4.2. Examples

Example 4.1. For $\phi \in L_b(\mathcal{B}(R^d))$, we set

$$\mathcal{T}^0_t\phi(x) := (2\pi t)^{-\frac{d}{2}} \int_{R^d} \phi(y) \exp\Big[-\frac{|y-x|^2}{2t}\Big] dy. \tag{4.3}$$

This semigroup is induced by a standard d-dimensional Brownian motion $(B_t)_{t\geq 0}$ by $\mathcal{T}^0_t\phi(x) = E[\phi(x+B_t)]$. It is well known that $u(t,x) := \mathcal{T}^0_t\phi(x)$ is the solution of the following heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x), \qquad (t,x) \in [0,\infty) \times R^d, \\ &u(0,\,\cdot\,) = \phi(\,\cdot\,) \in L_b(\mathcal{B}(R^d)). \end{aligned} \tag{4.4}$$

Example 4.2. (A Nonlinear Generalization of $\mathcal{T}^0$) For some fixed $\mu \in R$ and for each $\phi(\,\cdot\,) \in L_b(\mathcal{B}(R^d))$, we first solve the following nonlinear equation

$$\begin{aligned} &\frac{\partial u^\mu}{\partial t}(t,x) = \frac{1}{2}\Delta u^\mu(t,x) + \mu|\nabla u^\mu|, \qquad (t,x) \in [0,\infty) \times R^d, \\ &u^\mu(0,\,\cdot\,) = \phi(\,\cdot\,) \in L_b(\mathcal{B}(R^d)). \end{aligned} \tag{4.5}$$

Then we define

$$\mathcal{T}^\mu_t\phi(x) := u^\mu(t,x), \qquad x \in R^d.$$

It is also easy to check that $(\mathcal{T}^\mu_t)_{t\geq 0}$ is a semigroup defined on $L^\infty(R^n)$. By Comparison Theorem of Parabolic PDE, if $\mu \geq \nu$, then

$$\mathcal{T}^\mu_t\phi(x) \geq \mathcal{T}^\nu_t\phi(x).$$

When $\mu \geq 0$, it is not hard to check that $\mathcal{T}^{\mu}$ is convex

$$\mathcal{T}_t^{\mu}(\alpha\phi + (1-\alpha)\psi)(x) \geq \alpha(\mathcal{T}_t^{\mu}\phi)(x) + (1-\alpha)(\mathcal{T}_t^{\mu}\psi)(x)$$

as well as homogeneous

$$\mathcal{T}_t^{\mu}[\mu|\phi|](x) = \mu\mathcal{T}_t^{\mu}[|\phi|](x).$$

It is a strongly self-dominated nonlinear Markov chain.

Example 4.3. (A Generalization of $\mathcal{T}^{\mu}$) Let $g(x,z) : R^d \times R^d \longmapsto R$ be a given continuous function satisfying

$$\begin{cases} g(x,0) \equiv 0, & \forall\, x \in R^d, \\ |g(x,z_1) - g(x,z_2)| \leq \mu|z_1 - z_2|, & z_1, z_2 \in R^d. \end{cases} \tag{4.6}$$

By analogy to the previous example, we first solve the following nonlinear equation

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + g(x, \nabla u), \qquad (t,x) \in [0,\infty) \times R^d, \\ &u(0,\,\cdot\,) = \phi(\,\cdot\,) \in L^{\infty}(R^d). \end{aligned} \tag{4.7}$$

Then we define

$$\mathcal{T}_t^{g}\phi(x) := u(t,x), \qquad x \in R^d. \tag{4.8}$$

$\mathcal{T}_\cdot^{g}$ is strongly dominated by $\mathcal{T}_\cdot^{\mu}$.

The following example gives a fully nonlinear self-dominated Markov chain.

Example 4.4. Let $a(x,v) : R^n \times R^k \to R^{n\times n}$ and $b(x,v) : R^n \times R^k \to R^n$ be uniformly continuous and bounded functions such that $a_{ij} = a_{ji}$, a and b are uniformly Lipschitz functions of x. Let V be a closed and bounded subset of R^k. We consider the following fully nonlinear parabolic PDE

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \sup_{v\in V}\Big\{\sum_{i,j=1}^{n} a_{ij}(x,v)\partial_{x_i x_j}u + \sum_{i=1}^{n} b_i(x,v)\partial_{x_i}u\Big\}, \qquad (t,x) \in [0,\infty) \times R^d, \\ &u(0,\,\cdot\,) = \phi(\,\cdot\,) \in C_b(R^d). \end{aligned} \tag{4.9}$$

Under the notion of viscosity solution, this equation has a unique solution in $C_b(R^d)$. Then we define $\mathcal{T}_t^{*}\phi(x) := u(t,x)$, $x \in R^d$. This is a strongly self-dominated Markov chain defined on $C_b(R^d)$. We alos have $\mathcal{T}_t^{*}[\lambda\phi] = \lambda\mathcal{T}_t^{*}[\phi]$, for $\lambda \geq 0$.

Remark 4.1. Equation (4.9) is known as Hamilton-Jacobi-Bellman equation. It is a fully nonlinear equation. For detailed studies of this equation, we refer to [9], also [4, 21, 22, 27, 31, 32, 43].

Example 4.5. We can also consider a situation similar to Example 4 where (4.9) is replaced by

$$\frac{\partial u}{\partial t}(t,x) = \inf_{v\in V}\Big\{\sum_{i,j=1}^{n} a_{ij}(x,v)\partial_{x_i x_j}u + \sum_{i=1}^{n} b_i(x,v)\partial_{x_i}u\Big\}.$$

The corresponding nonlinear Markov chain is dominated by $\mathcal{T}_\cdot^{*}$.

Remark 4.2. We can also consider a combination of the last two examples (see [34]).

§ 5. Filtration-Consistent Expectation Generated by Nonlinear Markov Chain

We shall introduce a filtration in the canonical space $(\Omega, \mathcal{F})$. A typical example is for $(\Omega, \mathcal{F}) = (C^d(R^+), \mathcal{B}(C^d(R^+)))$. In this case we set

$$C^d_{0,t}(R^+) := \left\{\omega \in C^d(R^+);\ \omega(s) \equiv \omega(t),\ \forall\, s \geq t\right\}$$

and $\mathcal{F}_t := \mathcal{B}(C^d_{x,t}(R^+))$. It is clear that $(\mathcal{F}_t)_{t\geq 0}$ is a filtration and $\mathcal{F} = \sigma\Big\{\bigcup\limits_t \mathcal{F}_t\Big\}$. Similarly to the notions $L_b(\mathcal{F})$ and $L_0(\mathcal{F})$, we can define $L_b(\mathcal{F}_t)$ and $L_0(\mathcal{F}_t)$.

5.1. Nonlinear expectation generated by nonlinear Markov chains

Let $(\mathcal{T}_t)_{t\geq 0}$ be a given nonlinear Markov chain satisfying (m1), (m2) and (m3). For a fixed $x_0 \in R^d$, we can induce a family of finite-dimensional nonlinear distributions in the following way. For each given integer $m \geq 1$ and $\phi \in L_b(\mathcal{B}(R^{m\times d}))$ and $0 < t_1 < \cdot s < t_m$, we successively define functions $\phi_i \in L_b(R^{(m-i)\times d})$, $i = 1, \cdots, m$, by

$$\begin{aligned}
\phi_1(x_1, \cdots, x_{m-1}) &:= \mathcal{T}_{t_m - t_{m-1}}[\phi(x_1, \cdots, x_{m-1}, \cdot)](x_{m-1}),\\
\phi_2(x_1, \cdots, x_{m-2}) &:= \mathcal{T}_{t_{m-1} - t_{m-2}}[\phi_1(x_1, \cdots, x_{m-2}, \cdot)](x_{m-2}),\\
&\ \ \vdots\\
\phi_{m-1}(x_1) &:= \mathcal{T}_{t_2 - t_1}[\phi_{m-2}(x_1, \cdot)](x_1),\\
\phi_m(x_0) &:= \mathcal{T}_{t_1}[\phi_{m-1}(\cdot)](x_0).
\end{aligned} \tag{5.1}$$

We then set

$$\mathcal{T}^{x_0}_{t_1,\cdots,t_m}[\phi] := \phi_m(x_0) : L_b(\mathcal{B}[(R^d)^m]) \longrightarrow R. \tag{5.2}$$

We have the following lemmas.

Lemma 5.1. *The functional $\mathcal{T}^{x_0}_{t_1,\cdots,t_m}[\cdot]$ defined in* (5.2) *is a nonlinear pre-expectation on $L_b(\mathcal{B}[(R^d)^m])$.*

Proof. This assertion follows from the fact that $(\mathcal{T}_t)_{t\geq 0}$ are pre-expectations.

Lemma 5.2. *If $\phi \in L_b(\mathcal{B}[(R^d)^m])$ does not depend on x_i for some $1 \leq i \leq m$, i.e.,*

$$\phi = \phi(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots x_m),$$

then ϕ can also be treated as an element of $L_b(\mathcal{B}[(R^d)^{m-1}])$ and we can use $\mathcal{T}_t[c] = c$ to get

$$\mathcal{T}^{x_0}_{t_1,\cdots,t_m}[\phi] = \mathcal{T}^{x_0}_{t_1,\cdots t_{i-1},t_{i+1}\cdots,t_m}[\phi]. \tag{5.3}$$

From the Chapman relation of $\mathcal{T}_t$, it is easy to check

Lemma 5.3. *For each $\phi \in L_b(\mathcal{B}[(R^d)^{n+m}])$ and $0 < t_1 < \cdots < t_{m+n}$,*

$$\mathcal{T}^{x_0}_{t_1,\cdots,t_n} \circ \mathcal{T}^{\cdot}_{t_{n+1}-t_n,\cdots,t_{n+m}-t_n}[\phi] := \mathcal{T}^{x_0}_{t_1,\cdots,t_n}[\mathcal{T}^{\cdot}_{t_{n+1}-t_n,\cdots,t_{n+m}-t_n}[\phi]] = \mathcal{T}^{x_0}_{t_1,\cdots,t_{n+m}}[\phi].$$

Here the meaning of the left hand side is as follows. We first take $(x_1, \cdots, x_n)$ *as a fixed parameter and calculate* $\psi(x_1, \cdots, x_n) = \mathcal{T}^{x_n}_{t_{n+1}-t_n, \cdots, t_{n+m}-t_n}[\phi(x_1, \cdots, x_n, \cdot)]$. *Then we calculate*

$$\mathcal{T}^{x_0}_{t_1, \cdots, t_n}[\mathcal{T}^{\cdot}_{t_{n+1}-t_n, \cdots, t_{n+m}-t_n}\phi] = \mathcal{T}^{x_0}_{t_1, \cdots, t_n}[\psi(\,\cdot\,)]. \tag{5.4}$$

Lemma 5.4. *We define the family of nonlinear distributions related to the Markov chain* $\mathcal{T}_t$ *by*

$$\{\mathcal{T}_{\pi(t_1), \cdot\cdot, \pi(t_m)}[\phi^\pi] := \mathcal{T}^0_{t_1, \cdots, t_m}[\phi], \quad \forall\, m \geq 0,\ 0 < t_1 < \cdots < t_m,\ \phi \in L_b(\mathcal{B}((R^d)^m),\ \pi \in \Pi_m\}.$$

This family satisfies the generalized Kolmogorov consistence conditions (k0), (k1) *and* (k2).

Proof. (k0) is due to Lemma 5.1. (k1) property of $\mathcal{T}^0_{t_1, \cdots, t_m}$ is simply due to its definition (5). The (k2) property is from (5.3) of Lemma 5.2.

Lemma 5.5. *Let* $\mathcal{T}_\cdot$ *and* $\mathcal{T}'_\cdot$ *be two Markov chain defined on* $L_b(\mathcal{B}(R^d))$ *such that* $\mathcal{T}_\cdot$ *is dominated by* $\mathcal{T}'_\cdot$, *and let* $\mathcal{T}^{x_0}_{t_1, \cdots, t_m}[\,\cdot\,]$ *and* $\mathcal{T}'^{x_0}_{t_1, \cdots, t_m}[\,\cdot\,]$ *be the corresponding families of nonlinear distributions related respectively to* $\mathcal{T}_\cdot$ *and* $\mathcal{T}'_\cdot$ *Then for each* $t_1, \cdots, t_m \in R^+$, $\mathcal{T}^{x_0}_{t_1, \cdots, t_m}[\,\cdot\,]$ *is dominated by* $\mathcal{T}'^{x_0}_{t_1, \cdots, t_m}[\,\cdot\,]$.

Proof. Without loss of generality, we can suppose that $0 < t_1 < \cdots < t_m$,

$$\begin{aligned}\mathcal{T}^{x_0}_{t_1, \cdots, t_m}[\phi(\,\cdot\,)] - \mathcal{T}^{x_0}_{t_1, \cdots, t_m}[\psi(\,\cdot\,)] &= \mathcal{T}^{x_0}_{t_1}[\mathcal{T}^{\cdot}_{t_2-t_1, \cdots, t_m-t_1}[\phi]] - \mathcal{T}^{x_0}_{t_1}[\mathcal{T}^{\cdot}_{t_2-t_1, \cdots, t_m-t_1}[\psi]] \\ &\leq \mathcal{T}'^{x_0}_{t_1}[\mathcal{T}^{\cdot}_{t_2-t_1, \cdots, t_m-t_1}[\phi] - \mathcal{T}^{\cdot}_{t_2-t_1, \cdots, t_m-t_1}[\psi]].\end{aligned}$$

Repeating this procedure and applying (5.4) for $\mathcal{T}'$, we have $\mathcal{T}^{x_0}_{t_1, \cdots, t_m}[\phi(\,\cdot\,)] - \mathcal{T}^{x_0}_{t_1, \cdots, t_m}[\psi(\,\cdot\,)] \leq \mathcal{T}'^{x_0}_{t_1, \cdots, t_m}[\phi(\,\cdot\,) - \psi(\,\cdot\,)]$. This completes the proof.

Since the family $\mathcal{T}^0_{t_1, \cdots, t_m}$, $t_1, \cdots, t_m \in R^+$ of finite-dimensional nonlinear distributions satisfies the generalized Kolmogorov consistence conditions (k0), (k1) and (k2), from Nonlinear Kolmogorov Theorem 3.1, we have immediately

Theorem 5.1. (i) *Let* $\mathcal{T}_\cdot$ *be a Markov chain defined on* $L_b(\mathcal{B}(R^d))$. *Then there exists a unique nonlinear pre-expectation* $\mathcal{E}[\,\cdot\,]$ *defined on* $L_0(\mathcal{F})$ *such that the related family of finite-dimensional nonlinear distributions of the canonical process* $(\omega_t)_{t \in R^+}$ *is* $\{\mathcal{T}^0_{t_1, \cdots, t_m};\ t_1, \cdots, t_m \in R^+\}$ (*defined in* (5)). *Particularly, for any* $X \in L_0(\mathcal{F})$ *with* $X = \phi(\omega_{t_1}, \cdots, \omega_{t_m})$, $\phi \in L^\infty((R^d)^m)$ *we have*

$$\mathcal{E}[X] = \mathcal{T}^0_{t_1, \cdots, t_m}[\phi]. \tag{5.5}$$

(ii) *Let* $\mathcal{T}^1_\cdot$ *and* $\mathcal{T}^2_\cdot$ *be two Markov chains defined on* $L_b(\mathcal{B}(R^d))$ *such that* $\mathcal{T}^1$ *is dominated* (*resp. strongly dominated*) *by* $\mathcal{T}^2$ *and then the corresponding* $\mathcal{E}^1$ *is also dominated* (*resp. strongly dominated*) *by* $\mathcal{E}^2$. *In particular, if a Markov chain* $\mathcal{T}^*$ *is self-dominated* (*resp. strongly self-dominated*), *then the corresponding pre-expectation* $\mathcal{E}^*$ *on* $L_0(\mathcal{F})$ *satisfying* (5.5) *is also self-dominated* (*resp. strongly self-dominated*). *Consequently* $\mathcal{E}^*$ *can be defined on the completion of* $[L_0(\mathcal{F})]_*$ *under the quasi-norm* $\|X\|_* := \mathcal{E}^*[|X|]$.

(iii) *Moreover, if a Markov chain* $\mathcal{T}_\cdot$ *on* $L_b(\mathcal{B}(R^d))$ *is dominated* (*resp. strongly dominated*) *by the above* $\mathcal{T}^*_\cdot$, *then the corresponding nonlinear pre-expectation* $\mathcal{E}$ *can be also* *uniquely extended to* $[L_0(\mathcal{F})]_*$. *The extended nonlinear expectation* $\mathcal{E}$ *is still dominated* (*resp. strongly dominated*) *by* $\mathcal{E}^*$.

5.2. Conditional nonlinear expectations under $\mathcal{F}_t$

Let $t>0$ and let $X \in L_0(\mathcal{F})$ be given as

$$X = \phi(\omega_{t_1}, \cdots, \omega_{t_n}, \omega_{t_{n+1}}, \cdots, \omega_{t_{n+m}}), \qquad 0 < t_1 < \cdots < t_n < \cdots < t_{n+m}, \quad (5.6)$$

where $\phi \in L_b((R^d)^{n+m})$. Without loss of generality, we may assume $t_n = t$. We consider, for each fixed $(x_1, \cdots, x_n) \in (R^d)^n$, $\phi(x_1, \cdots, x_n, \cdot) \in L_b(\mathcal{B}[(R^d)^m])$ and set

$$\Phi(x_1, \cdot s, x_n) := \mathcal{T}^{x_n}_{t_{n+1}-t_n, \cdots, t_{n+m}-t_n}[\phi(x_1, \cdots, x_n, \cdot)](x_n). \quad (5.7)$$

Definition 5.1. *For any* $X \in L_0(\mathcal{F})$ *in form of* (5.6) *with* $t = t_n$, *the conditional nonlinear pre-expectation under* $\mathcal{F}_t$ *denoted by* $\mathcal{E}[\cdot/\mathcal{F}_t] : L_0(\mathcal{F}) \longrightarrow L_0(\mathcal{F}_t)$ *is defined by*

$$\mathcal{E}[X/\mathcal{F}_t] := \Phi(\omega_{t_1}, \cdots, \omega_{t_n}). \quad (5.8)$$

where $\Phi \in L_b(\mathcal{B}[(R^d)^m])$ *is given by* (5.7).

Remark 5.1. The above formulations suggest that, in contrast to one's intuition, an $\mathcal{F}_T$-consistent expectation is calculated in a backward manner: from the terminal point T to the initial time $t=0$. In fact we first have data $\mathcal{E}[X/\mathcal{F}_T] = X$, then $\mathcal{E}[X/\mathcal{F}_t]$ from $t<T$ till $t=0$. In some sense, we are calculating a kind of backward SDE of type [33].

Lemma 5.6. *For a given* $t = t_n > 0$ *we have*

$$\mathcal{E}[1_K(\omega_{t_1}, \cdots, \omega_{t_n})X/\mathcal{F}_t] = 1_K \mathcal{E}[X/\mathcal{F}_t](\omega_{t_1}, \cdots, \omega_{t_n}), \qquad \forall\, K \in \mathcal{B}((R^d)^n), \quad (5.9)$$

$$\mathcal{E}[\mathcal{E}[X/\mathcal{F}_t]/\mathcal{F}_s] = \mathcal{E}[X/\mathcal{F}_{t\wedge s}]. \quad (5.10)$$

Proof. When $(x_1, \cdots, x_n)$ is considered as a parameter, it is clear that

$$\begin{aligned}&\mathcal{T}^{x_n}_{t_{n+1}-t_n, \cdots, t_{n+m}-t_n}[1_K(x_1, \cdots, x_n)\phi(x_1, \cdots, x_n, \cdot)]\\ &= 1_K(x_1, \cdots, x_n)\mathcal{T}^{x_n}_{t_{n+1}-t_n, \cdots, t_{n+m}-t_n}[\phi(x_1, \cdots, x_n, \cdot)]. \end{aligned} \quad (5.11)$$

Thus (5.9) holds. (5.10) is simply from the definition.

Remark 5.2. The above definition (5.8) is applied for any $t \in R^+$. Indeed, if $t \notin \{t_1, \cdots, t_{n+m}\}$ then we can use Lemma 5.2 to treat ϕ as an element in $L^\infty((R^d)^{n+m+1})$.

The following result is a simple consequence of Lemma 5.5.

Lemma 5.7. *Let* $\mathcal{T}^1_\cdot$ *and* $\mathcal{T}^2_\cdot$ *be two Markov chains defined on* $L_b(\mathcal{B}(R^d))$ *such that* $\mathcal{T}^1$ *is dominated* (*resp. strongly dominated*) *by* $\mathcal{T}^2$ *and then the corresponding conditional nonlinear pre-expectations* $\mathcal{E}^1[\cdot/\mathcal{F}_t]$ *is also dominated* (*resp. strongly dominated*) *by* $\mathcal{E}^2[\cdot/\mathcal{F}_t]$, *i.e.,*

$$\mathcal{E}^1[X/\mathcal{F}_t](\omega) - \mathcal{E}^1[Y/\mathcal{F}_t](\omega) \le \mathcal{E}^2[|X-Y|/\mathcal{F}_t](\omega), \qquad \forall\, \omega \in \Omega$$

$$(\textit{resp. } \mathcal{E}^1[X/\mathcal{F}_t](\omega) - \mathcal{E}^1[Y/\mathcal{F}_t](\omega) \le \mathcal{E}^2[X-Y/\mathcal{F}_t](\omega), \qquad \forall\, \omega \in \Omega).$$

In particular, if $\mathcal{T}^*$ is a self-dominated (resp. strongly self-dominated) Markov chain, then the corresponding $\mathcal{E}^*[\cdot/\mathcal{F}_t]$ is also self-dominated (resp. strongly dominated).

Now let $\mathcal{E}[\cdot]$ and $\mathcal{E}[\cdot/\mathcal{F}_t]$ be induced by $\mathcal{T}$ which is dominated by the above $\mathcal{T}^*$. From

$$\mathcal{E}[X/\mathcal{F}_t]-\mathcal{E}[Y/\mathcal{F}_t]\le \mathcal{E}^*[|X-Y|/\mathcal{F}_t], \qquad \forall X,Y\in L_0(\mathcal{F}), \tag{5.12}$$

we have $|\mathcal{E}[X/\mathcal{F}_t]-\mathcal{E}[Y/\mathcal{F}_t]|\le \mathcal{E}^*[|X-Y|/\mathcal{F}_t]$, $\forall X,Y\in L_0(\mathcal{F})$. It follows from $\mathcal{E}^*[\mathcal{E}^*[\cdot/\mathcal{F}_t]] = \mathcal{E}^*[\cdot]$ that $\forall X,Y\in L_0(\mathcal{F})$,

$$\|\mathcal{E}[X/\mathcal{F}_t]-\mathcal{E}[Y/\mathcal{F}_t]\|_* \le \|X-Y\|_*. \tag{5.13}$$

We then have

Propsition 5.1. *Under the constraint* (5.13), *the conditional nonlinear pre-expectation $\mathcal{E}[\cdot/\mathcal{F}_t]$ defined on $L_0(\mathcal{F})$ can be uniquely extended to a continuous mapping*

$$\mathcal{E}[\cdot/\mathcal{F}_t]:[L_0(\mathcal{F})]_*\longrightarrow [L_0(\mathcal{F}_t)]_*.$$

We have, for each $X,\ X'\in[L_0(\mathcal{F})]_*$,

(i) $\mathcal{E}[X/\mathcal{F}_t]=X$, *if* $X\in[L_0(\mathcal{F}_t)]_*$;

(ii) *If* $X\ge X'$, *then* $\mathcal{E}[X/\mathcal{F}_t]\ge\mathcal{E}[X'/\mathcal{F}_t]$;

(iii) $\mathcal{E}[\mathcal{E}[X/\mathcal{F}_t]/\mathcal{F}_s]=\mathcal{E}[X/\mathcal{F}_{t\wedge s}]$, $\mathcal{E}[\mathcal{E}[X/\mathcal{F}_t]]=\mathcal{E}[X]$;

(iv) $\mathcal{E}[1_K(\omega_{t_1},\cdots,\omega_{t_n})X/\mathcal{F}_t]=1_K(\omega_{t_1},\cdots,\omega_{t_n})\mathcal{E}[X/\mathcal{F}_t]$;

(v) $\mathcal{E}[X/\mathcal{F}_t]-\mathcal{E}[Y/\mathcal{F}_t]\le\mathcal{E}^*[X-Y/\mathcal{F}_t]$, $\forall X,Y\in[L_0(\mathcal{F})]_*$.

Remark 5.3. From Example 4.4 we see that the $\mathcal{F}_t$-consistent nonlinear expectations introduced in this section can be fully nonlinear. Thus the framework of this section largely generalizes the notion of g-expectation.

§ 6. Backward SDE Under Nonlinear Expectations

We are in the framework of the previous section. For each $j=1,\cdots,m$, let $\mathcal{E}^{*,j}$ be a self-dominated nonlinear expectation defined on $[L_0(\mathcal{F})]_{*,j}$, the completion of $L_0(\mathcal{F})$ under $\|\cdot\|_{*,j}=\mathcal{E}^{*,j}[|\cdot|]$, and let $\mathcal{E}^j$ be $\mathcal{E}^{*,j}$-dominated nonlinear expectation on $[L_0(\mathcal{F})]_{*,j}$. We assume that these $\mathcal{E}^{*,j}$ and $\mathcal{E}^j$ are $\mathcal{F}_t$-consistent and satisfy all properties in Proposition 5.1. We also assume that $\mathcal{E}^{*,j}[\alpha X]=\alpha\mathcal{E}^{*,j}[X], \forall\alpha>0,\ X\in[L_0(\mathcal{F})]_{*,j}$. Under this assumption, $\|\cdot\|_{*,j}=\mathcal{E}^{*,j}[|\cdot|]$ becomes a norm. Thus $[L_0(\mathcal{F})]_{*,j}$ is a Banach spaces. We assume that, for each j, $[L_0(\mathcal{F})]_{*,j}$ is a separable space under $\|\cdot\|_{*,j}$. An R^m-valued random vector $Y=(Y^1,\cdots,Y^m)$ is said to be in $[L_0(\mathcal{F})]_*$ if $Y^j\in[L_0(\mathcal{F})]_{*,j}$ for each $j=1,\cdots,m$. The norm of $[L_0(\mathcal{F})]_*$ is defined by $\|\cdot\|_*:=\max\limits_{1\le j\le m}\|\cdot\|_{*,j}$. We also denote $\mathcal{E}[Y]:=(\mathcal{E}^1[Y_1],\cdots,\mathcal{E}^m[Y_m])$, $\mathcal{E}[Y/\mathcal{F}_t]:=(\mathcal{E}^1[Y_1/\mathcal{F}_t],\cdots,\mathcal{E}^m[Y_m/\mathcal{F}_t])$.

We are interested in R^m-valued stochastic processes $Y_t(\omega):t\in[0,T]\longrightarrow[L_0(\mathcal{F})]_*$. We first consider the following space of stochastic processes:

$$L_0(0,T;R^m)=\Big\{Y_t=\sum_{i=1}^N\xi_i\mathbf{1}_{A_i}(t),\ t\in[0,T];\ \forall N,\ \forall\xi_i\in[L_0(\mathcal{F})]_*,\ \forall\ \{A_i\}_{i=1}^N\Big\},$$

where $\{A_i\}_{i=1}^N$ is an arbitrary partition of $\mathcal{B}([0,T])$, i.e., $A_i \in \mathcal{B}([0,T])$, $A_i \cap A_j = \emptyset$, when $i \neq j$ with $\bigcup_{i=1}^N A_i = [0,T]$. A process is said to be $\mathcal{B}([0,T])$-strongly measurable if there exists a sequence $\{Y^i\}_{i=1}^\infty$ in $L_0(0,T;R^m)$ such that $\|Y_t^i - Y_t\|_*$ converges to zero for $m(dt)$-almost all $t \in [0,T]$. Here $m(dt) = dt$ denotes the Lebesgue measure.

For each $Y \in L_0(0,T;R^m)$ with $Y_t = \sum_{i=1}^N \xi_i \mathbf{1}_{A_i}(t)$, we define $\int_0^T Y_s ds := \sum_{i=1}^N \xi_i m(A_i)$. Since $\left\| \sum_{i=1}^N \xi_i m(A_i) \right\|_* \leq \sum_{i=1}^N \|\xi_i m(A_i)\|_* = \sum_{i=1}^N \|\xi_i t\|_* m(A_i)$, we then have

$$\Big\| \int_0^T Y_s ds \Big\|_* \leq \int_0^T \|Y_s\|_* ds, \qquad \forall\, Y \in L_0(0,T;R^m). \tag{6.1}$$

It is clear that $\int_0^T \|\cdot\|_* \, ds$ constitutes a norm on $L_0(0,T;R^m)$. The completion of $L_0(0,T;R^m)$ under this norm is a Banach space and is denoted by $L_*(0,T;R^m)$. For each $(Y_t)_{t\geq 0} \in L_*(0,T;R^m)$ we can define $\int_0^T Y_s ds := \lim_{i\to\infty} \int_0^T Y_s^i ds$, where $\{Y^i\}_{i=1}^\infty$ is a sequence in $L_0(0,T;R^m)$ which converges to Y under the norm $\int_0^T \|\cdot\|_* \, ds$. By (6.1) this integral is uniquely defined. Furthermore

$$\Big\| \int_0^T Y_s ds \Big\|_* \leq \int_0^T \|Y_s\|_* \, ds, \qquad \forall\, Y \in L_*(0,T;R^m). \tag{6.2}$$

$\int_0^T Y_s ds$ is called the Bochner's integral of $(Y_t)_{t\geq 0}$ (see e.g. [44]). It is easy to see that, for each $Y \in L_*(0,T;R^m)$, the process $\int_0^t Y_s ds := \int_0^T Y_s 1_{[0,t]}(s) ds$, $t \in [0,T]$ is still in $L_*(0,T;R^m)$. We also define a space of adapted processes

$$M_*(0,T;R^m) := \{Y \in L_*(0,T;R^m);\ \ Y_t \text{ is } \mathcal{F}_t\text{-measurable for each } t \in [0,T]\}.$$

We assume

$$\text{For each } X \in [L_0(\mathcal{F})]_*, \qquad \mathcal{E}[X/\mathcal{F}_t]_{t\in[0,T]} \in M_*(0,T;R^m), \tag{6.3}$$

$$\lim_{\delta \searrow 0} \|\mathcal{E}[X/\mathcal{F}_{t+\delta}] - \mathcal{E}[X/\mathcal{F}_t]\|_* = 0. \tag{6.4}$$

We also assume that

$$\mathcal{E}[X+\eta/\mathcal{F}_t] = \mathcal{E}[X/\mathcal{F}_t]+\eta, \qquad \forall\, X \in [L_0(\mathcal{F})]_*,\ \eta \in [L_0(\mathcal{F}_t)]_*,\ \forall\, t \geq 0. \tag{6.5}$$

Let a function $f : (\omega,t,y) \in \Omega \times [0,T] \times R^m \longmapsto R^m$ be given such that

$$\begin{cases} f(\,\cdot\,,y) \in M_*(0,T;R^m), & \forall\, y \in R^m; \\ |f(t,y_1) - f(t,y_2)| \leq C_1|y_1 - y_2|, & \forall\, y_1,\ y_2 \in R, \end{cases} \tag{6.6}$$

where C_1 is a fixed constant. For a given terminal data $X \in [L_0(\mathcal{F})]_*$, we consider the following type of Backward Stochastic Differential Equation (BSDE):

$$Y_t = \mathcal{E}\Big[X + \int_t^T f(s,Y_s) ds | \mathcal{F}_t\Big]. \tag{6.7}$$

Theorem 6.1. *We assume* (6.3)–(6.6). *Then there exists a unique process* $Y \in M_*(0,T;R^m)$ *solution of* (6.7). *Moreover,* Y_t *is continuous in* t *in the following sense:*

$$\lim_{\delta \searrow 0} \|Y_{t+\delta} - Y_t\|_* = 0, \qquad \forall\, t \in [0,T).$$

Proof. We first consider a special situation of (6.7) when $f = \phi \in M_*(0,T;R^m)$. It is clear that for any $0 \le a < b \le T$, $X + \int_a^b \phi_s ds \in [L_0(\mathcal{F})]_*$. Moreover we have

$$Y_t = \mathcal{E}\Big[X + \int_t^T \phi_s ds|\mathcal{F}_t\Big] = \mathcal{E}\Big[X + \int_0^T \phi_s ds|\mathcal{F}_t\Big] + \int_0^t \phi_s ds.$$

Thus $Y \in M_*(0,T;R^m)$. The first term of the right hand side is continuous in time, so is the second term since

$$\Big\| \int_0^{t+\delta} \phi_s ds - \int_0^t \phi_s ds \Big\|_* \le \int_t^{t+\delta} \|\phi_s\|_* \, ds \searrow 0, \qquad \text{as } \delta \searrow 0.$$

We now consider the general situation. By the above discussion, we define a mapping $\Lambda_t(y_\cdot) : L^{1*}_{\mathcal{F}}(0,T;R^m) \longmapsto L^{1*}_{\mathcal{F}}(0,T;R^m)$ by $\Lambda_t(y_\cdot) = \mathcal{E}\big[X + \int_t^T f(s,y(s))ds/\mathcal{F}_t\big]$. For each t, we have

$$\begin{aligned}\|\Lambda_t(y_\cdot) - \Lambda_t(y'_\cdot)\|_* &\le \Big\| \int_t^T [f(s,y_s) - f(s,y'_s)]ds \Big\|_* \\ &\le \int_t^T \|f(s,y_s) - f(s,y'_s)\|_* \, ds \le C_1 \int_t^T \|y_s - y'_s\|_* \, ds.\end{aligned}$$

We observe that, for any finite number β, the following two norms are equivalent in $M_*(0,T;R^m)$ $\int_0^T \|\phi_s\|_* \, dt \sim \int_0^T \|\phi_s\|_* \, e^{\beta t} dt$. Thus we multiply $e^{2C_1 t}$ on both sides of the above inequality and then integrate them on $[0,T]$. It follows that

$$\begin{aligned}&\int_0^T \|\Lambda_t(y_\cdot) - \Lambda_t(y'_\cdot)\|_* e^{2C_1 t} dt \le C_1 \int_0^T e^{2C_1 t} \int_t^T \|y_s - y'_s\|_* \, ds \\ &= C_1 \int_0^T \int_0^s e^{2C_1 t} dt \, \|y_s - y'_s\|_* \, ds = (2C_1)^{-1} C_1 \int_0^T (e^{2C_1 s} - 1) \, \|y_s - y'_s\|_* \, ds.\end{aligned}$$

We then have

$$\int_0^T \|\Lambda_t(y_\cdot) - \Lambda_t(y'_\cdot)\|_* \, e^{2C_1 t} dt \le \frac{1}{2} \int_0^T \|y_t - y'_t\|_* \, e^{2C_1 t} dt.$$

Namely, Λ is a contract mapping on $M_*(0,T;R^m)$. It follows that this mapping has a unique fixed point Y: $Y_t = \mathcal{E}[X + \int_t^T f(s,Y_s)ds|\mathcal{F}_t]$.

We now consider the difference of the solution of BSDE (6.7) and the one of the following BSDE:

$$Y'_t = \mathcal{E}\Big[X' + \int_t^T [f(s,Y'_s) + \phi_s]ds|\mathcal{F}_t\Big]. \tag{6.8}$$

where $X' \in [L_0(\mathcal{F})]_*$ and $\phi \in M_*(0,T;R^m)$ are given. The following continuous dependence theorem estimates the distance between the solutions of (6.7) and (6.8).

Proposition 6.1. *We have*

$$\int_0^T \|Y_t - Y_t'\|_* e^{2C_1 t} dt \le C_0 \|X - X'\|_* + C_0 \int_0^T e^{2C_1 t} \|\phi_t\|_* dt, \tag{6.9}$$

where the constant C_0 depends only on C_1, the Lipschitz constant of $f(t, y)$ with respect to y and T.

Proof. We have

$$\begin{aligned}
\|Y_t - Y_t'\|_* &\le \left\| X - X' + \int_t^T [f(s, Y_s) - f(s, Y_s')] ds \right\|_* \\
&\le \|X - X'\|_* + \int_t^T \|f(s, Y_s) - f(s, Y_s')\|_* ds \\
&\le \|X - X'\|_* + \int_t^T [C_1 \|Y_s - Y_s'\|_* + \|\phi_s\|_*] ds.
\end{aligned}$$

As in the previous proof, we multiply $e^{2C_1 t}$ on both sides of the above inequality and then integrate them on $[0, T]$:

$$\begin{aligned}
&\int_0^T \|Y_t - Y_t'\|_* e^{2C_1 t} dt \\
&\le \|X - X'\|_* \int_0^T e^{2C_1 t} dt + \int_0^T e^{2C_1 t} \int_t^T [C_1 \|Y_s - Y_s'\|_* + \|\phi_s\|_*] ds dt \\
&\le \|X - X'\|_* (e^{2C_1 T} - 1) + \int_0^T e^{2C_1 t} 2^{-1} C_1^{-1} [C_1 \|Y_t - Y_t'\|_* + \|\phi_t\|_*] dt.
\end{aligned}$$

Thus we have (6.8).

Remark 6.1. Unlike the classical theorem of BSDE, the above result of existence and uniqueness does not require the conditions for the $\mathcal{F}_T$-measurability of X. If X is assumed to be $\mathcal{F}_T$-measurable, then we have $Y_T = X$.

§ 7. Nonlinear Expectations, Nonlinear Expected Utilities and Risk Measures

To measure the preference of an agent $\mathcal{A}$, a fundamental tool in economics is the utility functional of $\mathcal{A}$. Under this framework, $\mathcal{A}$ prefers a random choice X than Y is formulated by $\mathcal{U}(X) \ge \mathcal{U}(Y)$. We shall work in $L_*^\infty(\mathcal{F})$ space introduced in Subsection 2.4. A utility functional of the agent $\mathcal{A}$ is a real functional $\mathcal{U}(\cdot) : L_*^\infty(\mathcal{F}) \longrightarrow R$. This functional satisfies the following obvious axioms:

(u1) Monotonicity: if $X \ge Y$ in $L_*^\infty(\mathcal{F})$, then $\mathcal{U}(X) \ge \mathcal{U}(Y)$, and if $X \ge Y$ and $\|X - Y\|_* > 0$, then $\mathcal{U}(X) > \mathcal{U}(Y)$;

(u2) Continuity: if $\|X_i - X\|_{*\infty} \to 0$, then $\mathcal{U}(X_i) \to \mathcal{U}(X)$.

We observe that if we assume moreover that $\mathcal{U}$ is constant-preserving, then it is a nonlinear expectation defined on $L_*^\infty(\mathcal{F})$. In general, a utility is not constant-preserving. But we have the following nonlinear expected utility theorem which generalized the well-known von Neuman-Morgenstern's axiom on expected utility.

Proposition 7.1. *Let $\mathcal{E}[\,\cdot\,]$ be a strictly monotonic expectation satisfying* (E1), (E2) *in Definition* 2.1. *We assume that $\mathcal{E}$ is continuous in $L^\infty_*(\mathcal{F})$ and let u be a continuous and strictly increasing function $u(\,\cdot\,): R \longrightarrow R$. Then the functional $\mathcal{U}$ defined by*

$$\mathcal{U}(X) := \mathcal{E}[u(X)] \tag{7.1}$$

is a utility functional satisfying (u1) *and* (u2).

Conversely, for each given utility $\mathcal{U}(\,\cdot\,)$ satisfying (u1) *and* (u2), *there exist a strict monotonic nonlinear expectation $\mathcal{E}[\,\cdot\,]$ and a continuous and strictly increasing function $u(\,\cdot\,): R \longrightarrow R$ such that* (7.1) *holds.*

Proof. The first claim is easy. For the second one, we set

$$u(x) := \mathcal{U}(x), \qquad \forall\, x \in R. \tag{7.2}$$

By (u1) and (u2) it is clear that $u(\,\cdot\,): R \longrightarrow R$ is continuous and strictly increasing, so is its inverse u^{-1}. It follows that for each $X \in L^\infty_*(\mathcal{F})$, $u^{-1}(X)$ is also a bounded element in $L^\infty_*(\mathcal{F})$. We then can set

$$\mathcal{E}[X] := \mathcal{U}(u^{-1}(X)). \tag{7.3}$$

Obviously (7.1) holds for this functional. It remains to prove that $\mathcal{E}$ is a nonlinear expectation defined on $L^\infty_*(\mathcal{F})$. It is clear that this functional $\mathcal{E}[\,\cdot\,]$ also satisfies the same properties (u1) and (u2) for that of $\mathcal{U}(\,\cdot\,)$. Moreover, according to the definition of U, $\mathcal{E}$ is constant-preserving: $\mathcal{E}[c] = \mathcal{U}(u^{-1}(c)) = u(u^{-1}(c)) = c$. Thus $\mathcal{E}[\,\cdot\,]$ is a nonlinear expectation in the sense of Definition 2.1.

Remark 7.1. In [42] von Neumann and Morgenstern have introduced the well-known expected utility and the related axiomatic system. It is widely used in economics, e.g., financial economics. They claimed that $\mathcal{U}$ can be characterized by $\mathcal{U}(X) = E[U(X)]$. Here $U: R \to R$ is a continuous and strictly increasing function. E is the (linear) expectation in some probability space $(\Omega, \mathcal{F}, P)$. It is clear that an expected utility satisfies (u1) and (u2).

But some real world utilities can not be represented by this expected utility. A well-known counterexample is the well-known Allais paradox (see [1]). If an agent $\mathcal{A}$ equipped with an expected utility has the following four random choices ξ_a, ξ_b, ξ_c and ξ_d with the following distributions:

$P(\xi_a = 100m) = 1;$
$P(\xi_b = 500m) = 0.10,$
$P(\xi_b = 100m) = 0.89, \qquad P(\xi_b = 0m) = 0.01;$
$P(\xi_c = 100m) = 0.11, \qquad P(\xi_c = 0m) = 0.89;$
$P(\xi_d = 500m) = 0.10, \qquad P(\xi_d = 0m) = 0.90;$

then it is easy to check that, for any function U, we always have $\mathcal{U}(\xi_a) - \mathcal{U}(\xi_b) = \mathcal{U}(\xi_c) - \mathcal{U}(\xi_d)$. But most people tested in experiments prefer to choose ξ_a than ξ_b, and to choose ξ_d than ξ_c. This contradicts the above equality. The notion of nonlinear expected utility of form (7.1) can overcome this paradox.

Remark 7.2. (Dynamical Risk Measures in Finance) In quantitative risk management of finance, risk measure is a central issue. Axiomatic definitions of measures of risk, called coherent risk, were introduced in [2]. A more general type, called convex risk measures, was then introduced in [23] (see also [24]). Recently Rosazza Gianin [41] considered a type of dynamic risk measures induced from g-expectations, defined in Example 2.11. This g-expectations provide naturally an $\mathcal{F}_t$-consistent measure of risk (see [3, 5, 38–40]).

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