

DYNAMIC BIFURCATION OF NONLINEAR EVOLUTION EQUATIONS***

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Abstract

The authors introduce a notion of dynamic bifurcation for nonlinear evolution equations, which can be called attractor bifurcation. It is proved that as the control parameter crosses certain critical value, the system bifurcates from a trivial steady state solution to an attractor with dimension between m and $m + 1$, where $m + 1$ is the number of eigenvalues crossing the imaginary axis. The attractor bifurcation theory presented in this article generalizes the existing steady state bifurcations and the Hopf bifurcations. It provides a unified point of view on dynamic bifurcation and can be applied to many problems in physics and mechanics.

Keywords Attractor bifurcation, Steady state bifurcation, Dynamic bifurcation, Hopf bifurcation, Nonlinear evolution equation

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§ 1. Introduction

A key problem in the study of problems in mathematical physics and mechanics is to understand and predict patterns and their transitions/evolutions. In fluid mechanics, for instance, it is important to study the periodic, quasi-periodic, aperiodic, and fully turbulent characteristics of flows; bifurcation theory enables one to determine how qualitatively different flow regimes appear and disappear as control parameters vary; it provides us, therefore, an important method to explore the theoretical limits of predicting these flow regimes.

As we know, the steady state bifurcation to multiple equilibria and to periodic solutions (the Hopf bifurcation) are two typical bifurcations for nonlinear evolution equations (see among others, [1, 9]). In this article, we introduce a new notion of dynamic bifurcation, which we call attractor bifurcation. We show that as the control parameter crosses certain critical value when there are $m + 1$ ($m \geq 0$) eigenvalues crossing the imaginary axis, the system bifurcates from a trivial steady state solution to an attractor with dimension between m and $m + 1$, provided the critical state is asymptotically stable. This new bifurcation concept generalizes the aforementioned known bifurcation concepts.

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There are a few important features of the attractor bifurcation. First, the bifurcation attractor does not include the trivial steady state, and is stable, hence it is physically important.

Second, the attractor contains a collection of solutions of the evolution equation, including possibly steady states, periodic orbits, as well as homoclinic and heteroclinic orbits.

Third, it provides a unified point of view on dynamic bifurcation and can be applied to many problems in physics and mechanics. One of the main advantages of the main theorems, Theorems 3.1 and 4.3, is that the verification of the eigenvalue conditions is much easier than classical bifurcation theorems in many application problems, including applications to Bénard problem in fluid mechanics, Ginzburg-Landau equations of superconductivity, and the Kuramoto-Sivashinsky equation for combustion theory, which will be reported in subsequent papers.

Fourth, from the application point of view, the Krasnoselskii-Rabinowitz theorem requires the number of eigenvalues $m + 1$ crossing the imaginary axis being an odd integer, and the Hopf bifurcation is for the case where $m + 1 = 2$. However, the new attractor bifurcation theorem obtained in this article can be applied to cases for all $m \geq 0$. In addition, the bifurcated attractor, as mentioned earlier, is stable, which is another subtle issue for other known bifurcation theorems. Of course, the price to pay here is the verification of the asymptotic stability of the critical state, in addition to the analysis needed for the eigenvalue problems in the linearized problem. On the other hand, as we shall see in subsequent articles, both the asymptotic stability of the critical state and the attractor bifurcation can be derived for many important physical problems including the Bénard problem and the Taylor problem in fluid mechanics, Ginzburg-Landau equations of superconductivity, and the Kuramoto-Sivashinsky equation (see [7, 8]).

Fifth, the main idea of the bifurcation theory presented in this article comes from an orbit stability lemma, Lemma 3.1, first introduced and used by the authors in [6], which plays also important roles in authors' recent work on boundary layer separations. The main results are achieved by using the center manifold theorem for both finite and infinite dimensional systems.

This article is organized as follows. Section 2 provides some preliminaries on invariant sets, attractors, center manifolds and stable manifolds. Sections 3 and 4 provide the bifurcation theory in both finite and infinite dimensional cases.

§ 2 . Preliminaries

2.1. Invariant sets and attractors

Let H and H_1 be two Hilbert spaces, and $H_1 \hookrightarrow H$ be dense inclusion. A linear mapping $L : H_1 \rightarrow H$ is called a completely continuous field if $L = A + B$ where $A : H_1 \rightarrow H$ is a linear homeomorphism and $B : H_1 \rightarrow H$ is a linear compact operator.

Consider the following abstract nonlinear evolution equation

$$\begin{cases} \frac{du}{dt} = L_\lambda u + G(u, \lambda), \\ u(0) = \varphi, \end{cases} \quad (2.1)$$

where $\lambda \in \mathbb{R}$ is a parameter, and

$L_\lambda : H_1 \rightarrow H$ is a linear completely continuous field,

$G(\cdot, \lambda) : H_1 \rightarrow H$ is a continuous operator,

$G(x, \lambda) = o(\|x\|_{H_1}), \quad \forall \lambda \in \mathbb{R}.$

Let $\{S(t)\}_{t \geq 0}$ be an operator semigroup generated by Equation (2.1), which enjoys the following properties:

(1) For any $t \geq 0$, $S(t) : H_1 \rightarrow H$ is a continuous linear operator;

(2) $S(0) = I$, the identity on H_1 ;

(3) $S(t+s) = S(t) \cdot S(s)$, for any $t, s \geq 0$.

Then the solution of (2.1) can be expressed as

$$u(t, \varphi) = S(t)\varphi, \quad \forall t \geq 0.$$

Definition 2.1. A set $\Sigma \subset H_1$ is called an invariant set of (2.1) if $S(t)\Sigma = \Sigma$ ($\forall t \geq 0$). An invariant set $\Sigma \subset H_1$ of (2.1) is called an attractor if Σ is compact and there exists a neighborhood $U \subset H_1$ of Σ such that for any $\varphi \in U$, we have

$$\text{dist}(u(t, \varphi), \Sigma) \longrightarrow 0 \text{ in } H\text{-norm as } t \rightarrow \infty. \quad (2.2)$$

The largest open set U satisfying (2.2) is called the basin of attraction of Σ .

For a set $\Sigma \subset H_1$, we define the ω -limit set of Σ by

$$\omega(\Sigma) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)\Sigma},$$

where the closures are taken in H . Likewise, when it exists, the α -limit set of $\Sigma \subset H_1$ is defined by

$$\alpha(\Sigma) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(-t)\Sigma}.$$

The following lemmas can be found in [2] or [11].

Lemma 2.1. Suppose that for some subset $\Sigma \subset H_1$, $\Sigma \neq \emptyset$, and for some $t_0 > 0$, the set $\bigcup_{t \geq t_0} S(t)\Sigma$ is relatively compact in H . Then $\omega(\Sigma)$ is nonempty, compact, and invariant. Similarly, if the sets $S(-t)\Sigma$ ($t \rightarrow 0$) are nonempty and for some $t_0 > 0$, $\bigcup_{t \geq t_0} S(-t)\Sigma$ is relatively compact, then $\alpha(\Sigma)$ is nonempty, compact, and invariant.

Definition 2.2. Let $\Sigma \subset H_1$ be a subset and U be an open set containing Σ . U is called an absorbing set of Σ if the orbit of any bounded set of U enters into Σ after a certain time (which may depend on the set). Namely, for any bounded $B_0 \subset U$, there exists a $t_0 = t_0(B_0)$ such that for all $t \geq t_0(B_0)$, $S(t)B_0 \subset \Sigma$.

Lemma 2.2. Suppose that the semigroup $\{S(t)\}_{t \geq 0}$ is uniformly compact for t large, i.e., for any bounded set B there exists t_0 such that $\bigcup_{t \geq t_0} S(t)B$ is relatively compact in H .

We also assume that there exist an open set U and a bounded set B of U such that B is absorbing in U . Then the ω -limit set of B , $\Sigma = \omega(B)$ is an attractor which attracts the bounded sets of U , and it is the maximum attractor in U . Furthermore, if U is connected, then Σ is connected, too.

2.2. Center manifolds and stable manifolds in \mathbb{R}^n

Consider a system of ordinary differential equations

$$\begin{cases} \frac{dx}{dt} = Ax + G_1(x, y, \lambda), \\ \frac{dy}{dt} = By + G_2(x, y, \lambda), \end{cases} \quad (2.3)$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^{n-m}$ ($0 < m \leq n$), A and B are the $m \times m$ and $(n-m) \times (n-m)$ matrices respectively, $G_i(x, y, \lambda)$ ($i = 1, 2$) are continuous on λ , and C^r ($r \geq 1$) on $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. Moreover

$$G_i(x, y, \lambda) = o(|x|, |y|), \quad \forall \lambda \in \mathbb{R} \quad (i = 1, 2). \quad (2.4)$$

The following is the center manifold theorem (see among others [1]).

Theorem 2.1. *Suppose that all eigenvalues of A have non-negative (resp. non-positive) real parts, and all eigenvalues of B have negative (resp. positive) real parts. Then for the system (2.3) with the condition (2.4), there exists a C^r function*

$$h(\cdot, \lambda) : D \longrightarrow \mathbb{R}^{n-m}, \quad D \subset \mathbb{R}^m \text{ a neighborhood of } x = 0,$$

such that $h(x, \lambda)$ is continuous on λ , and

- (i) $h(0, \lambda) = 0$, $h'_x(0, \lambda) = 0$;
- (ii) the set

$$M_\lambda = \{(x, y) \mid x \in D \subset \mathbb{R}^m, y = h(x, \lambda)\},$$

called the center manifold, is a local invariant manifold of (2.3);

(iii) if M_λ is positive invariant (resp. negative invariant), namely $z(t, \varphi) \in M_\lambda$ (resp. $z(-t, \varphi) \in M_\lambda$), $\forall t \geq 0$, then M_λ is an attracting set of (2.3) (resp. a repelling set), i.e., there is a neighborhood $U \subset \mathbb{R}^n$ of M_λ such that for $\varphi \in U$ we have

$$\lim_{t \rightarrow \infty} \text{dist}(z(t, \varphi), M_\lambda) = 0 \quad (\text{resp. } \lim_{t \rightarrow \infty} \text{dist}(z(-t, \varphi), M_\lambda) = 0)$$

where $z(t, \varphi) = (x(t, \varphi), y(t, \varphi))$ is the solution of (2.3) with the initial value $z(0, \varphi) = \varphi$.

Property (i) means that the center manifold $M_\lambda \subset \mathbb{R}^n$ is tangent to the eigenspace \mathbb{R}^m of A at $z = (x, y) = 0$.

Although, as we know, the local center manifold M_λ may not be unique, the following theorem makes it applicable (see [1]).

Theorem 2.2. *There is a neighborhood $U \subset \mathbb{R}^n$ of $z = 0$ such that every invariant set of (2.3) in U belongs to the intersection of all local center manifolds in U .*

In the following, we introduce the stable manifold theorem (see [4]), which will be used in the attractor bifurcation theorems.

Theorem 2.3. *Let all eigenvalues of A have positive real parts, and all eigenvalues of B have negative real parts. Then there exist two unique manifolds W^u and W^s , called the unstable manifold and stable manifold of (2.3) respectively, which are characterized by*

$$W^u = \left\{ z \in \mathbb{R}^n \mid \lim_{t \rightarrow \infty} S(-t)z = 0 \right\},$$

$$W^s = \left\{ z \in \mathbb{R}^n \mid \lim_{t \rightarrow \infty} S(t)z = 0 \right\},$$

where $S(t)$ is the semigroup generated by (2.3). Moreover, W^u and W^s are tangent to the eigenspaces of A and B respectively at $z = 0$:

$$T_{z=0}W^u = \mathbb{R}^m, \quad T_{z=0}W^s = \mathbb{R}^{n-m},$$

therefore, the stable manifold W^s and the unstable manifold W^u are transversal at $z = 0$.

§ 3. Finite Dimensional Vector Fields

3.1. Attractor bifurcation

We consider the dynamic bifurcation problem for the finite dimensional systems given by

$$\frac{dx}{dt} = A_\lambda x + G(x, \lambda), \quad \lambda \in \mathbb{R}, x \in \mathbb{R}^n \quad (n \geq 2), \quad (3.1)$$

where $G: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is C^r ($r \geq 1$) on $x \in \mathbb{R}^n$, and continuous on $\lambda \in \mathbb{R}$,

$$G(x, \lambda) = o(|x|), \quad \forall \lambda \in \mathbb{R}, \quad (3.2)$$

and

$$A_\lambda = \begin{pmatrix} a_{11}(\lambda) & \cdots & a_{1n}(\lambda) \\ \vdots & & \vdots \\ a_{n1}(\lambda) & \cdots & a_{nn}(\lambda) \end{pmatrix} \quad (3.3)$$

is an $n \times n$ matrix, $a_{ij}(\lambda)$ are continuous functions of λ .

Let all eigenvalues (counting the multiplicities) of (3.3) be denoted by

$$\beta_1(\lambda), \dots, \beta_n(\lambda). \quad (3.4)$$

We know that the eigenvalues $\beta_i(\lambda)$ ($1 \leq i \leq n$) are continuous on λ .

Definition 3.1. (1) *We say that the system (3.1) bifurcates from $(x, \lambda) = (0, \lambda_0)$ an invariant set Ω_λ with $0 \notin \Omega_\lambda$, if there exists a sequence of invariant sets $\{\Omega_{\lambda_n}\}$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda_0$, $\lim_{n \rightarrow \infty} d(\Omega_{\lambda_n}, 0) = \lim_{n \rightarrow \infty} \max_{x \in \Omega_{\lambda_n}} |x| = 0$.*

(2) *If the invariant sets Ω_λ are attractors of (3.1), then we call attractor bifurcation.*

(3) *If the invariant sets Ω_λ are attractors of (3.1), which are homotopic to an m -dimensional sphere, then we say that the system (3.1) has an S^m -attractor bifurcation at $(0, \lambda_0)$.*

We remark here that the classical definition of bifurcation from the trivial solution $u = 0$ of an evolution equation (3.1) at λ_0 is as follows. The evolution equation (3.1) has a

bifurcation from the trivial solution $u = 0$ at $\lambda = \lambda_0$ if there are a sequence $\lambda_n \neq \lambda_0$ with $\lambda_n \rightarrow \lambda_0$ and a sequence of solutions Σ_n of (3.1) such that $0 \notin \Sigma_n$ and

$$d(\Sigma_n, 0) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

In the classical steady state bifurcation, Σ_n contains steady state solutions, and in the Hopf bifurcation, Σ_n contains a periodic solution. It is easy to see that our definition of bifurcation, Definition 3.1, fits this classical definition, and the corresponding Σ_n is an invariant set of (3.1). It is this new way of looking at Σ_n that leads to wide applications of the theory, which will appear in subsequent articles.

Assume

$$\operatorname{Re} \beta_i(\lambda) \begin{cases} < 0, & \lambda < \lambda_0, \\ = 0, & \lambda = \lambda_0, \\ > 0, & \lambda > \lambda_0, \end{cases} \quad \text{if } 1 \leq i \leq m+1, \quad (3.5)$$

$$\operatorname{Re} \beta_j(\lambda_0) < 0, \quad \text{if } m+2 \leq j \leq n, \quad (3.6)$$

where $0 \leq m \leq n-1$.

We denote the eigenspace of A_λ at λ_0 by

$$E_0 = \{x \in \mathbb{R}^n \mid A_{\lambda_0}^k x = 0, \ k = 1, 2, \dots\}.$$

It is known that $\dim E_0 = m+1$.

The main results in this section are the following S^m -attractor bifurcation theorems for the finite dimensional system (3.1).

Theorem 3.1. (Attractor Bifurcation Theorem) *Assume (3.2)–(3.6). If $x = 0$ is locally asymptotically stable for (3.1) at $\lambda = \lambda_0$, then the following assertions hold true.*

- (1) *The system (3.1) bifurcates from $(0, \lambda_0)$ an attractor Ω_λ with $m \leq \dim \Omega_\lambda \leq m+1$, which is connected when $m \geq 1$;*
- (2) *Ω_λ is a limit of a sequence of $(m+1)$ -dimensional annulus M_k with $M_{k+1} \subset M_k$, i.e., $\Omega_\lambda = \bigcap_{k=1}^{\infty} M_k$;*
- (3) *If Ω_λ is a finite simplicial complex, then Ω_λ is a deformation retract of $(m+1)$ -annulus; hence Ω_λ has the homotopy type of S^m ;*
- (4) *For any $x_\lambda \in \Omega_\lambda$, x_λ can be expressed as*

$$x_\lambda = z_\lambda + o(|z_\lambda|), \quad z_\lambda \in E_0;$$

- (5) *If the singular points of (3.1) in Ω_λ are finite, then we have the following index formula*

$$\sum_{x \in \Omega_\lambda} \operatorname{ind}[-(A_\lambda + G), x] = \begin{cases} 2, & m = \text{even}, \\ 0, & m = \text{odd}. \end{cases}$$

Theorem 3.2. *Assume (3.2)–(3.6). If $x = 0$ is globally asymptotically stable for (3.1) at $\lambda = \lambda_0$, then for any bounded open set $U \subset \mathbb{R}^n$ with $0 \in U$, there is an $\varepsilon > 0$ such that*

as $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, the attractor Ω_λ of (3.1) bifurcated from $(0, \lambda_0)$ attracts U/Γ , i.e., for any $x_0 \in U/\Gamma$ the solution $x(t, x_0)$ with initial value $x(0, x_0) = x_0$ satisfies

$$\lim_{t \rightarrow \infty} \text{dist}(x(t, x_0), \Omega_\lambda) = 0,$$

where $\Gamma \subset \mathbb{R}^2$ is the stable manifold of $x = 0$ with $\dim \Gamma = n - m - 1$. Furthermore, if (3.1) has global attractor for any λ , then we can take $U = \mathbb{R}^n$.

Remark 3.1. Theorem 3.1 can be regarded as a united version of the two types of static and dynamic bifurcations. When $m = \text{even}$, the S^m -attractor Ω_λ must contain singular points of the vector field $A_\lambda + G$, which is an assertion of the Krasnoselskii theorem. When $m = 1$ and $\beta_1(\lambda_0) = \bar{\beta}_2(\lambda_0) = i$, this theorem gives the classical Hopf bifurcation.

Remark 3.2. In general, the attractors Ω_λ in Theorem 3.2 are not global. For example, the system

$$\frac{dx}{dt} = \lambda x - x^3(1 - e^{-1/x^4}), \quad x \in \mathbb{R}^1,$$

satisfies the conditions of Theorem 3.2 with $\lambda_0 = 0$. But, for any $\lambda > 0$ sufficiently small, Ω_λ contains two singular points $x_{1,2} \sim \pm \lambda^{1/2}$ of the function $\lambda x - x^3(1 - e^{-1/x^4})$. Then it is easy to see that Ω_λ is certainly not a global attractor since there are two more singular points $\bar{x}_{1,2} \sim \pm \lambda^{-1/2}$, bifurcated from infinity, which are not in Ω_λ .

3.2. Stability of attractors

In order to prove the attractor bifurcation theorems, we first introduce the stability theorem of attractors, which is crucial in the proof of Theorems 3.1 and 3.2. To this end, we start with a technical lemma on stability of extended orbits for vector fields.

Let $v \in C^r(U, \mathbb{R}^n)$ be a vector field where $U \subset \mathbb{R}^n$ is an open set. A curve $\gamma \subset U$ is called an extended orbit of v , if γ is a union of curves $\gamma = \bigcup_{i=1} \gamma_i$ such that either γ_i is an orbit of v , or γ_i consists of singular points of v , and if γ_i and γ_{i+1} are orbits of v , then the ω -limit set of γ_i is the α -limit set of γ_{i+1} ,

$$\omega(x) = \alpha(y), \quad \forall x \in \gamma_i, y \in \gamma_{i+1}.$$

Namely, endpoints of γ_i are singular points of v , and the starting endpoint of γ_{i+1} is the finishing endpoint of γ_i (see Fig. 3.1).

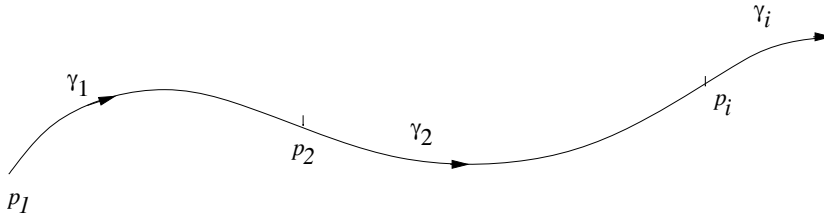


Fig. 3.1

Then we have the following stability lemma of extended orbits. The result of this lemma has been proved and used in Step 2 of the proof of Lemma 4.5 in [6]. Here we only state the result as a lemma.

Lemma 3.1. (Stability of Extended Orbits) (cf. [6]) *Let $v_k \in C^r(U, \mathbb{R}^n)$ be a sequence of vector fields with $\lim_{k \rightarrow \infty} v_k = v_0 \in C^r(U, \mathbb{R}^n)$. Suppose that $\gamma_k \subset U$ is an extended orbit of v_k with finite length uniformly with respect to k , and the starting points p_1^k of γ_k converge to p_1 , then the extended orbits γ_k of v_k converge, up to taking a subsequence, to an extended orbit γ of v_0 with starting point p_1 .*

Remark 3.3. The stability lemma of extended orbits is useful for orbit analysis of vector fields, which is a basic tool to solve some problems in fluid dynamics.

The following is the stability theorem of attractors.

Theorem 3.3. *Let $v_n \in C^r(U, \mathbb{R}^n)$ ($r \geq 1$) be a sequence of vector fields such that $\lim_{n \rightarrow \infty} v_n = v_0 \in C^r(U, \mathbb{R}^n)$. Let $\Sigma_0 \subset U$ be an attractor of v_0 and $D \subset U$ be the basin of attraction for Σ_0 . Then the following assertions hold true.*

(1) *For each n sufficiently large, v_n has an attractor $\Sigma_n \subset D$, and*

$$\lim_{n \rightarrow \infty} d(\Sigma_n, \Sigma_0) = \lim_{n \rightarrow \infty} \sup_{x \in \Sigma_n} \text{dist}(x, \Sigma_0) = 0.$$

(2) *If the basin of attraction $D \subset U$ of Σ_0 is unbounded, then for any bounded open set $O \subset D$ there is an N sufficiently large such that O is in the basins of attraction of Σ_n for all $n > N$.*

Proof. Let $D_0 \subset\subset D$ be a bounded open set with $\Sigma_0 \subset D_0$. Then the ω -limit set for v_0 of D_0 is the attractor Σ_0 , i.e.,

$$\Sigma_0 = \omega(D_0). \quad (3.7)$$

To complete the proof, by Lemma 2.2 it suffices to prove that

$$\begin{cases} \lim_{n \rightarrow \infty} d(\omega_n(D_0), \Sigma_0) = \lim_{n \rightarrow \infty} \sup_{x \in \omega_n(D_0)} \text{dist}(x, \Sigma_0) = 0, \\ \omega_n(D_0) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S_n(t)D_0}, \end{cases} \quad (3.8)$$

where $S_n(t)$ is the operator semigroup generated by v_n .

By Lemma 2.1, $\omega_n(D_0)$ is an invariant set of v_n . If (3.8) is false, then at least one of the following two cases must occur:

(a) There exist a number $\delta > 0$ and points $p_n \in \omega_n(D_0)$ such that $p_n \rightarrow p_0 \in \Sigma_0$ ($n \rightarrow \infty$) and the extended orbits $\Gamma_n \subset \omega_n(D_0)$ of v_n starting at p_n with bounded length satisfy that

$$d(\Gamma_n, \Sigma_0) = \sup_{x \in \Gamma_n} \text{dist}(x, \Sigma_0) > \delta. \quad (3.9)$$

(b) There exist extended orbits $\Gamma_n \subset \omega_n(D_0)$ and a number $\delta > 0$ such that

$$\text{dist}(\Gamma_n, \Sigma_0) > \delta, \quad \forall n \in N. \quad (3.10)$$

For the case (a), by Lemma 3.1 the extended orbits Γ_n converge, up to taking a subsequence, to an extended orbit Γ_0 of v_0 with the starting point $p_0 \in \Sigma_0$:

$$\Gamma_n \longrightarrow \Gamma_0 \quad (n \rightarrow \infty). \quad (3.11)$$

From (3.9) and (3.11) it follows that

$$d(\Gamma_0, \Sigma_0) = \sup_{x \in \Gamma_0} \text{dist}(x, \Sigma_0) \geq \delta > 0. \quad (3.12)$$

On the other hand, Σ_0 is an invariant set of v_0 ; therefore $\Gamma_0 \subset \Sigma_0$, a contradiction to (3.12).

If only the case (b) occurs, then by the definition of ω -limit set, we infer from (3.7) and (3.10) that there exist a number $\delta_1 > 0$ and points $p_n \in D_0$ with $p_n \rightarrow p_0 \in D_0$ ($n \rightarrow \infty$) such that the orbits $x_n(t)$ of v_n with the initial value $x_n(0) = p_n$ satisfy

$$\text{dist}(x_n(t), \Sigma_0) > \delta_1, \quad \forall 0 \leq t < \infty. \quad (3.13)$$

By Lemma 3.1, from (3.13) we derive that the (extended) orbit $x_0(t)$ of v_0 starting at $p_0 \in D_0$ satisfies

$$\text{dist}(x_0(t), \Sigma_0) \geq \delta_1, \quad \forall 0 \leq t < \infty.$$

Here the inequality holds true for all $t \in [0, \infty)$ is due to the fact that for any time $T > 0$, the arc-lengths of $x_n(t)$

$$\alpha_n(T) = \int_0^T |v_n(t)| dt$$

is uniformly bounded independent of n .

It is a contradiction to (3.7).

Hence the equality (3.8) holds true. The proof is complete.

3.3. Proofs of Theorems 3.1 and 3.2

It is easy to see that Theorem 3.2 is a direct corollary of Theorem 3.1 and Assertion 2 of Theorem 3.3. Hence we only need to prove Theorem 3.1. We proceed in several steps as follows.

Step 1. Under a proper coordinate transformation, the system (3.1) can be rewritten as follows

$$\begin{cases} \frac{dx}{dt} = B_\lambda x + g_1(x, y, \lambda), \\ \frac{dy}{dt} = C_\lambda y + g_2(x, y, \lambda), \end{cases} \quad (3.14)$$

where $x \in \mathbb{R}^{m+1}$, $y \in \mathbb{R}^{n-m-1}$, and B_λ is the $(m+1) \times (m+1)$ matrix with eigenvalues $\beta_1(\lambda), \dots, \beta_{m+1}(\lambda)$, C_λ is the $(n-m-1) \times (n-m-1)$ matrix with eigenvalues $\beta_{m+2}(\lambda), \dots, \beta_n(\lambda)$, and

$$g_i(x, y, \lambda) = o(|x|, |y|), \quad \forall \lambda \in \mathbb{R} \quad (i = 1, 2). \quad (3.15)$$

For simplicity, we assume that $\lambda_0 = 0$, i.e.,

$$\begin{aligned} \operatorname{Re} \beta_i(\lambda) & \begin{cases} < 0, & \lambda < 0, \\ = 0, & \lambda = 0, \\ > 0, & \lambda > 0, \end{cases} & \text{if } 1 \leq i \leq m+1, \\ \operatorname{Re} \beta_j(0) & < 0, & \text{if } m+2 \leq j \leq n. \end{aligned}$$

Let $h(x, \lambda)$ be the center manifold function defined as in Theorem 2.1, and $M_\lambda = \{(x, y) \mid y = h(x, \lambda), x \in D \subset \mathbb{R}^{m+1}\}$ be a center manifold of (3.14). It is known that the topological structure of the orbits of (3.14) in M_λ is equivalent to that of the following system in $D \subset \mathbb{R}^{m+1}$,

$$\frac{dx}{dt} = B_\lambda x + f(x, \lambda), \quad x \in D \subset \mathbb{R}^{m+1}, \quad (3.16)$$

where

$$f(x, \lambda) = g_1(x, h(x, \lambda), \lambda).$$

By (3.15) and Theorem 2.1, we have

$$f(x, \lambda) = o(|x|), \quad \forall \lambda \in \mathbb{R}.$$

Step 2. Proof of Assertion (1)

Let $v_\lambda = B_\lambda + f(\cdot, \lambda)$. By assumption, $z = (x, y) = 0$ is asymptotically stable for (3.14) at $\lambda_0 = 0$. Hence $x = 0$ is an attractor of v_0 .

By Theorem 3.3, there exist constants $r, \lambda_1 > 0$ such that for all $0 < \lambda < \lambda_1$, the set $\mathcal{B}_r = \{x \in \mathbb{R}^{m+1} \mid |x| < r\}$ is an absorbing set of v_λ , and the ω -limit set

$$\mathcal{A}_\lambda = \omega_\lambda(\overline{\mathcal{B}_r}) \subset \mathcal{B}_r \quad (0 < \lambda < \lambda_1) \quad (3.17)$$

is an attractor of v_λ in some open set $D \subset \mathbb{R}^{m+1}$ ($\overline{\mathcal{B}_r} \subset D$).

In addition, by the stable manifold theorem (Theorem 2.3), the unstable manifold W_λ^u of v_λ contains an open neighborhood of $x = 0$ in \mathbb{R}^{m+1} for all $0 < \lambda < \lambda_1$. From (3.17) we see that

$$W_\lambda^u \subset \mathcal{A}_\lambda \subset \mathcal{B}_r, \quad \forall 0 < \lambda < \lambda_1.$$

By the definition of unstable manifolds, we obtain that

$$\mathcal{A}_\lambda \setminus W_\lambda^u \subset D \subset \mathbb{R}^{m+1} \text{ is an attractor of (3.16) in } D \setminus \{0\},$$

which implies that

$$\begin{cases} \Omega_\lambda \text{ is an attractor of (3.14),} \\ m \leq \dim \Omega_\lambda \leq m+1, \\ 0 \notin \Omega_\lambda, \\ \lim_{\lambda \rightarrow 0^+} d(\Omega_\lambda, 0) = \lim_{\lambda \rightarrow 0^+} \sup_{x \in \Omega_\lambda} |x| = 0, \end{cases} \quad (3.18)$$

where Ω_λ is defined by

$$\Omega_\lambda = \{(x, y) \in \mathbb{R}^n \mid x \in \mathcal{A}_\lambda \setminus W_\lambda^u, y = h(x, \lambda)\}.$$

Here $m \leq \dim \Omega_\lambda$ follows from Corollary 2 on p.46 in [5]¹. Therefore, Assertion (1) is proved.

Step 3. Proof of Assertion (2)

Let $\Sigma_\lambda = \mathcal{A}_\lambda \setminus W_\lambda^u$ be the attractor of (3.16) in $D \setminus \{0\}$. Obviously, there is an $r_\lambda > 0$ sufficiently small such that the disk $\overline{B}_{r_\lambda} \subset W_\lambda^u$, and the annulus $R_\lambda = B_r \setminus B_{r_\lambda}$ is absorbing in $D \setminus \{0\}$. Hence, by Lemma 2.2, the attractor Σ_λ of (3.16) in $D \setminus \{0\}$ is given by

$$\Sigma_\lambda = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S_\lambda(t) R_\lambda}. \quad (3.19)$$

As $\Sigma_\lambda \subset R_\lambda$ is invariant under the action of the semigroup $\{S_\lambda(t)\}_{t \geq 0}$, for any $t_0 \geq 0$,

$$\Sigma_\lambda \subset S_\lambda(t_0) R_\lambda. \quad (3.20)$$

It follows from (3.19) and (3.20) that there is a sequence $\{t_n\}$ with $t_{n+1} > t_n > 0$ and $t_n \rightarrow \infty$ ($n \rightarrow \infty$) such that

$$\begin{cases} S_\lambda(t_{n+1}) R_\lambda \subset S_\lambda(t_n) R_\lambda, \\ \Sigma_\lambda = \lim_{n \rightarrow \infty} S_\lambda(t_n) R_\lambda = \bigcap_{n=1}^{\infty} S_\lambda(t_n) R_\lambda. \end{cases}$$

We know that $S_\lambda(t) R_\lambda$ is homeomorphic to an $(m+1)$ -annulus for any $t \in \mathbb{R}^1$; hence Assertion (2) is proved.

Step 4. Proof of Assertion (3)

Let $M \subset \mathbb{R}^{m+1}$ be an $(m+1)$ -dimensional smooth manifold with boundary. For each point $x \in \partial M$ we define

$z(x, s)$ = the point $z \in M$, which lies on the inward normal line
starting at x , and the arc length from z to x is s ($s \geq 0$).

Obviously, $z(x, 0) = x$.

By Assertion (2), we can take a sequence of smooth $(m+1)$ -dimensional annulus $\{M_n\}$ in $R_\lambda = B_r \setminus B_{r_\lambda}$ such that

$$\begin{cases} \Sigma_\lambda \subset M_{n+1} \subset M_n \subset R_\lambda, & \forall n \geq 1, \\ M_n \text{ are deformation retracts of } R_\lambda, \text{ and} \\ \lim_{n \rightarrow \infty} M_n = \Sigma_\lambda. \end{cases} \quad (3.21)$$

Moreover, the sequence $\{M_n\}$ enjoys the following properties:

(1) for any point $x \in \partial M_n$, there exists a number $\lambda_n(x) \geq 0$ such that for all $x, y \in \partial M_n$, $x \neq y$, the line segment

$$\ell_x = \{z(x, \lambda) \mid 0 \leq \lambda \leq \lambda_n(x)\}$$

¹Here one can view the dimension as either the topological dimension or the Hausdorff dimension, thanks to

$$\text{Hausdorff dim } X \geq \dim X$$

given on p.107 in [5].

does not intersect with the line segment ℓ_y ;

(2) the points in the line segment ℓ_x ($x \in \partial M_n$) satisfy

$$\begin{aligned} z(x, \lambda) &\overline{\in} M_{n+1}, & 0 \leq \lambda < \lambda_n(x) \quad \text{as } \lambda_n(x) > 0; \text{ and} \\ z(x, \lambda_n(x)) &\in \partial M_{n+1} & (\text{if } \lambda_n(x) = 0, \text{ then } x \in \partial M_n \cap \partial M_{n+1}). \end{aligned}$$

These properties can be achieved by letting the smooth manifold $M_i = R_\lambda$ shrink along its inward normal direction, and by taking properly the intercepted manifold M_{i+1} , inductively.

Thus, $\forall x \in \partial M_1$ we can define a curve

$$L_x = \bigcup_{n=1}^{\infty} \ell_{x_n}, \quad x_1 = x, x_{n+1} = z(x_n, \lambda_n(x_n)).$$

Namely, L_x is the union of the line segments ℓ_{x_n} , where the endpoint x_{n+1} of ℓ_x is the starting point of $\ell_{x_{n+1}}$.

Since Σ_λ is a finite simplicial complex, the length of L_x is finite for any $x \in \partial M_1$; otherwise the number of simplices in Σ_λ can not be finite. It is easy to see that

$$L_x \cap L_y = \emptyset, \quad \forall x \neq y, x, y \in \partial M_1,$$

and by (3.21) the endpoint q_x of $L_x = \bigcup_{n=1}^{\infty} \ell_{x_n}$ satisfies

$$\lim_{n \rightarrow \infty} y_n = q_x \in \Sigma_\lambda \quad (y_n \in \ell_{x_n}),$$

since L_x has finite length.

On the other hand, we see that

$$M_1 = \Sigma_\lambda \bigcup \left(\bigcup_{x \in \partial M_1} L_x \right).$$

Then we define a mapping $H : M_1 \times [0, 1] \rightarrow M_1$ by

$$H(y, t) = \begin{cases} y, & y \in \Sigma_\lambda, \\ p(y, t), & y \in L_x, \end{cases}$$

where $p(y, t)$ is the point $p \in L_x$ such that the arc length along L_x from y to p is $t \times r(y)$ where $r(y)$ is the length of the arc in L_x from y to $q_x \in \Sigma_\lambda$ the endpoint of L_x . It is clear that H is continuous, and

$$H(\cdot, 0) = \text{id} : M_1 \longrightarrow M_1,$$

$$H(\cdot, 1) : M_1 \longrightarrow \Sigma_\lambda,$$

$$H \circ i = \text{id} : \Sigma_\lambda \longrightarrow \Sigma_\lambda,$$

where $i : \Sigma_\lambda \rightarrow M_1$ is an inclusion mapping. Hence Σ_λ is a deformation retract of $M_1 = R_\lambda$. Assertion (3) is proved.

Step 5. Proof of Assertion (4)

By (3.18) we see that the attractor Ω_λ of (3.14) is in the center manifold M_λ for $0 < \lambda < \lambda_1$. Since the eigenspace E_0 is tangent to M_λ ($0 < \lambda < \lambda_1$) at $x = 0$, Assertion (4) follows.

Step 6. Proof of Assertion (5)

By the topological degree theory, the degree of the vector fields in (3.1) satisfies

$$\deg(-(A_\lambda + G), \mathcal{B}_r, 0) = 1, \quad \forall -\lambda_1 < \lambda < \lambda_1, \quad (3.22)$$

where $\lambda_1 > 0$ and \mathcal{B}_r are defined as in (3.17).

Since Ω_λ is the maximum attractor of (3.1) in $\mathcal{B}_r \setminus \{0\}$, all nonzero singular points of $A_\lambda + G_\lambda$ in \mathcal{B}_r are in Ω_λ . Then we have

$$\deg(-(A_\lambda + G_\lambda), \mathcal{B}_r, 0) = \text{ind}(-(A_\lambda + G_\lambda), 0) + \sum_{x_i \in \Omega_\lambda} \text{ind}(-(A_\lambda + G_\lambda), x_i). \quad (3.23)$$

On the other hand, by (3.5) and (3.6) we obtain

$$\text{ind}(-(A_\lambda + G_\lambda), 0) = \begin{cases} 1, & m = \text{odd}, \\ -1, & m = \text{even}, \end{cases} \quad (0 < \lambda < \lambda_1). \quad (3.24)$$

Hence Assertion (5) follows from (3.22)–(3.24).

The proof of Theorem 3.1 is complete.

3.4. Pitchfork bifurcation

A related interesting question is to address that under what conditions the attractors Ω_λ in Theorem 3.1 are homeomorphic to an m -dimensional sphere. In this subsection, we consider the case where $m = 0$, which corresponds to the classical pitchfork bifurcation.

Theorem 3.4. *Assume that the conditions (3.5) and (3.6) with $m = 0$ hold true, $G(x, \lambda)$ is analytic at $x = 0$, and $x = 0$ is locally asymptotically stable for (3.1) at $\lambda = \lambda_0$. Then there exists an open set $U \subset \mathbb{R}^n$ with $0 \in U$, such that as $\lambda > \lambda_0$ the system (3.1) bifurcates from $(0, \lambda_0)$ exactly two equilibrium points $x_1, x_2 \in U$, and the open set U is decomposed into two open sets U_λ^1 and U_λ^2 ,*

$$\overline{U} = \overline{U}_\lambda^1 + \overline{U}_\lambda^2, \quad U_\lambda^1 \cap U_\lambda^2 = \emptyset \quad \text{and} \quad 0 \in \overline{U}_\lambda^1 \cap \overline{U}_\lambda^2$$

with $x_i \in U_\lambda^i$ ($i = 1, 2$), such that

$$\lim_{t \rightarrow \infty} x(t, \varphi) = x_i \quad \text{as} \quad \varphi \in U_\lambda^i \quad (i = 1, 2),$$

where $x(t, \varphi)$ is the solution of (3.1) with $x(0, \varphi) = \varphi$.

Proof. System (3.1) can be rewritten as follows

$$\begin{cases} \frac{dz}{dt} = \beta_1(\lambda)z + g_1(z, y, \lambda), \\ \frac{dy}{dt} = B_\lambda y + g_2(x, y, \lambda), \end{cases} \quad (3.25)$$

where $z \in \mathbb{R}^1$, $y \in \mathbb{R}^{n-1}$, B_λ is an $(n-1) \times (n-1)$ matrix, and

$$g_i(x, y, \lambda) = o(|z|, |y|), \quad \forall \lambda \in \mathbb{R}^1 \quad (i = 1, 2). \quad (3.26)$$

By Theorem 2.1, the center manifold function $h(\cdot, \lambda) : I = (-a, a) \rightarrow \mathbb{R}^{n-1}$ with $a > 0$ is C^∞ , and the bifurcation problem of (3.25) is equivalent to the bifurcation of the following equation

$$\frac{dz}{dt} = \beta_1(\lambda)z + g_1(z, h(z, \lambda), \lambda), \quad z \in I = (-a, a), \quad (3.27)$$

where $\beta_1(\lambda)$ satisfies that

$$\beta_1(\lambda) \begin{cases} < 0, & \lambda < \lambda_0, \\ = 0, & \lambda = \lambda_0, \\ > 0, & \lambda > \lambda_0. \end{cases}$$

We need to show that there is an integer $k \geq 2$ and some $\alpha \neq 0$ such that

$$g_1(z, h(z, \lambda_0), \lambda_0) = \alpha z^k + o(|z|^k). \quad (3.28)$$

By assumption, $z = 0$ is asymptotically stable for (3.27) at $\lambda = \lambda_0$. Hence we have

$$g_1(z, h(z, \lambda_0), \lambda_0) \neq 0, \quad \forall |z| > 0 \text{ sufficiently small.} \quad (3.29)$$

For simplicity, we drop the dependence on λ_0 , e.g., we write $g_i(z, y) = g_i(z, y, \lambda_0)$, $h(z) = h(z, \lambda_0)$, and $B = B_{\lambda_0}$. We know that the center manifold function $h(z)$ satisfies

$$\frac{dh}{dz} = \frac{Bh + g_2(z, h)}{g_1(z, h)}, \quad |z| < a. \quad (3.30)$$

If (3.28) is not true, then

$$\left. \frac{d^k(g_1(z, h(z)))}{dz^k} \right|_{z=0} = 0, \quad \forall k < \infty,$$

i.e.,

$$g_1(z, h(z)) = o(|z|^\infty). \quad (3.31)$$

We infer from (3.30) that

$$Bh(z) + g_2(z, h) = o(|z|^\infty). \quad (3.32)$$

In the following, we shall show from (3.32) that there exists an analytic $h_1(z)$ ($h_1(0) = 0$, $h_1'(0) = 0$) satisfying

$$Bh_1(z) + g_2(z, h_1(z)) = 0 \quad \text{for } |z| < \varepsilon \text{ sufficiently small,} \quad (3.33)$$

$$h(z) = h_1(z) + o(|z|^\infty). \quad (3.34)$$

Since B is a nondegenerate matrix and $g_2(z, y)$ is analytic, by the implicit function theorem and (3.26), there is a unique analytic function $h_1(z)$ with $h_1(0) = 0$, $h_1'(0) = 0$

satisfying (3.33) for $|z| \leq \varepsilon$ sufficiently small. We define a mapping $B + g : C^1[-\varepsilon, \varepsilon] \rightarrow C^1[-\varepsilon, \varepsilon]$ by

$$(B + g)y = By + g_2(z, y)$$

for any $y \in C^1[-\varepsilon, \varepsilon]$. Thanks to $h_1(0) = 0$ and $h'_1(0) = 0$, we can take $\varepsilon > 0$ sufficiently small such that the Frechét derivative of $B + g$ at $h_1(z) \in C^1[-\varepsilon, \varepsilon]$ given by

$$B + Dg(h_1) = B + \frac{\partial}{\partial y} g_2(z, h_1(z)) : C^1[-\varepsilon, \varepsilon] \rightarrow C^1[-\varepsilon, \varepsilon]$$

is a linear homeomorphism. Hence, by the inverse function theorem it follows from (3.33) that there exists a unique function

$$h(z) = h_1(z) + h_2(z) \in C^1[-\varepsilon, \varepsilon]$$

satisfying (3.32). Moreover h_2 is C^∞ at $x = 0$ and $h_2(z) = o(|z|^\infty)$. Thus, the center manifold function $h(z)$ satisfies (3.33) and (3.34). Since $g_1(z, y)$ is analytic, from (3.31) and (3.34) we can deduce that

$$g_1(z, h_1(z)) = 0, \quad \forall |z| < \varepsilon.$$

It means that all points on the curve $\{(z, h_1(z)) \mid |z| < \varepsilon\}$ are the singular points of (3.25) at $\lambda = \lambda_0$, a contradiction to the assumption that $x = 0$ is asymptotically stable for (3.25) at $\lambda = \lambda_0$.

Therefore, the equality (3.28) holds true. By the stability of $z = 0$ for (3.27) at $\lambda = \lambda_0$, we infer that the constants in (3.28) are as follows

$$k = \text{an odd number, and } \alpha < 0. \quad (3.35)$$

From (3.28) and (3.35) it follows that the equation

$$\begin{cases} \beta_1(\lambda)z + g_2(z, h(z), \lambda) = \beta_1(\lambda)z + \alpha(\lambda)z^k + o(|z|^k) = 0, \\ \alpha(\lambda_0) = \alpha < 0, \quad k = \text{odd}, \end{cases}$$

bifurcates from $(0, \lambda_0)$ exactly two singular points $z_1, z_2 \in (-a, a)$ for $\lambda > \lambda_0$, i.e., the equation (3.26) bifurcates from $(0, \lambda_0)$ exactly two asymptotically stable equilibrium points z_1 and z_2 . Therefore, the two points $x_i = (z_i, h(z_i)) \in \mathbb{R}^n$ are the attractors of (3.25) in an open set $U \subset \mathbb{R}^n$. By the stable manifold theorem (Theorem 2.3), there is an $(n - 1)$ -dimensional stable manifold W^s of (3.25) at $x = 0$ dividing the open set U into two parts U_λ^1 and U_λ^2 such that $x_i \in U_\lambda^i$ and x_i attracts U_λ^i ($i = 1, 2$). This proof is complete.

Remark 3.4. It is still an open question whether the analytic condition of vector fields is sufficient for the bifurcated attractor Ω_λ of (3.1) to be homeomorphic to an m -dimensional sphere with $m \geq 1$.

3.5. Minimal attractors

It is easy to see that a subset $\Gamma_\lambda \subset \Omega_\lambda$ may be an attractor as well. The minimal attractor Γ_λ contained in Ω_λ is called the bifurcated minimal attractor of (3.1). In general,

the bifurcated attractor Ω_λ may have no minimal attractors. We are interested here in the existence problem of minimal attractors.

In this subsection, we always assume that the conditions of Theorem 3.1 hold. If the bifurcated attractor Ω_λ of (3.1) is homeomorphic to a sphere S^m , we denote it by $\Omega_\lambda = S^m$ for simplicity.

Theorem 3.5. *Assume that $m = 1$ and $\Omega_\lambda = S^1$ contains singular points of (3.1) which are all nondegenerate. Then the following assertions hold true.*

(1) *The number of singular points on $\Omega_\lambda = S^1$ is $2k$ for some integer $k \geq 1$, and there are exactly k singular points $\{x_i \mid 1 \leq i \leq k\} \subset \Omega_\lambda$ such that each x_i forms a bifurcated minimal attractors of (3.1);*

(2) *There is an open set $D \subset \mathbb{R}^n$ which can be decomposed into k open sets D_i ($1 \leq i \leq k$) such that $\Omega_\lambda \cup \{0\} \subset \overline{D}$, $\overline{D} = \sum_{i=1}^k \overline{D}_i$, $D_i \cap D_j = \emptyset$ ($i \neq j$), $0 \in \bigcap_{j=1}^k \overline{D}_j$ and $x_i \in D_i$ attracts D_i . Moreover the projection of D and D_i on the center manifold is as shown in Fig. 3.2 for $k = 2$.*

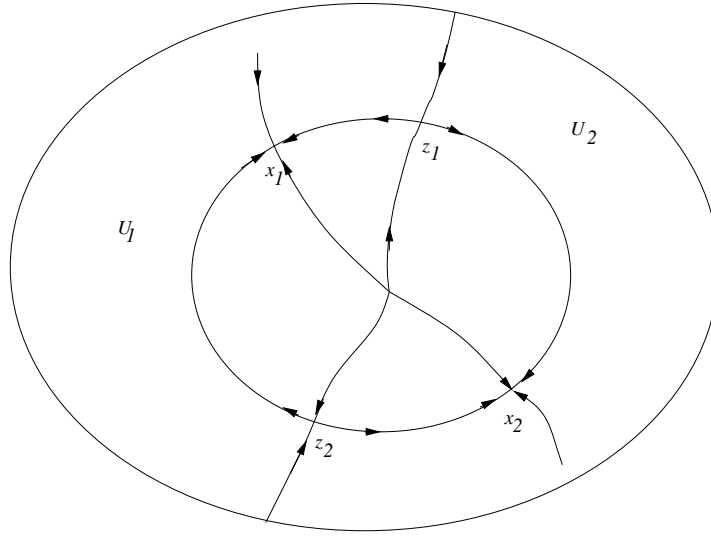


Fig. 3.2

Proof. Since all singular points in $\Omega_\lambda = S^1$ are nondegenerate, the singular points in Ω_λ are either attractors or repellers. The attractors and repellers on Ω_λ are alternately positioned, hence the number of singular points in Ω_λ is even, i.e., $2k$ ($k \geq 1$), with k attractors and k repellers.

Let $\{x_i \mid 1 \leq i \leq k\}$ be the attractors, and $\{z_i \mid 1 \leq i \leq k\}$ be the repellers. Obviously, $\{x_i \mid 1 \leq i \leq k\}$ are the minimal attractors in Ω_λ of (3.1), and $\{z_i \mid 1 \leq i \leq k\}$ are the saddle points of (3.1). It is easy to see that in the two dimensional center manifold, there is an open set U which can be decomposed into k open sets U_i ($1 \leq i \leq k$) such that (see Fig. 3.2)

- (1) $\overline{U} = \sum_{i=1}^k \overline{U}_i$, $U_i \cap U_j = \emptyset$ ($i \neq j$), $0 \in \bigcap_{i=1}^k \overline{U}_i$,
- (2) $\bigcup_{i=1}^k \partial U_i \cap U$ consists of the stable manifolds of the saddles z_j ($1 \leq j \leq k$), and
- (3) $x_i \in U_i$ and attracts U_i ($1 \leq i \leq k$).

Therefore, the open sets $D = \{(x, h(x)) \mid x \in U\}$ and $D_i = \{(x, h(x)) \mid x \in U_i\}$ are as desired by this theorem. This proof is complete.

Theorem 3.6. *Let $m = 2$ in (3.5). If the bifurcated attractor Ω_λ of (3.1) contains exactly two singular points which are nondegenerate, then only one of the following two assertions holds true.*

- (1) Ω_λ contains at least a periodic orbit.
- (2) The set of minimal attractors consists of one singular point $x_0 \in \Omega_\lambda$, and there is an open set $D \subset \mathbb{R}^n$ with $\Omega_\lambda \cup \{0\} \subset \overline{D}$ such that x_0 attracts D .

Epecially, if the vector field in (3.1) is odd, i.e., $G(-x, \lambda) = -G(x, \lambda)$, then Ω_λ must contain a periodic orbit.

Proof. By Theorem 3.1, Ω_λ is a limit of a sequence of three dimensional annulus. Hence, Ω_λ must contain a two dimensional invariant set Σ , where the Poincaré-Bendixon theorem is valid. Thus, by the regularity of singular points in Ω_λ , for any point $x \in \Sigma$, the ω -limit set $\omega(x)$ is either a singular point or a periodic orbit.

If Σ contains no periodic orbits, then as a two dimensional invariant closed set, all points in Σ are interior points of Σ . Hence $\Sigma = S^2$. By assumption we can derive that $\Omega_\lambda = \Sigma = S^2$, otherwise by the Poincaré-Hopf index theorem, Ω_λ must contain more than two singular points. Since $\Omega_\lambda = S^2$ contains only two nondegenerate singular points, by Theorem 3.1, their indices are $+1$. Therefore, the two singular points are either attractors or repellers in $\Omega_\lambda = S^2$ (since Ω_λ contains no periodic orbits). Due to the Poincaré-Bendixon theorem, we can deduce that one of the two singular points is an attractor and another is a repeller, furthermore, the attractor attracts $\Omega_\lambda \setminus \{x_1\}$, where x_1 is the repeller in Ω_λ .

When the vector field $G(x, \lambda)$ in (3.1) is odd, the two singular points in Ω_λ have the same eigenvalues. Hence they have the same local topological structure, which implies, by the above conclusion, that Ω_λ must contain a periodic orbit. The proof is complete.

3.6. Generalized Hopf bifurcation

Now, let us consider the more general bifurcation. Let the eigenvalues (3.4) satisfy that

$$\operatorname{Re} \beta_i(\lambda) \begin{cases} < 0 \text{ (or } > 0), & \lambda < \lambda_0, \\ = 0, & \lambda = \lambda_0, \\ > 0 \text{ (or } < 0), & \lambda > \lambda_0, \end{cases} \quad \forall 1 \leq i \leq m+1, \quad (3.36)$$

$$\operatorname{Re} \beta_j(\lambda_0) \neq 0, \quad \forall m+2 \leq j \leq n. \quad (3.37)$$

It is known that if $m = \text{even}$, the system (3.1) must bifurcate from $(0, \lambda_0)$ a singular point. When $m = 1$, the Hopf bifurcation amounts to saying that if $\beta_1(\lambda) = \bar{\beta}_2(\lambda)$ with $\operatorname{Im} \beta_1(\lambda_0) \neq 0$, then under the conditions (3.36) and (3.37) the system (3.1) bifurcates from

$(0, \lambda_0)$ a periodic orbit. Our next question is to see whether (3.1) bifurcates from $(0, \lambda_0)$ an invariant set assuming only (3.36) and (3.37) with $m = \text{odd}$ are valid, i.e., without the asymptotic stability assumption. In general, we shall see later, this statement is not true. However, we can still derive a generalized version of the Hopf bifurcation as follows.

Under the conditions (3.36) and (3.37) we know that there exists an $(m+1)$ -dimensional center manifold of (3.1) at $\lambda = \lambda_0$,

$$M_c^{m+1} = \{(x, y) \in \mathbb{R}^n \mid x \in \Omega \subset \mathbb{R}^{m+1}, y = h(x, \lambda_0)\}$$

which is invariant under the flows of (3.1) at $\lambda = \lambda_0$.

We say the center manifold M_c^{m+1} is stable (resp. is unstable) if the ω -limit set (resp. the α -limit set) of M_c^{m+1} is

$$\omega(M_c^{m+1}) = 0 \quad (\text{resp. } \alpha(M_c^{m+1}) = 0).$$

Theorem 3.7. *Let the conditions (3.36) and (3.37) hold true. If the center manifold M_c^{m+1} of (3.1) at $\lambda = \lambda_0$ is stable or unstable, then (3.1) must bifurcate from $(0, \lambda_0)$ an invariant set Γ_λ with $\lambda \neq \lambda_0$, and Γ_λ has the homotopy type of S^m provided Γ_λ being a finite simplicial complex. Especially, as $m = 1$, if there are no singular points in Γ_λ , then Γ_λ must contain a periodic orbit.*

The proof of Theorem 3.7 is similar to that of Theorem 3.1, so we omit the details.

Remark 3.5. If $m = 1$ in (3.36) and (3.37) and $\beta_1(\lambda) = \bar{\beta}_2(\lambda)$ with $\text{Im } \beta_1(\lambda_0) \neq 0$, then the center manifold M_c^2 of (3.1) must be one of the three cases: (i) stable, (ii) unstable, (iii) containing infinite periodic orbits which implies the bifurcation of periodic orbits. Hence, the Hopf bifurcation is included in Theorem 3.7.

The following flows as shown in Fig. 3.3 (a)–(c) provide an example which shows schematically that under only conditions (3.36) and (3.37), bifurcations to invariant sets may not occur.

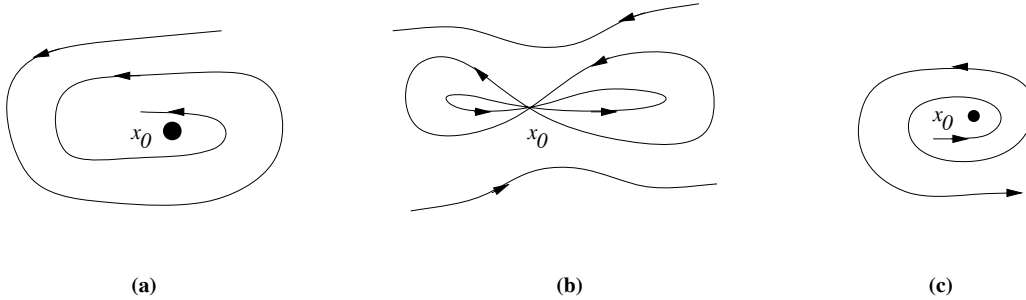


Fig. 3.3. (a) $\lambda < \lambda_0$ (b) $\lambda = \lambda_0$ (c) $\lambda > \lambda_0$

The parametrized vector field $v(x, \lambda)$ illustrated by Fig. 3.3(a) has a stable focus x_0 at $\lambda < \lambda_0$, and $v(x, \lambda)$ illustrated by Fig. 3.3(c) has an unstable focus x_0 at $\lambda > \lambda_0$, which is nondegenerate. $v(x, \lambda_0)$ as shown in Fig. 3.3(b) has a degenerate singular point x_0 with the

elliptic-hyperbolic structure. The flows of $v(x, \lambda)$ in a neighborhood of x_0 has no invariant sets except the singular point x_0 .

§ 4. Infinite Dimensional Systems

In this section, we generalize the attractor bifurcation theorems for finite dimensional vector fields to infinite dimensional systems defined in Hilbert spaces. These generalizations can be applied to bifurcation problems of various types for nonlinear time dependent partial differential equations in physics and mechanics.

4.1. Center manifold theorems

First, we recall an infinite dimensional version of the center manifold theorem without proof (see among others [3]). Let H and H_1 be two Hilbert spaces, and $H_1 \hookrightarrow H$ be a dense inclusion embedding. We consider the nonlinear evolution equations given by

$$\begin{cases} \frac{du}{dt} = L_\lambda u + G(u, \lambda), & u \in H_1, \lambda \in \mathbb{R}^1, \\ u(0) = \varphi, \end{cases} \quad (4.1)$$

where $L_\lambda : H_1 \rightarrow H$ are parameterized linear completely continuous fields continuously depending on $\lambda \in \mathbb{R}^1$, which satisfy

$$\begin{cases} L_\lambda = -A + B_\lambda & \text{is a sectorial operator,} \\ A : H_1 \rightarrow H & \text{a linear homeomorphism,} \\ B_\lambda : H_1 \rightarrow H & \text{the parameterized linear compact operators.} \end{cases} \quad (4.2)$$

It is easy to see (see [3, 10]) that L_λ generates an analytic semi-group $\{e^{-tL_\lambda}\}_{t \geq 0}$. Then we can define fractional power operators L_λ^α for any $0 \leq \alpha \leq 1$ with domain $H_\alpha = D(L_\lambda^\alpha)$ such that $H_{\alpha_1} \subset H_{\alpha_2}$ if $\alpha_1 > \alpha_2$, and $H_0 = H$.

Furthermore, we assume that the nonlinear terms $G(\cdot, \lambda) : H_\alpha \rightarrow H$ for some $1 > \alpha \geq 0$ are a family of parameterized C^r bounded operators ($r \geq 1$) continuously depending on the parameter $\lambda \in \mathbb{R}^1$, such that

$$G(u, \lambda) = o(\|u\|_{H_\alpha}), \quad \forall \lambda \in \mathbb{R}^1. \quad (4.3)$$

We assume that the spaces H_1 and H can be decomposed into

$$\begin{cases} H_1 = E_1^\lambda \oplus E_2^\lambda, \dim E_1^\lambda < \infty, \text{ near } \lambda_0 \in \mathbb{R}^1, \\ H = \tilde{E}_1^\lambda \oplus \tilde{E}_2^\lambda, \tilde{E}_1^\lambda = E_1^\lambda, \tilde{E}_2^\lambda = \text{closure of } E_2^\lambda \text{ in } H, \end{cases} \quad (4.4)$$

where E_1^λ and E_2^λ are the invariant subspaces of L_λ , i.e., L_λ can be decomposed into $L_\lambda = \mathcal{L}_1^\lambda \oplus \mathcal{L}_2^\lambda$ such that for any λ near λ_0 ,

$$\begin{cases} \mathcal{L}_1^\lambda = L_\lambda|_{E_1^\lambda} : E_1^\lambda \rightarrow \tilde{E}_1^\lambda, \\ \mathcal{L}_2^\lambda = L_\lambda|_{E_2^\lambda} : E_2^\lambda \rightarrow \tilde{E}_2^\lambda, \end{cases} \quad (4.5)$$

where the eigenvalues of \mathcal{L}_2^λ possess the negative real parts, and the eigenvalues of \mathcal{L}_1^λ possess the non-negative real parts at $\lambda = \lambda_0$.

Thus, for λ near λ_0 , Equation (4.1) can be rewritten as

$$\begin{cases} \frac{dx}{dt} = \mathcal{L}_1^\lambda x + G_1(x, y, \lambda), \\ \frac{dy}{dt} = \mathcal{L}_2^\lambda y + G_2(x, y, \lambda), \end{cases} \quad (4.6)$$

where $u = x + y \in H_1$, $x \in E_1^\lambda$, $y \in E_2^\lambda$, $G_i(x, y, \lambda) = P_i G(u, \lambda)$, and $P_i : H \rightarrow \tilde{E}_i$ are canonical projections.

Let $S_\lambda(t) : \tilde{E}_2^\lambda \rightarrow \tilde{E}_2^\lambda$ be the analytic semi-groups generated by \mathcal{L}_2^λ . We have the following center manifold theorem for (4.1) (see among others [3, 11]).

Theorem 4.1. *Assume (4.2)–(4.5). Then there exist a neighborhood of λ_0 given by $|\lambda - \lambda_0| < \delta$ for some $\delta > 0$, a neighborhood $B_\lambda \subset E_1^\lambda$ of $x = 0$, and a C^1 function $h(\cdot, \lambda) : B_\lambda \rightarrow E_2^\lambda(\alpha)$ depending continuously on λ , where $E_2^\lambda(\alpha)$ is the completion of E_2^λ in the H_α -norm ($0 \leq \alpha < 1$ as in (4.3)), such that*

(1) $h(0, \lambda) = 0$, $D_x h(0, \lambda) = 0$;

(2) the set

$$M_\lambda = \{(x, y) \in H_1 \mid x \in B_\lambda, y = h(x, \lambda) \in E_2^\lambda\},$$

called center manifold, is locally invariant for (4.1), i.e., $\forall \varphi \in M_\lambda$,

$$u_\lambda(t, \varphi) \in M_\lambda, \quad \forall 0 \leq t < t_\varphi,$$

for some $t_\varphi > 0$, where $u_\lambda(t, \varphi)$ is the solution of (4.1);

(3) if $(x_\lambda(t), y_\lambda(t))$ is a solution of (4.6), then there is a $\beta_\lambda > 0$ and $k_\lambda > 0$ with k_λ depending on $(x_\lambda(0), y_\lambda(0))$ such that

$$\|y_\lambda(t) - h(x_\lambda(t), \lambda)\|_H \leq k_\lambda e^{-\beta_\lambda t}.$$

If we only consider the existence of the local center manifold, then the conditions in (4.5) can be modified in the following fashion. Let the operator $L_\lambda = \mathcal{L}_1^\lambda \oplus \mathcal{L}_2^\lambda$ and \mathcal{L}_2^λ be decomposed into

$$\begin{cases} \mathcal{L}_2^\lambda = \mathcal{L}_{21}^\lambda \oplus \mathcal{L}_{22}^\lambda, \\ E_2^\lambda = E_{21}^\lambda \oplus E_{22}^\lambda, \quad \tilde{E}_2^\lambda = \tilde{E}_{21}^\lambda \oplus \tilde{E}_{22}^\lambda, \\ \dim E_{21}^\lambda = \dim \tilde{E}_{21}^\lambda < \infty, \\ \mathcal{L}_{2i}^\lambda : E_{2i}^\lambda \rightarrow \tilde{E}_{2i}^\lambda \text{ are invariant } (i = 1, 2), \end{cases} \quad (4.7)$$

such that at $\lambda = \lambda_0$,

$$\begin{cases} \text{eigenvalues of } \mathcal{L}_1^\lambda : E_1^\lambda \rightarrow \tilde{E}_1^\lambda \text{ have zero real parts;} \\ \text{eigenvalues of } \mathcal{L}_{21}^\lambda : E_{21}^\lambda \rightarrow \tilde{E}_{21}^\lambda \text{ have positive real parts;} \\ \text{eigenvalues of } \mathcal{L}_{22}^\lambda : E_{22}^\lambda \rightarrow \tilde{E}_{22}^\lambda \text{ have negative real parts.} \end{cases} \quad (4.8)$$

Then we have the following center manifold theorem.

Theorem 4.2. *Assume (4.2)–(4.4), (4.7) and (4.8). Then the conclusions (1) and (2) in Theorem 4.1 hold true.*

4.2. Attractor bifurcation for infinite dimensional systems

We consider the attractor bifurcation of (4.1). Let the inclusion $H_1 \hookrightarrow H$ be compact. For the linear operator $L_\lambda = -A + B_\lambda$ we assume that there exists a real eigenvalue sequence $\{\rho_k\} \subset \mathbb{R}^1$ and an eigenvector sequence $\{e_k\} \subset H_1$ of A :

$$\begin{cases} Ae_k = \rho_k e_k, \\ 0 < \rho_1 \leq \rho_2 \leq \cdots, \\ \rho_k \rightarrow \infty \ (k \rightarrow \infty) \end{cases} \quad (4.9)$$

such that $\{e_k\}$ is an orthogonal basis of H .

For the compact operator $B_\lambda : H_1 \rightarrow H$, we also assume that there is a constant $0 < \theta < 1$ such that

$$B_\lambda : H_\theta \longrightarrow H \text{ bounded,} \quad \forall \lambda \in \mathbb{R}^1. \quad (4.10)$$

A number $\lambda = \alpha + i\beta \in \mathbb{C}$ is called an eigenvalue of a linear mapping $L : H_1 \rightarrow H$ if there exist $x, y \in H_1$ with $x \neq 0$ such that $Lz = \lambda z$, $z = x + iy$; and the space $E_\lambda = \{x, y \in H_1 \mid (L - \lambda I)^n z = 0, z = x + iy\}$ is called an eigenspace of L corresponding to λ , and the elements $x, y \in E_\lambda$ are called eigenvectors of L .

Let the eigenvalues (counting multiplicity) of L_λ be $\{\beta_1(\lambda), \beta_2(\lambda), \dots\}$ with $\beta_k(\lambda) \in \mathbb{C}$ such that

$$\begin{aligned} \operatorname{Re} \beta_i(\lambda) &\begin{cases} < 0, & \lambda < \lambda_0, \\ = 0, & \lambda = \lambda_0, \\ > 0, & \lambda > \lambda_0, \end{cases} \quad \forall 1 \leq i \leq m+1, \\ \operatorname{Re} \beta_j(\lambda_0) &< 0, \quad \forall m+2 \leq j. \end{aligned} \quad (4.11)$$

Now we are in position to state the attractor bifurcation theorem for the infinite dimensional system (4.1).

Theorem 4.3. *Assume conditions (4.3) and (4.9)–(4.11). If $u = 0$ is a locally asymptotically stable equilibrium point of (4.1) at $\lambda = \lambda_0$, then the following assertions hold true.*

(1) *The system (4.1) bifurcates from $(0, \lambda_0)$ an attractor Ω_λ with $m \leq \dim \Omega_\lambda \leq m+1$, which is connected when $m > 0$.*

(2) *Ω_λ is a limit of a sequence of $(m+1)$ -dimensional annulus M_k with $M_{k+1} \subset M_k$; especially, if Ω_λ is a finite simplicial complex, then Ω_λ has the homotopy type of S^m .*

(3) *For any $u_\lambda \in \Omega_\lambda$, u_λ can be expressed as $u_\lambda = v_\lambda + o(\|v_\lambda\|_1)$, $v_\lambda \in E_0 = \{v \in H_1 \mid L_{\lambda_0}^k v = 0, k = 1, 2, \dots\}$.*

(4) *If $G : H_1 \rightarrow H$ is compact, and the number of equilibrium points of (4.1) in Ω_λ is finite, then we have the index formula*

$$\sum_{u_i \in \Omega_\lambda} \operatorname{ind}[-(L_\lambda + G), u_i] = \begin{cases} 2, & m = \text{even}, \\ 0, & m = \text{odd}. \end{cases}$$

(5) If $u = 0$ is globally stable, then for any bounded open set $U \subset H_1$ with $0 \in U$ there is an $\varepsilon > 0$ such that as $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, the attractor Ω_λ attracts $U \setminus \Gamma$, where Γ is the stable manifold of $u = 0$ with codimension $m + 1$. Furthermore, if (4.1) has global attractor for any λ , then we can take $U = H$.

Proof. Under (4.11), the linear completely continuous field L_λ can be decomposed as follows: $L_\lambda = \mathcal{L}_1^\lambda \oplus \mathcal{L}_2^\lambda$, for λ near λ_0 , $H_1 = E_1^\lambda \oplus E_2^\lambda$, $H = \tilde{E}_1^\lambda \oplus \tilde{E}_2^\lambda$, $\dim E_1^\lambda = \dim \tilde{E}_1^\lambda = m + 1$, $\mathcal{L}_1^\lambda : E_1^\lambda \rightarrow \tilde{E}_1^\lambda$, $\mathcal{L}_2^\lambda : E_2^\lambda \rightarrow \tilde{E}_2^\lambda$. Moreover, \mathcal{L}_1^λ has the eigenvalues $\beta_1(\lambda), \dots, \beta_{m+1}(\lambda)$, and \mathcal{L}_2^λ have the eigenvalues $\beta_j(\lambda)$ ($m + 2 \leq j$).

By (4.9) and (4.10), it is known that L_λ generates an analytic semigroup $T_\lambda(t)$ which can be decomposed into $T_\lambda(t) = S_1^\lambda(t) \oplus S_2^\lambda(t)$, where $S_i^\lambda(t)$ is generated by \mathcal{L}_i^λ . Therefore $S_2^\lambda(t)$ is also an analytic semigroup.

Thus the system (4.1) can be decomposed into the form (4.6), and the center manifold theorem is valid for (4.6). Let $h_\lambda : B_\lambda \rightarrow E_2^\lambda$ be a center manifold function of (4.6), $B_\lambda \subset E_1^\lambda$ is a neighborhood of $x = 0$. Then the bifurcation problem of (4.1) is equivalent to that of the following equations

$$\frac{dx}{dt} = \mathcal{L}_1^\lambda x + G_1(x, h_\lambda(x), \lambda), \quad x \in B_\lambda \subset E_1^\lambda. \quad (4.12)$$

Since $u = 0$ is asymptotically stable for (4.1), $x = 0$ is asymptotically stable for (4.12). Then this theorem follows from Theorems 3.1 and 3.2, and the attractor stability theorem in [11]. This proof is complete.

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