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ON QUASI-TORAL RESTRICTED LIE ALGEBRAS****

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Abstract

This paper gives some sufficient conditions for the commutativity of quasi-toral restricted Lie algebras and characterizes some properties on semisimple quasi-toral restricted Lie algebras.

Keywords Quasi-toral restricted Lie algebras, Quasi-toral elements, Torus algebras, Characteristic ideals, Ad-semisimple
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§ 1 . Introduction

In the theory of associative ring there are a number of known sufficient conditions for the commutativity of an associative ring R, some in the form of a polynomial identity. For instance, it is a well-known result that if every element x of R satisfies $x^{n(x)} = x$ for some $n(x) > 1$, then R is abelian (see [1, p.73]). It is a remarkably wide generalization of Wedderburn's theorem.

The concept of a restricted Lie algebra is attributable to N. Jacobson [8] in 1943. It is well known that [7] the Lie algebras associated with algebraic groups over a field of characteristic p are restricted Lie algebras. According to [15], the classification of the simple restricted Lie algebras is equivalent to the classification of the simple Lie algebras. Moreover, Terrell L. Hodge [17] defined a restricted structure for Lie triple systems in the characteristic $p > 2$ setting, akin to the restricted structure for Lie algebras, and initiated a study of theory of restricted modules in 2001. Now, restricted Lie algebras attract more and more attentions.

The presence of a p-power operator in theory of restricted Lie algebras motivates a study of analogous conditions within this category. Jacobson conjectured (see [2, p.196]) that every restricted Lie algebra $(L, [p])$ satisfying the requirement $x^{[p]^{n(x)}} = x$ for any $x \in L$ is abelian, where $n(x) \in N$. An early result relating the commutativity of a restricted Lie algebra to conditions imposed on the p-mapping was obtained by Chew [3]. Rolf Farsteiner [4, 5] gave some proofs of Jacobson's conjecture in some special cases. Now, the conjecture

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is still an open problem. In fact, the conjecture is a conjecture on properties of restricted Lie algebras with quasi-toral elements. So the study of quasi-toral elements will be useful for solving Jacobson's conjecture.

It is well known that solvable Lie algebras have played an important role over the last years both in the domain of algebras and in the domain of differential geometry. In this paper, we give a necessary and sufficient condition of solvable Lie algebras and obtain some sufficient conditions for the commutativity of restricted Lie algebras. Moreover, we give some elementary proofs of Jacobson's conjecture in some special cases and generalize Rolf Farnstener's results. In addition, we characterize some properties on semisimple quasi-toral restricted Lie algebras.

Throughout this paper, all algebras are assumed to be finite-dimensional over a field $\mathbf{F}$ of positive characteristic $p \geq 3$.

Definition 1.1. (cf. [7]) *Let L be a Lie algebra over $\mathbf{F}$. A mapping $[p]: L \to L, a \to a^{[p]}$ is called a p-mapping if*

(1) $\mathrm{ad}a^{[p]} = (\mathrm{ad}a)^p$ *for any $a \in L$;*

(2) $(\alpha a)^{[p]} = \alpha^p a^{[p]}$ *for any $a \in L, \alpha \in \mathbf{F}$;*

(3) $(a+b)^{[p]} = a^{[p]} + b^{[p]} + \sum\limits_{i=1}^{p-1} s_i(a,\ b)$ *for any $a,\ b \in L$, where s_i satifies* $(\mathrm{ad}\ (a \otimes x + b \otimes 1))^{p-1}(a \otimes 1) = \sum\limits_{i=1}^{p-1} i s_i(a,\ b) \otimes x^{i-1}$ *in $L \otimes_{\mathbf{F}} \mathbf{F}[x]$ for any $a,\ b \in L$.*

The pair $(L, [p])$ is referred to as a restricted Lie algebra.

Moreover, it is well known (see [4, p.42]) that if $(L, [p])$ is a restricted Lie algebra, then

$$(a+b)^{[p]^n} = a^{[p]^n} + b^{[p]^n} + \sum_{i=1}^{p^n-1} s_i^{(n)}(a,b), \qquad \text{for any} \quad a, b \in L,$$

where $(\mathrm{ad}\ (a \otimes x + b \otimes 1))^{p^n-1}(a \otimes 1) = \sum\limits_{i=1}^{p^n-1} i s_i^{(n)}(a,b) \otimes x^{i-1}$ in $L \otimes_{\mathbf{F}} \mathbf{F}[x]$ for any $a, b \in L$.

Definition 1.2. (cf. [4]) *Let $(L, [p])$ be a restricted Lie algebra over $\mathbf{F}$. A mapping $[p]^n : L \to L$ is called p-semilinear if it is p-semilinear with respect to the homomorphism $\alpha \to \alpha^{p^n}$.*

Definition 1.3. (cf. [5]) *A restricted Lie algebra $(L,\ [p])$ is called quasi-toral if there exists a positive integer $n(x)$ such that $x^{[p]^{n(x)}} = x$ for any element $x \in L$.*

Definition 1.4. (cf. [8]) *L is called a torus algebra if $\mathrm{ad}x$ is semisimple for all $x \in L$.*

Definition 1.5. (cf. [7]) *Let $(L, [p])$ be a restricted Lie algebra over $\mathbf{F}$. An element $x \in L$ is called semisimple if $x = \sum\limits_{i=1}^{m} \alpha_i x^{[p]^i},\ \alpha_i \in \mathbf{F}$.*

Definition 1.6. (cf. [7]) *Let $(L, [p])$ be a restricted Lie algebra over $\mathbf{F}$. A subalgebra $H \subseteq L$ is called a torus if*

(1) *T is an abelian p-subalgebra;*

(2) *x is semisimple for all $x \in T$.*

We now state some results used in the paper.

Lemma 1.1. (cf. [4]) *Let $(L, [p])$ be a restricted Lie algebra over $\mathbf{F}$ such that* card$(\mathbf{F}) \geq p^n$. *If $[p]^n$ is a p-semilinear mapping, then $L^{[p]^n} \subseteq C(L)$.*

Lemma 1.2. (cf. [5]) *Let $(L, [p])$ be a restricted Lie algebra. Let $G \subseteq L$ be quasi-toral and let x be an element of G such that $\mathbf{F}_{p^{n(x)}}$ is contained in $\mathbf{F}$. Then x lies centrally in G.*

Lemma 1.3. (cf. [4]) *Let $(L, [p])$ be a restricted Lie algebra over $\mathbf{F}$ of positive characteristic $p \geq 3$. If L is quasi-toral, then $(L, [p])$ does not contain a three-dimensional simple Lie algebra.*

Lemma 1.4. (cf. [7]) *Let T be a torus of the finite-dimensional restricted Lie algebra $(L, [p])$. Then any T-invariant subspace $W \subseteq L$ decomposes into $W = C_W(T) + [T,\ W]$.*

Lemma 1.5. (cf. [7]) *Let $(L, [p])$ be a finite dimensional restricted Lie algebra over a field $\mathbf{F}$. Then the following statements are equivalent:*
(1) *$[p]$ is nonsingular and $\mathbf{F}$ is perfect;*
(2) *$[p]$ is injective and $\mathbf{F}$ is perfect;*
(3) *$[p]$ is surjective.*

§ 2 . Main Results

Theorem 2.1. *If $x = x^{[p]^n} + (\mathrm{ad}y)^{p^n-1}(x)$, $(\mathrm{ad}x)^{p^n}(y) = 0$ and x is a quasi-toral element, then y is a quasi-toral element if and only if $x + y$ is a quasi-toral element.*

Proof. Since x is quasi-toral, there exists $m \geq n$ such that $x^{[p]^m} = x$. Then

$$[x,\ y] = [x^{[p]^m},\ y] = (\mathrm{ad}x)^{p^m}(y) = (\mathrm{ad}x)^{p^m - p^n}(\mathrm{ad}x)^{p^n}(y) = 0.$$

$\Rightarrow$. Since y is quasi-toral, there is $k \in N$ such that $y^{[p]^k} = y$. By virtue of a routine computation, we obtain $x^{[p]^{km}} = x$, $y^{[p]^{mk}} = y$. Then

$$(x + y)^{[p]^{km}} = x^{[p]^{km}} + y^{[p]^{km}} + \sum_{i=1}^{p^{km}-1} s_i^{(km)}(x,\ y) = x + y,$$

by means of $[x,\ y] = 0$. So $x + y$ is a quasi-toral element.

$\Leftarrow$. Since $x + y$ is quasi-toral, there is $u \in N$ such that $(x + y)^{[p]^u} = x + y$. Then

$$(x + y)^{[p]^u} = x^{[p]^u} + y^{[p]^u} + \sum_{i=1}^{p^u-1} s_i^{(u)}(x,\ y) = x^{[p]^u} + y^{[p]^u},$$

by virtue of $[x,\ y] = 0$. So we obtain $x + y = (x + y)^{[p]^{um}} = x^{[p]^{um}} + y^{[p]^{um}}$ and $x^{[p]^{um}} = x$ since $x^{[p]^m} = x$. Hence $y^{[p]^{um}} = y$, i.e., y is a quasi-toral element.

Remark 2.1. (1) By means of Theorem 2.1, if x and y are quasi-toral elements such that $[x,\ y] = 0$, then $x + y$ is quasi-toral.

(2) By virtue of the remark in [5, p.1480], a quasi-toral subalgebra of a restricted subalgebra is not necessarily a p-subalgebra.

Theorem 2.2. *Let $(L,[p])$ be a finite-dimensional restricted Lie algebra over a finite field* $\mathbf{F}$ *satisfying the requirement* $x^{[p]^{n(x)}}=x$ *for any* $x\in L$, *where* $n(x)\in N$. *Then there is* $n\in N$ *such that* $x^{[p]^n}=x$ *for any* $x\in L$.

Proof. By virtue of our present assumption, the number of elements in the Lie algebra is finite. Let $x^{[p]^{n(x)}}=x$ and $y^{[p]^{n(y)}}=y$. Obviously, if z is quasi-toral, then $z^{[p]^{k\cdot n(z)}}=z$ for any $k\in N$. So $x^{[p]^{n(x)\cdot n(y)}}=x$ and $y^{[p]^{n(y)\cdot n(x)}}=y$. Let $m=n(x)\cdot n(y)$. Then $x^{[p]^m}=x$ and $y^{[p]^m}=y$. Using induction on the number of elements in the Lie algebra, we obtain the desired result.

We will give some sufficient conditions for the commutativity of quasi-toral restricted Lie algebras.

Theorem 2.3. *Let $(L,[p])$ be a quasi-toral restricted Lie algebra over* $\mathbf{F}$. *If there is* $m\in N$ *such that* $L_{(m)}:=[L^{[p]^m},\ L^{[p]^m}]=\{0\}$, *then the following statements hold:*

(1) *L is abelian;*

(2) $L^{[p]}=L$. *In particular, if* $e_1,\ e_2,\ \cdots,\ e_n$ *is a basis of* L, *then* $e_1^{[p]},\ e_2^{[p]},\ \cdots,\ e_n^{[p]}$ *is a basis of* L.

Proof. (1) Since there is $m\in N$ such that $L_{(m)}=[L^{[p]^m},L^{[p]^m}]=\{0\}$, if $x^{[p]^{n(x)}}=x$ for all $x\in L$, then $x^{[p]^{tn(x)}}=(\cdots(x^{[p]^{n(x)}})\cdots)^{[p]^{n(x)}}=x$ for all $x\in L$. As there is $k\in N$ such that $1\le n(x)\le k$, $x^{[p]^{k!}}=x$ for all $x\in L$.

(i) If $m\le k!$, then $[L^{[p]^{k!}},L^{[p]^{k!}}]=\{0\}$ by $L_{(m)}=[L^{[p]^m},L^{[p]^m}]=\{0\}$.

Since $1\le n(x)\le k$ and $x^{[p]^{k!}}=x$ for all $x\in L$, $[L,L]=\{0\}$, i.e., L is abelian.

(ii) If $m>k!$, then there exist u,v such that $m=uk!+v,\ 0\le v<k!$.

Since $1\le n(x)\le k$ and $x^{[p]^{k!}}=x$ for all $x\in L$, $L^{[p]^{k!}}=L$, $\{0\}=L_{(m)}=[L^{[p]^m},L^{[p]^m}]=[L^{[p]^{uk!+v}},L^{[p]^{uk!+v}}]=[L^{[p]^v},L^{[p]^v}]$. Since $0\le v<k!$, we have $[L,L]=\{0\}$ by virtue of (i), i.e., L is abelian.

(2) Let x be a nonzero element of L. Since x is quasi-toral, $\alpha x=(\alpha x)^{[p]^{n(x)}}=\alpha^{p^{n(x)}}x^{[p]^{n(x)}}=\alpha^{p^{n(x)}}x$, i.e., $\alpha^{n(x)}=\alpha$ for all $\alpha\in\mathbf{F}$. So $\mathbf{F}$ is perfect. Obviously, $[p]$ on L is nonsingular. By virtue of Lemma 1.5, $[p]$ is injective and surjective. Hence $L^{[p]}=L$.

Suppose that $\sum c_ie_i^{[p]}=0$ where $c_i\in\mathbf{F}\,(1\le i\le n)$. Since L is abelian and $\mathbf{F}$ is perfect, there is $d_i\in\mathbf{F}$ such that $d_i^p=c_i$ for $1\le i\le n$ and $0=\sum d_i^pe_i^{[p]}=(\sum d_ie_i)^{[p]}$. By the preceding proof, $\sum d_ie_i=0$ and $d_i=0$ for $1\le i\le n$. Thus $c_i=0$ and $e_1^{[p]},\ e_2^{[p]},\ \cdots,\ e_n^{[p]}$ is a basis of L.

Lemma 2.1. *Let $(L,[p])$ be a quasi-toral restricted Lie algebra over* $\mathbf{F}$. *Then L is a torus algebra.*

Proof. Since L is quasi-toral, there is $n(x)\in N$ such that $x^{[p]^{n(x)}}=x$ for all $x\in L$. Then $(\mathrm{ad}x)^{p^{n(x)}}=\mathrm{ad}x$. Let $m_x(X)\in F[X]$ be the minimum polynomial of $\mathrm{ad}x$. So there is $f(X)\in F[X]$ such that $f(X)\cdot m_x(X)=X^{p^{n(x)}}-X$. Taking the derivative we obtain $f(X)'m_x(X)+f(X)m_x(X)'=-1$ since $\mathbf{F}$ is of characteristic p, which means $\mathrm{ad}x$ is semisimple for any $x\in L$. Hence L is a torus algebra.

Theorem 2.4. *Let $(L,[p])$ be a quasi-toral restricted Lie algebra over* $\mathbf{F}$. *Then L is solvable if and only if L is abelian.*

Proof. $\Leftarrow$. It is obvious.

$\Rightarrow$. Let J be an ideal of L such that $[J, J] \subseteq C(L)$. For any $x \in J$ we have $(\mathrm{ad}x)^3 = 0$. By virtue of Lemma 2.1, we therefore have $\mathrm{ad}x = 0$, then $J \subseteq C(L)$, i.e., if J is an ideal of L such that $[J, J] \subseteq C(L)$, then $J \subseteq C(L)$. Since L is solvable, there is $k \in N$ such that $L^{(k)} = [L^{(k-1)}, L^{(k-1)}] = \{0\}$. Obviously, $L^{(m)}$ is an ideal of L for any $1 \leq m \leq k-1$. Since $[L^{(k-1)}, L^{(k-1)}] = \{0\}$, $L^{(k-1)} = [L^{(k-2)}, L^{(k-2)}] \subseteq C(L)$. Then $L^{(k-2)} \subseteq C(L)$, i.e., $L^{(k-2)} = [L^{(k-3)}, L^{(k-3)}] \subseteq C(L)$. So $L^{(k-3)} \subseteq C(L)$. Using the same methods, we have $L^{(0)} \subseteq C(L)$, i.e., L is abelian.

Corollary 2.1. *Let $(L, [p])$ be a finite-dimensional solvable restricted Lie algebra over $\mathbf{F}$. If L is not abelian, then there is $x \in L$ such that x is not a quasi-toral element of L.*

Proof. If every element $x \in L$ is a quasi-toral element, then L is abelian since L is solvable by virtue of Theorem 2.4. We have arrived at a contradiction since L is not abelian.

Remark 2.2. It is well known that if $(L, [p])$ is a finite-dimensional restricted Lie algebra over $\mathbf{F}$, then for all $x \in L$ there is $k \in N$ such that $x^{[p]^k}$ is semisimple. But the theorem does not hold for quasi-toral elements by Corollary 2.1, i.e., if $(L, [p])$ is a finite-dimensional restricted Lie algebra over $\mathbf{F}$, then there is $x \in L$ for any $k \in N$ such that $x^{[p]^k}$ is not quasi-toral.

Corollary 2.2. *Let $(L, [p])$ be a nonsolvable quasi-toral restricted Lie algebra over $\mathbf{F}$. Then $C(L)$ is a maximal solvable ideal of L and $L/C(L)$ is semisimple.*

Proof. Let A be an abelian ideal of L. For any $x \in A$, $y \in L$, we have $[x^{[p]}, y] = (\mathrm{ad}x)^p(y) = [x,\ \cdots,\ [x,\ [x,\ y]],\ \cdots,\ ]$. Since A is an abelian ideal of L, we have $[x, y] \in A$ and $[x[x,\ y]] = 0$. Then $[x^{[p]}, y] = 0$. As $x^{[p]^{n(x)}} = x$ for any $x \in L$, we have $[x, y] = [x^{[p]^{n(x)}}, y] = 0$. So $A \subseteq C(L)$. If A is a maximal abelian ideal of L, then $C(L) \subseteq A$. Hence $A = C(L)$. Obviously, $C(L)$ is restricted. By virtue of Theorem 2.4, $C(L)$ is a maximal solvable ideal of L, then $L/C(L)$ is semisimple.

Corollary 2.3. *Let $(L, [p])$ be a quasi-toral restricted Lie algebra over $\mathbf{F}$. If H is a Cartan subalgebra of L such that each* $\mathrm{ad}h$ *$(h \in H)$ has all its characteristic roots in $\mathbf{F}$, then L is not semisimple.*

Proof. Since H is a Cartan subalgebra of L such that each $\mathrm{ad}h$ $(h \in H)$ has all its characteristic roots in F, H is a maximal abelian subalgebra of L by virtue of Theorem 2.4 and $L = H \bigoplus \sum L_\phi,\ \phi \in \triangle$, where $\triangle$ is a root system of L with respect to H.

If d is the dimension of L_ϕ, then for any $x \in L_\phi,\ y \in H$, $(\mathrm{ad}y - \phi(y)I)^d x = 0$. Choose k such that $p^k \geq d$; then $(\mathrm{ad}y)^{p^{nk}} x - \phi(y)^{p^{nk}} x = 0$ since $y^{[p]^{nk}} = y$.

By virtue of the proof of Theorem 2.3(2), $\mathbf{F}$ is perfect. So there is $\phi(h) = \phi(y)^{p^{nk}}$. Since $y^{[p]^{nk}} = y$, we have $[y,\ x] = \phi(h)x$. For any $x \in L_\phi,\ y \in H$, $[x,\ y] = [x^{[p]^{nk}},\ y] = 0$. Then we have $[H,\ L] = [H,\ H \bigoplus \sum L_\phi] = \{0\}$. Hence H is an abelian ideal of L. We obtain the desired result.

The following theorem will generalize Rolf Farnstener's result.

Theorem 2.5. *Let $(L, [p])$ be a restricted Lie algebra over $\mathbf{F}$ such that* $\mathrm{card}\,(\mathbf{F}) \geq \mathrm{p}^{\mathrm{n}}$.

If L satisfies the requirement $x^{[p]^n} = \beta x$ for any $x \in L$, $\beta \in \mathbf{F}$ fixed and $\neq 0$, then L is abelian.

Proof. Since $x^{[p]^n} = \beta x$ for any $x \in L$, $\alpha x = (\alpha x)^{[p]^n} = \alpha^{p^n} \beta x$. Then for any $\alpha \in \mathbf{F}$, we have $\alpha^{p^n} = \alpha$. Since $(\alpha x + y)^{[p]^n} = \beta(\alpha x + y) = \alpha^{p^n} \beta x + \beta y = (\alpha x)^{[p]^n} + y^{[p]^n}$, $[p]^n$ is a p-semilinear mapping. By virtue of Lemma 1.1, we have $L^{[p]^n} \subseteq C(L)$, i.e., L is abelian since $x^{[p]^n} = \beta x$.

By Theorem 2.5, we obtain some sufficient conditions for the commutativity of restricted Lie algebras.

Corollary 2.4. *Every restricted Lie algebra $(L, [p])$ over an infinite field $\mathbf{F}$ satisfying the requirement $x^{[p]^{n(x)}} = \alpha x$, $\alpha \in \mathbf{F}$ fixed and $\neq 0$, for any $x \in L$, is abelian.*

Corollary 2.5. *Every restricted Lie algebra $(L, [p])$ over $\mathbf{F}$ satisfying the requirement $x^{[p]} = \alpha x$, $\alpha \in \mathbf{F}$ fixed and $\neq 0$, for any $x \in L$, is abelian.*

Corollary 2.6. *Every restricted Lie algebra $(L, [p])$ over $\mathbf{F}$ satisfying the requirement $x^{[p]^2} = x$, for any $x \in L$, is abelian.*

Proof. If there is a nonzero element u of L such that $x + x^{[p]} = u$, it is clear that $u^{[p]} = u$ by a routine computation. Since $\mathbf{F}_p$ is contained in $\mathbf{F}$, u lies centrally in L by virtue of Lemma 1.2. Since u is not zero, we have $C(L) \neq 0$. If L is not abelian, then L is nonsolvable and $\overline{L} = L/C(L) \neq \{0\}$ by Theorem 2.4. By Corollary 2.2, $\overline{L} = L/C(L)$ is semisimple. Since $y^{[p]^2} = y$ for any $y \in L$, $(y + C(L))^{[p]^2} = y + C(L)$, i.e., $\bar{y}^{[p]^2} = \bar{y}$ and $\bar{y} + \bar{y}^{[p]} = 0$ since $x + x^{[p]} = u$. Then $\bar{y}^{[p]} = -\bar{y}$ for any $\bar{y} \in \overline{L}$. Owing to Corollary 2.5, $\overline{L} = L/C(L)$ is abelian. We have arrived at a contradiction. Hence L is abelian.

If for any $x \in L$, $x + x^{[p]} = 0$ holds, i.e., $x^{[p]} = -x$ for any $x \in L$, then L is abelian by means of Corollary 2.5. We obtain the desired result.

Theorem 2.6. *Let $(L, [p])$ be a four-dimensional restricted Lie algebra over $\mathbf{F}$ such that* (i) *$x^{[p]^{n(x)}} = x$ for any $x \in L$,* (ii) *there is a nonzero element u of L such that $x + x^{[p]} + \cdots + x^{[p]^{n(x)-1}} = u$. Then L is abelian.*

Proof. Owing to Theorem 2.4, the statement holds for 1-dimensional and 2-dimensional restricted Lie algebra over $\mathbf{F}$. It is known that 3-dimensional restricted Lie algebra is simple or solvable (cf. [6, p.34]). By virtue of Lemma 1.3, 3-dimensional restricted Lie algebra such that $x^{[p]^{n(x)}} = x$ for any $x \in L$ is solvable. Owing to Theorem 2.4, 3-dimensional restricted Lie algebra such that $x^{[p]^{n(x)}} = x$ for any $x \in L$ is abelian.

Since there is a nonzero element u of L such that $x + x^{[p]} + \cdots + x^{[p]^{n-1}} = u$, it is clear that $u^{[p]} = u$ by a routine computation. Since F_p is contained in F, u lies centrally in L by virtue of Lemma 1.1. Since u is not zero, we have $C(L) \neq 0$. If L is not abelian, then L is nonsolvable and $\overline{L} = L/C(L) \neq \{0\}$ by Theorem 2.4. By Corollary 2.2, $\overline{L} = L/C(L)$ is semisimple. Since $y^{[p]^{n(y)}} = y$ for any $y \in L$, $(y + C(L))^{[p]^{n(y)}} = y + C(L)$, i.e., $\bar{y}^{[p]^{n(y)}} = \bar{y}$ for any $\bar{y} \in \overline{L}$ and $\dim(L/C(L)) \leq 3$, $L/C(L)$ is abelian. we have arrived at a contradiction. As a result, L is ablian.

Remark 2.3. There are no restricted Lie algebras $(L, [p])$ over $\mathbf{F}$ satisfying the requirement $x^{[p]^2} = -x$ for any $x \in L$.

Proof. If there is a nonzero element u of L such that $x+x^{[p]} = u$, it is clear that $u^{[p]} = u$ by a routine computation. Then there is a nonzero element u of L such that $u^{[p]^2} = u$. This contradicts our present assumption. If $x+x^{[p]} = 0$ for any $x \in L$, then $x^{[p]} = -x$. Obviously, $x^{[p]^2} = x$ for any $x \in L$. This also contradicts our present assumption. We obtain the desired result.

Owing to Theorem 2.5, every finite-dimensional quasi-toral restricted Lie algebra over an infinite field $\mathbf{F}$ is abelian. The following major results illustrate some properties on quasi-toral restricted Lie algebras over a finite field $\mathbf{F}$. The following $\mathbf{F}$ is assumed to be a finite field of positive characteristic $p \geq 3$.

Theorem 2.7. *Let $(L, [p])$ be a minimal-dimensional restricted Lie algebra over $\mathbf{F}$ such that* (i) *L is quasi-toral and* (ii) *$C(L) = \{0\}$. If L is not simple. Then the following statements hold:*

(1) *Let J be a proper ideal of L. Then J is characteristic and semisimple, and L/J is abelian.*

(2) *Let J be a maximal ideal of L. Then* codimension $J = 1$.

(3) $[L, L]$ *is simple and* $\mathrm{Der}_{\mathbf{F}}[L, L]$ *is simple complete.*

(4) *L has more than one proper ideals.*

Proof. (1) Assume that H is a proper restricted ideal of L. Since L is a minimal-dimensional restricted Lie algebra over $\mathbf{F}$ such that (i) and (ii) hold, $C(H) \neq \{0\}$. By virtue of Corollary 2.2, $\overline{H} := H/C(H)$ is semisimple, i.e., the central of $\overline{H}$ is zero. Let $\bar{x} \in \overline{H} = H/C(H)$. Since $x^{[p]^{n(x)}} = x$ for any $x \in L$, where $n(x) \in N$, $(x + C(H))^{[p]^{n(x)}} = x + C(H)$, i.e., $\bar{x}^{[p]^{n(x)}} = \bar{x}$ for any $\bar{x} \in \overline{L}$. Then $\overline{H} = H/C(H) \neq \{0\}$ such that (i) and (ii) hold. Since $\mathrm{dim}\overline{H} < \mathrm{dim}L$, this contradicts the choice of $\mathrm{dim}L$. Consequently, L has not any proper restricted ideal.

Let J be a proper ideal of L. Since J is not restricted, there is $x_1 \in J$ such that $x_1^{[p]} \notin J$. Since $[J \dot{+}\, Fx_1^{[p]}, L] \subseteq J \subseteq J \dot{+}\, Fx_1^{[p]}$, $J \dot{+}\, Fx_1^{[p]}$ is an ideal of L. If $J \dot{+}\, Fx_1^{[p]} \neq L$, then $J \dot{+}\, Fx_1^{[p]}$ is not restricted, i.e., there is $x_2 \in J \dot{+}\, Fx_1^{[p]}$ such that $x_2^{[p]} \notin J \dot{+}\, Fx_1^{[p]}$. Then $J \dot{+}\, Fx_1^{[p]} \dot{+}\, Fx_2^{[p]}$ is an ideal of L. By using the same methods, there are $x_1 \in J$, $x_2 \in J, \cdots,$ $x_n \in J$ such that $x_1^{[p]} \notin J$, $x_2^{[p]} \notin J, \cdots, x_n^{[p]} \notin J$ and $L = J \dot{+}\, Fx_1^{[p]} \dot{+}\, Fx_2^{[p]} \dot{+} \cdots \dot{+}\, Fx_n^{[p]}$ since L is finite dimensional. By a routine computation, we obtain $[L, L] \subseteq J$. Hence L/J is abelian.

Let $D \in \mathrm{Der}L$ and $a, x \in L$. If A is the transformation $x \to [a, x]$ and B is the transformation $x \to [D(a), x]$, then $A = \mathrm{ad}a$, $B = \mathrm{ad}(D(a))$. We can prove $(\mathrm{ad}A)^k(B) = \sum_{i=0}^{k}(-1)^{k-i}C_k^i A^i B A^{k-i}$ by induction on k.

Then by the result, we have

$$(\mathrm{ad}A)^{p-1}(B) = \sum_{i=0}^{p-1}(-1)^{p-1-i}C_{p-1}^i A^i B A^{p-1-i}.$$

Since

$$C_{p-1}^{i}=\frac{(p-1)(p-2)\cdots(p-i)}{i\cdot(i-1)\cdot\cdots\cdot 1}=\frac{(-1)(-2)\cdots(-i)}{i\cdot(i-1)\cdot\cdots\cdot 1}=(-1)^{i},$$

we have $(-1)^{p-1-i}C_{p-1}^{i}=(-1)^{p-1}=1$. So $BA^{p-1}+ABA^{p-2}+\cdots+A^{p-1}B=[A,\cdots[A,B]\cdots]$.

Then

$$\begin{aligned}D[a^{[p]},x]&=D[a,\cdots[a,x]\cdots]\\&=[D(a),\cdots[a,x]\cdots]+\cdots+[a,\cdots[a,D(x)]\cdots]\\&=[a^{[p]},D(x)]+[a,a\cdots[a,D(a)],\cdots x].\end{aligned}$$

On the other hand, we have $D[a^{[p]},x]=[D(a^{[p]}),x]+[a^{[p]},D(x)]$ since D is a derivation. Hence $[D(a^{[p]}),x]=[a,a\cdots[a,D(a)],\cdots x]$ for all $x\in L$. Since $C(L)=\{0\}$, we obtain $(\mathrm{ad}a)^{p-1}(D(a))-D(a^{[p]})=0$. Thus $(\mathrm{ad}a)^{p-1}(D(a))=D(a^{[p]})$ and every $D\in\mathrm{Der}L$ is a restricted derivations of L. Let J be a proper ideal of L.

If $D\in\mathrm{Der}L,\ a\in J$, for all $n(a)\in N$, then $D(a^{[p]^{n(a)}})=(\mathrm{ad}a)^{p^{n(a)}-1}(D(a))=[a,\cdots[a,[a,D(a)]],\cdots]\in J$. So we can obtain $D(J)\subseteq J$, i.e., J is a characteristic ideal of L.

Let J be a proper ideal of L and I be an abelian ideal of J. Then there are $x_1\in J,\ x_2\in J,\ \cdots,\ x_n\in J$ such that $x_1^{[p]}\notin J,\ x_2^{[p]}\notin J,\ \cdots,\ x_n^{[p]}\notin J$ and $L=J\dotplus Fx_1^{[p]}\dotplus Fx_2^{[p]}\dotplus\cdots\dotplus Fx_n^{[p]}$. So $[I,\ L]=[I,\ J\dotplus Fx_1^{[p]}\dotplus Fx_2^{[p]}\dotplus\cdots\dotplus Fx_n^{[p]}]\subseteq I$, i.e., I is also an abelian ideal of L. By means of Corollary 2.2, $C(L)$ is a maximal abelian ideal of L. Then $I=C(L)=\{0\}$. Hence J is semisimple.

(2) Let J be a maximal ideal of L. Since J is not a proper restricted ideal of L, there is $x\in J$ such that $x^{[p]}\notin J$. According to $\mathrm{ad}x^{[p]}(J)\subseteq(\mathrm{ad}x)^{p}(J)=[x,\cdots,[x,[x,J]],\cdots]\subseteq J$, we obtain $[Fx^{[p]}\dotplus J,\ Fx^{[p]}\dotplus J]\subseteq J\subseteq Fx^{[p]}\dotplus J$. Then $Fx^{[p]}\dotplus J$ is a subalgebra of L. As $[Fx^{[p]}\dotplus J,\ L]\subseteq J$, $Fx^{[p]}\dotplus J$ is a nontrivial ideal of L. If J is a maximal ideal of L, then $L=Fx^{[p]}\dotplus J$. So $\dim J=\dim L-1=n-1$, i.e., codimension$J=1$.

(3) Due to the proof of (1), $[L,L]$ is a minimal proper nonrestricted ideal of L. Let I be a nonzero ideal of $[L,L]$. Then there are $x_1\in[L,L],x_2\in[L,L],\cdots,x_n\in[L,L]$ such that $x_1^{[p]}\notin[L,L],x_2^{[p]}\notin[L,L],\cdots,x_n^{[p]}\notin[L,L]$ and $L=[L,L]\dotplus Fx_1^{[p]}\dotplus Fx_2^{[p]}\dotplus\cdots\dotplus Fx_n^{[p]}$.

So $[I,L]=[I,[L,L]\dotplus Fx_1^{[p]}\dotplus Fx_2^{[p]}\dotplus\cdots\dotplus Fx_n^{[p]}]\subseteq I$, i.e., I is an ideal of L. According to the proof of (1), we obtain $I\supseteq[L,L]$ and $I=[L,L]$. As a result, $[L,L]$ is simple. In the light of this (cf. [9]), it is obvious that $\mathrm{Der}_{\mathbf{F}}[L,L]$ is simple complete.

(4) By virtue of (1) and (3), $[L,L]$ is a minimal simple proper ideal of L. If L has a unique proper ideal, then $[L,L]$ is uniquely an ideal of L. By means of (2), codimension$([L,L])=1$. So there exists $x\notin[L,L]$ such that $L=[L,L]\dotplus Fx$. Hence $[L,L]=[[L,L]\dotplus Fx,[L,L]\dotplus Fx]=L^{(2)}\dotplus[Fx,[L,L]]$. Since $[L,L]$ is simple, $L^{(1)}=L^{(2)}$ and $[Fx,[L,L]]=\{0\}$. Then $[Fx,L]=[Fx,[L,L]\dotplus Fx]=\{0\}$. Thus $Fx\subseteq C(L)$ and we have arrived at a contradiction since $C(L)=\{0\}$. Hence L has necessarily more than one proper ideals.

Theorem 2.8. *Let* $(L,[p])$ *be a minimal-dimensional restricted Lie algebra over* $\mathbf{F}$ *such that* (i) L *is quasi-toral and* (ii) $C(L)=\{0\}$*. If* L *is simple, then the following statements hold:*

(1) *All proper subalgebras of L are abelian.*

(2) $D \in \mathrm{Der}_{\mathbf{F}} L$ *is semisimple.*

(3) *If H is a maximal proper subalgebra of L, then H is a Cartan subalgebra of L.*

(4) *If H is a maximal proper subalgebra of L, then $L = H + [H,\ L]$.*

Proof. (1) Let H be a maximal subalgebra of L. If H is not a restricted subalgebra of L, then there is $x \in H$ such that $x^{[p]} \notin H$. Since H is a maximal subalgebra of L, we have $L = \langle x^{[p]}, H\rangle$.

For any $h \in H$, we have $[x^{[p]}, h] = (\mathrm{ad}x)^p(h) = [x, \cdots, [x, [x, h]], \cdots] \in H$. So $[Fx^{[p]} \dot{+} H,\ Fx^{[p]} \dot{+} H] \subseteq H \subseteq Fx^{[p]} \dot{+} H$. Then $Fx^{[p]} \dot{+} H$ is a subalgebra of L. According to the maximality of H, we have $L = \langle x^{[p]}, H\rangle = Fx^{[p]} \dot{+} H$. As $[H, L] = [H, Fx^{[p]} \dot{+} H] \subseteq H$, H is a nonzero maximal ideal of L. We have arrived at a contradiction since L has not nontrivial ideal. Hence H is a restricted subalgebra of L.

We claim that all the proper restricted subalgebras of L are abelian. Let J be a proper restricted subalgebra of L. Then J is a quasi-toral restricted subalgebra of L. If J is nonabelian, then J is nonsolvable by Theorem 2.4. Since J is a restricted proper subalgebra of L and L is minimal-dimensional such that (i) and (ii) hold, $C(J) \neq \{0\}$. According to Corollary 2.2, $J/C(J) \neq \{0\}$ is semisimple. Then the central of $J/C(J) \neq \{0\}$ is zero.

Let $\bar{x} \in \overline{J} = J/C(J)$. Since $x^{[p]^{n(x)}} = x$ for any $x \in L$, where $n(x) \in N$, $(x+C(J))^{[p]^{n(x)}} = x + C(J)$, i.e., $\bar{x}^{[p]^{n(x)}} = \bar{x}$ for any $\bar{x} \in \overline{L}$, where $n(x) \in N$. So $\overline{J} = J/C(J) \neq \{0\}$ satisfies $\bar{x}^{[p]^{n(x)}} = \bar{x}$ for any $\bar{x} \in \overline{L}$, where $n(x) \in N$, and the central of $J/C(J)$ is zero. Since $\dim\overline{J} < \dim L$, this contradicts the choice of $\dim L$. Hence J is abelian.

We claim that all proper nonrestricted subalgebra of L are abelian. Let M be a proper nonrestricted subalgebra of L. Then M is not a maximal subalgebra of L since all maximal subalgebras of L is a restricted subalgebra of L. So there exists $x_1 \in M$ such that $x_1^{[p]} \notin M$. Let $M_1 = M \dot{+} Fx_1^{[p]}$. It is clear that M_1 is a subalgebra of L. If M_1 is restricted, then M_1 is abelian by the above proof. So it is clear that M is abelian. If M_1 is nonrestricted, then there exists x_2 of M such that $x_2^{[p]} \notin M$. Let $M_2 = M_1 \dot{+}\ Fx_2^{[p]}$. Obviously, M_2 is a subalgebra of L. Using the same methods, there exist $x_1, x_2, \cdots, x_k \in M$ such that $x_1^{[p]} \notin M$, $x_2^{[p]} \notin M, \cdots, x_k^{[p]} \notin M$ and $M_k = M \dot{+}\ Fx_1^{[p]} \dot{+}\ , \cdots, \dot{+}\ Fx_k^{[p]}$ is a maximal subalgebra of L if $M_{k-1} = M \dot{+}\ Fx_1^{[p]} \dot{+}\ , \cdots, \dot{+}\ Fx_{k-1}^{[p]}$ is nonrestricted for some $k \in N$. As every maximal subalgebra is restricted, M_k is also restricted. So M_k is abelian. It is clear that M is abelian.

Thus all proper subalgebras of L are abelian.

(2) Let $D \in \mathrm{Der}_{\mathbf{F}}(L)$ be a nonzero nilpotent derivation of L and let $D^n = 0$, $D^{n-1} \neq 0$. Let $V_m := \ker D^m,\ 1 \leq m \leq n$.

Since $D^{m+1}(V_{m+1}) = \{0\}$, i.e., $D^m(D(V_{m+1})) = \{0\}$, $D(V_{m+1}) \subseteq V_m$.

Obviously, $V_1\ \subseteq V_2 \subseteq \cdots \subseteq V_n = L$.

We will show that for all $m \in N$ $(1 \leq m \leq n-1)$, V_m is a proper subalgebra of L by induction on m.

For the case $m = 1$.

Since $D[V_1, V_1] = [D(V_1), V_1] + [V_1, D(V_1)]$ and $V_1 = \ker D$, $D[V_1, V_1] = \{0\}$. Then $[V_1, V_1] \subseteq \ker D = V_1$, i.e., V_1 is a proper subalgebra of L. So $[V_1, V_1] = \{0\}$ by means of (1).

Suppose that the statement holds for the case $m = k$, i.e., V_k is a proper subalgebra of L of L for any $k \in N$, then $[V_k, V_k] = \{0\}$ by virtue of (1).

For the case $m = k+1$, we obtain

$$\begin{aligned} & D^{k+1}[V_{k+1},\ V_{k+1}] \\ &= \sum_{i=0}^{k+1} C_{k+1}^i [D^i(V_{k+1}),\ D^{k+1-i}(V_{k+1})] \\ &= \sum_{i=1}^{k} C_{k+1}^i [D^i(V_{k+1}),\ D^{k+1-i}(V_{k+1})] + [D^{k+1}(V_{k+1}),\ V_{k+1}] + [V_{k+1},\ D^{k+1}(V_{k+1})]. \end{aligned}$$

Since $D(V_{m+1}) \subseteq V_m$ and $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = L$, $D^i(V_{k+1}) \subseteq V_{k+1-i}$ and D^{k+1-i} $(V_{k+1}) \subseteq V_i$.

Then

$$\sum_{i=1}^{k} C_{k+1}^i [D^i(V_{k+1}), D^{k+1-i}(V_{k+1})] \subseteq \sum_{i=1}^{k} C_{k+1}^i [V_{k+1-i}, V_i] \subseteq \sum_{i=1}^{k} C_{k+1}^i [V_k, V_k] = \{0\}.$$

Since

$$V_{k+1} = \ker D^{k+1},$$

we have

$$[D^{k+1}(V_{k+1}),\ V_{k+1}] + [V_{k+1},\ D^{k+1}(V_{k+1})] = \{0\}.$$

So

$$\begin{aligned} D^{k+1}[V_{k+1},\ V_{k+1}] &= \sum_{i=1}^{k} C_{k+1}^i [D^i(V_{k+1}),\ D^{k+1-i}(V_{k+1})] \\ &\quad + [D^{k+1}(V_{k+1}),\ V_{k+1}] + [V_{k+1},\ D^{k+1}(V_{k+1})] \\ &= \{0\}. \end{aligned}$$

It is clear that $D^{k+1}[V_{k+1},\ V_{k+1}] = \{0\}$, i.e., $[V_{k+1},\ V_{k+1}] \subseteq \mathrm{Ker} D^{k+1} = V_{k+1}$. So V_{k+1} is a proper subalgebra of L and $[V_{k+1},\ V_{k+1}] = \{0\}$ by means of (1). Thus we have proved that V_m is a proper subalgebra of L for any $m \in N$ $(1 \le m \le n-1)$.

It is obvious that $V_{n-1} \subset L$ is a proper subalgebra of L and $[V_{n-1}, V_{n-1}] = \{0\}$. Since $V_n = L$ and $D(V_n) \subseteq V_{n-1}$,

$$[D(L), D(L)] \subseteq [V_{n-1}, V_{n-1}] = \{0\}.$$

Since

$$\sum_{i=1}^{m-1} C_m^i [D^i(L), D^{m-i}(L)] \subseteq \sum_{i=1}^{m-1} C_m^i [D(L), D(L)] = \{0\},$$

we have

$$\begin{aligned}
D^m[L,L] &= \sum_{i=0}^{m} C_m^i[D^i(L), D^{m-i}(L)] \\
&= \sum_{i=1}^{m-1} C_m^i[D^i(L), D^{m-i}(L)] + [D^m(L), L] + [L, D^m(L)] \\
&= [D^m(L), L] + [L, D^m(L)].
\end{aligned}$$

So D^m is a derivation of L for any $1 \leq m \leq n-1$.

As

$$\begin{aligned}
D^{n-1}[V_{n-1}, L] &= \sum_{i=0}^{n-1} C_{n-1}^i[D^i(V_{n-1}), D^{n-1-i}(L)] \\
&= \sum_{i=1}^{n-2} C_{n-1}^i[D^i(V_{n-1}), D^{n-1-i}(L)] \\
&\quad + [D^{n-1}(V_{n-1}), L] + [V_{n-1}, D^{n-1}(L)]
\end{aligned}$$

and

$$\sum_{i=1}^{n-2} C_{n-1}^i[D^i(V_{n-1}), D^{n-1-i}(L)] \subseteq \sum_{i=1}^{n-2} C_{n-1}^i[D(L), D(L)] = \{0\},$$

we have $D^{n-1}[V_{n-1}, L] = \{0\}$ since $V_{n-1} = \ker D^{n-1}$, i.e., $[V_{n-1}, L] \subseteq \ker D^{n-1} = V_{n-1}$. Hence $V_{n-1} \neq L$ is an ideal of L. Then $V_{n-1} = \ker D^{n-1} = \{0\}$ since L is simple. But V_{n-1} is a proper subalgebra of L. We have arrived at a contradiction. Thus $\mathrm{Der}_{\mathrm{F}} L$ does not contain any nonzero nilpotent elements. By virtue of Jordan-Chevalley decomposition Theorem, $D \in \mathrm{Der}_{\mathrm{F}} L$ is semisimple.

(3) Let H be a maximal proper subalgebra of L. It is clear that H is a maximal abelian subalgebra of L by means of (1). Let $\mathrm{Nor}_L(H)$ be a normalizer of H in L. Suppose that $x \in \mathrm{Nor}_L(H)$ and $x \notin H$. Since $y^{[p]^{n(y)}} = y$ for any $y \in H$,

$$[y, x] = [y^{[p]^{n(y)}}, x] = [y, [y, \cdots, [y, x], \cdots]] = 0.$$

Then $[x, H] = \{0\}$. So $[H \dotplus Fx, H \dotplus Fx] = \{0\}$, i.e., $H \dotplus Fx$ is a subalgebra of L. Then $L = Fx \dotplus H$ by the maximality of H. Hence

$$[Fx, L] = [H \dotplus Fx, Fx] = \{0\}, i.e., Fx \subseteq C(L).$$

This contradicts $C(L) = \{0\}$. So

$$H \supseteq \mathrm{Nor}_L(H).$$

It is clear that

$$H \subseteq \mathrm{Nor}_L(H).$$

Then

$$\mathrm{Nor}_L(H) = H.$$

Therefore, H is a Cartan subalgebra of L.

(4) If H is a maximal proper subalgebra of L, then H is a maximal proper abelian restricted subalgebra of L by virtue of (1). Since L is quasi-toral, every element of L is semisimple. Then H is a torus of L. It is clear that L is a H-invariant subspace. So $L = C_L(H) + [H,\ L]$ by means of Lemma 1.4.

Since H is abelian, $H \subseteq C_L(H)$. If there is $x \in C_L(H)$ and $x \notin H$, then $[H \dotplus Fx, H \dotplus Fx] = \{0\}$, i.e., $H \dotplus Fx$ is an abelian restricted subalgebra of L. This contradicts the maximality of H. So $H = C_L(H)$ and we obtain the desired result.

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