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ON THE EMBEDDING OF TOP IN THE CATEGORY OF STRATIFIED L-TOPOLOGICAL SPACES***

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Abstract

Let L be a meet continuous lattice. It is proved that the category Top of topological spaces can be embedded in the category of stratified L-topological spaces as a concretely both reflective and coreflective full subcategory if and only if L is a continuous lattice.

Keywords Continuous lattice, L-topological space, Reflective subcategory, Coreflective subcategory
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§ 1 . Introduction and Preliminaries

Let L be a complete lattice. An L-topology on a set X is a subset τ of L^X closed with respect to finite meets and arbitrary joins. (X, τ) is called an L-topological space. The L-topology τ is called stratified if it contains all the constant maps from X to L, and in this case (X, τ) is called a stratified L-topological space. A continuous map between two L-topological spaces (X, τ_X) and (Y, τ_Y) is a function $f : X \longrightarrow Y$ such that $f^{\leftarrow}(\lambda) = \lambda \circ f \in \tau_X$ for each $\lambda \in \tau_Y$. The category of L-topological spaces is denoted by L-Top and the full subcategory of stratified L-topological spaces is denoted by SL-Top. Both L-Top and SL-Top are topological categories over Set.

Lattice valued topology, or fuzzy topology, is intended to be an extension of classical topology. This makes sense in two quite different way. The first is that by replacing 2 $=\{0, 1\}$ by an arbitrary complete lattice L (sometimes with extra structures) we obtain a new category of a topological nature; and when L reduces to 2, we come back to the classical topology. So every theorem about L-topological spaces is a theorem about classical topological spaces. The second way is much subtler and more interesting. In order to explain it, we recall the Lowen functors at first.

In 1976 Lowen [10] introduced a pair of functors: ω : Top $\longrightarrow$ S$[0,1]$-Top and ι : S$[0,1]$-Top$\longrightarrow$Top as follows: ω maps every topological space $(X, \mathcal{T})$ to $(X, \omega(\mathcal{T}))$, where $\omega(\mathcal{T})$

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denotes the collection of the lower semicontinuous functions from X to $[0,1]$; and the functor ι takes every [0,1]-topological space (X,Δ) to $(X,\iota(\Delta))$, where $\iota(\Delta)$ is the coarsest topology on X making all $\lambda\in\Delta$ lower semicontinuous. These two functors play a prominent role in fuzzy topology, and are called the Lowen functors in the literature. It is well known that ι is a right adjoint of ω and that ω also has a concrete left adjoint. So, ω embeds Top in the category S[0,1]-Top as a simultaneously reflective and coreflective full subcategory (see [13, 10, 14, 16]).

Besides Top, S[0,1]-Top contains other full subcategories which are both concretely reflective and coreflective. The importance of the existence of such subcategories lies in that, as observed by Lowen and Wuyts [14], each such subcategory is closed under the formation of initial and final structures in S[0,1]-Top, hence gives rise to a perfectly viable and natural autonomous theory of fuzzy topology. The best example is the theory for the category of fuzzy neighborhood spaces initiated by Lowen [11, 12]. This phenomenon sharply distinguishes fuzzy topology from classical topology on the categorical level, since the category Top of topological spaces contains no nontrivial such subcategories (see [7]).

Therefore, fuzzy topology should consist of a system of theories corresponding to the both initially and finally closed subcategories of S[0,1]-Top with the classical theory of topology corresponding to the subcategory $\omega(\mathrm{Top})\cong\mathrm{Top}$. This is the second way that fuzzy topology can be regarded as an extension of classical topology. This also explains to some extent why one can find very often in the literature that there are many quite different generalizations of the same concept in topology to the fuzzy setting, they are invented for different subcategories of S[0,1]-Top which are both initially and finally closed. We refer the reader to [17] for an analysis in this spirit of the interrelationship between different notions of uniformity in fuzzy set theory.

Then, a natural question is: For a complete lattice L, is the theory of L-topological spaces an extension of classical topology in the second sense? Clearly, the answer to this question reduces to that whether Top can be embedded in the category SL-Top as a both initially and finally closed full subcategory.

In 1990, replacing the lower semicontinuous functions by Scott continuous functions, Warner [15] generalized the construction of the Lowen functors to the case L is a distributive continuous lattice. In 1992, Kubiak [9], Kotzé and Kubiak [8] generalized the construction to a general setting when L is a complete lattice by considering a suitable topology on L.

Let L be a continuous lattice. Write $\omega_L:\mathrm{Top}\longrightarrow \mathrm{S}L\text{-Top}$ for the functor which maps every topological space $(X,\mathcal{T})$ to $(X,\omega_L(\mathcal{T}))$, where $\omega_L(\mathcal{T})$ denotes the collection of the continuous functions from X to L with respect to the Scott topology. Then ω_L is a full embedding and has simultaneously a concrete left adjoint and a concrete right adjoint. Therefore, Top can be embedded in the category SL-Top as a both initially and finally closed full subcategory.

In this note, we show that if L is a meet continuous lattice, then Top can be embedded in SL-Top as a both initially and finally closed full subcategory if and only if L is continuous.

This note is arranged as follows. In Section 2 we prove a general result about Galois connections between categories of L-topological spaces. This result will be employed in

Section 3 to prove our main result in this note.

We refer to [1] for category theory and to [5] for lattice theory. However, we recall here some basic ideas about continuous lattices.

Let L be a complete lattice and $a, b \in L$. We say that a is way below b, in symbols, $a \ll b$, or $b \gg a$, if for every directed subset $D \subseteq L, \bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. A complete lattice L is said to be continuous if every element in L is the join of all the elements way below it. An upper set U in L is called Scott open if for every directed subset $D \subseteq L, \bigvee D \in U$ implies $U \cap D \neq \emptyset$. All the Scott open sets form a topology on L, called the Scott topology on L, denoted by $\sigma(L)$. It is well known that in a continuous lattice L, $\{\Uparrow a \mid a \in L\}$ is a base for the Scott topology, where $\Uparrow a = \{b \in L \mid b \gg a\}$.

Suppose X is a set and $U \subseteq X, a \in L$. We define a function $a \wedge U : X \longrightarrow L$ by $a \wedge U(x) = a$ if $x \in U$ and $a \wedge U(x) = 0$ if $x \notin U$. Functions of this form will be called one step functions in short.

§ 2. Galois Connections

By a Galois connection between two concrete categories we mean a Galois connection of the third kind introduced in [6] or a Galois correspondence in [1]. Precisely, a Galois connection between two concrete categories $\boldsymbol{A}$ and $\boldsymbol{B}$ is a pair of concrete functors $F : \boldsymbol{A} \longrightarrow \boldsymbol{B}$, $G : \boldsymbol{B} \longrightarrow \boldsymbol{A}$ such that $\{\mathrm{id}_Y : FG(Y) \longrightarrow Y \mid Y \in \boldsymbol{B}\}$ is a natural transformation from the functor $F \circ G$ to the identity functor on $\boldsymbol{B}$ and $\{\mathrm{id}_X : X \longrightarrow GF(X) \mid X \in \boldsymbol{A}\}$ is a natural transformation from the identity functor on $\boldsymbol{A}$ to $G \circ F$. We refer the reader to [1, 6] for more about Galois connections between concrete categories.

Suppose L_1, L_2 are complete lattices and $\Delta \subseteq L_1^{L_2}$ is a stratified L_1-topology on L_2. Given a stratified L_1-topological space (X, τ), let $\omega_\Delta(\tau)$ be the stratified L_2-topology on X generated as a subbase by

$$\{f : (X, \tau) \longrightarrow (L_2, \Delta) \mid f \text{ is continuous}\}.$$

Clearly, in this way we obtain a concrete functor $\omega_\Delta : \mathrm{S}L_1\text{-Top} \longrightarrow \mathrm{S}L_2\text{-Top}$. Conversely, given a stratified L_2-topological space (X, η), let $\iota_\Delta(\eta)$ be the stratified L_1-topology on X generated as a subbase by $\{\delta \circ \lambda \mid \delta \in \Delta, \lambda \in \eta\}$. Then, we obtain a concrete functor $\iota_\Delta : \mathrm{S}L_2\text{-Top} \longrightarrow \mathrm{S}L_1\text{-Top}$.

Theorem 2.1. (cf. [18]) (1) *The pair* $(\omega_\Delta, \iota_\Delta)$ *is a Galois connection if and only if for each stratified* L_1*-topological space* (X, τ), $\omega_\Delta(\tau) = \{f : (X, \tau) \longrightarrow (L_2, \Delta) \mid f$ *is continuous*$\}$, *that is to say, the collection of the continuous functions from* (X, τ) *to* (L_2, Δ) *is already an* L_2*-topology on* X.

(2) *Suppose that F:* $\mathrm{S}L_1\text{-Top} \longrightarrow \mathrm{S}L_2\text{-Top}$, $G : L_2\text{-Top} \longrightarrow L_1\text{-Top}$ *are concrete functors and that* (F, G) *is a Galois connection. Then there is a unique stratified* L_1*-topology* Δ *on* L_2 *such that* $F = \omega_\Delta, G = \iota_\Delta$.

Corollary 2.1. *Suppose that the stratified* L_1*-topology* Δ *on* L_2 *has a subbase consisting of maps which preserve binary meets and non-empty joins. Then* $(\omega_\Delta, \iota_\Delta)$ *is a Galois connection.*

Note 2.1. Suppose Δ is an L_1-topology on L_2. It is possible that $(\omega_\Delta, \iota_\Delta)$ is not a Galois connection, but ω_Δ is the left adjoint part of a Galois connection. For example, let L be a continuous lattice and Γ be a topology on L finer than the Scott topology Σ. If every element in Γ is an upper set, it is routine to check that $\omega_\Sigma = \omega_\Gamma$. Thus, $(\omega_\Gamma, \iota_\Sigma)$ is a Galois connection, but $(\omega_\Gamma, \iota_\Gamma)$ is not whenever $\Gamma \neq \Sigma$ (Theorem 2.1).

Galois connections abound in fuzzy topology, with the Lowen adjunction (ω, ι) (see [10]) being the primary example, and they play an important role in the investigation of the relationship between different categories of L-topological spaces. Several examples were listed in [18]. Here we present some Galois connections between the category of [0, 1]-topological spaces and the category of intuitionistic fuzzy topological spaces which receive attention recently in the literature. Note that Theorem 2.1 is also valid for the Galois connections between categories of L-topological spaces which are not necessarily stratified. Of course, we do not require the L-topology Δ to be stratified in this case.

Example 2.1. Let X be a set. An intuitionistic fuzzy set (see [2]) A of X is a pair (μ_A, γ_A) of ([0, 1]-)fuzzy subsets of X such that $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$, or equivalently, $\mu_A(x) \leq 1 - \gamma_A(x)$ for all $x \in X$. $\mu(x)$ is interpreted as the degree that x has the property A; $\gamma(x)$ as the degree that x does not have the property A; and $1 - \mu(x) - \gamma(x)$ as the 'hesitation degree' (see [4]). Suppose $A = (\mu_A, \gamma_A), B = (\mu_B, \gamma_B)$ are two intuitionistic fuzzy sets of X. Define $A \leq B$ iff $\mu_A \leq \mu_B, \gamma_A \geq \gamma_B$. Then under the partial ordering $\leq$, all the intuitionistic fuzzy sets of X form a completely distributive lattice with an order-reversing involution $(\mu, \gamma) \mapsto (\gamma, \mu)$. An intuitionistic fuzzy topology (see [3]) on X is a subset of intuitionistic fuzzy sets on X which are closed under finite meets and arbitrary joins.

Let $L = \{(a, b) \in [0,1] \times [0,1] | a \leq b\}$. Clearly L is a completely distributive lattice with an order-reversing involution $(a, b) \mapsto (1-b, 1-a)$. For each intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$, let $e(A) \in L^X$ be defined by $e(A)(x) = (\mu_A(x), 1 - \gamma_A(x))$. Then e establishes a lattice isomorphism between the complete lattice of intuitionistic fuzzy sets of X and the complete lattice L^X (see [4]). Thus, an intuitionistic fuzzy topology on X is just an L-topology on X.

Let $f_1, f_2, f_3 : [0,1] \longrightarrow L$ be defined by $f_1(a) = (a, a); f_2(a) = (0, a); f_3(a) = (a, 1)$. Let $\Delta_1 = \{0, f_1, 1\}; \Delta_2 = \{0, f_2, 1\}; \Delta_3 = \{0, f_3, 1\}$. Then all of $(\omega_{\Delta_i}, \iota_{\Delta_i}), i \leq 3$, are Galois connections between the category of [0, 1]-topological spaces and the category of intuitionistic fuzzy topological spaces (= L-topological spaces). And all of $\omega_{\Delta_i}, i \leq 3$, are full embeddings.

It is showed in [18] that if L_1 is a complete Heyting algebra and $(\omega_\Delta, \iota_\Delta)$ is a Galois connection between L_1-Top and L_2-Top, then every function $f \in \Delta$ is necessarily order-preserving. The following question is raised in [18].

Question. Let L_1, L_2 be complete lattices and Δ be an L_1-topology on L_2 such that $(\omega_\Delta, \iota_\Delta)$ is a Galois connection. Then, is every element λ in Δ necessarily a Scott continuous function $L_2 \longrightarrow L_1$?

In the following we will see that the answer to this question is positive if L_1 is meet continuous.

Let $2=\{0,1\}$ be the complete chain of two elements and L be a complete lattice. A function $f:2\longrightarrow L$ can be simply denoted as a pair (a,b) of elements in L, precisely, $f(0)=a, f(1)=b$. Thus, an L-topology on 2 can be regarded as a subset $\Gamma\subseteq L\times L$ closed under finite meets and arbitrary joins.

Let L be a complete lattice and $\Lambda=\{(a,b)\in L\times L\mid a\leq b\}$. Then Λ is a stratified L-topology on 2 and by Corollary 2.1, $(\omega_\Lambda,\iota_\Lambda)$ is a Galois connection between SL-Top and Top. Thus, ω_Λ : SL-Top$\longrightarrow$Top is a left adjoint of ι_Λ : Top $\longrightarrow$ SL-Top. Clearly, for each topological space $(X,\mathcal{T})$, $\iota_\Lambda(\mathcal{T})$ is the stratified L-topology on X generated as a subbase by $\{a\wedge U\mid a\in L, U\in\mathcal{T}\}$. We will write π_L for the functor ι_Λ : Top $\longrightarrow$ SL-Top in the sequel.

Lemma 2.1. *A complete lattice L is meet continuous if and only if for every topological space $(X,\mathcal{T})$, the collection $\{a\wedge U\mid a\in L, U\in\mathcal{T}\}$ is a base for $\pi_L(\mathcal{T})$. In this case, π_L : Top $\longrightarrow$ SL-Top is a right inverse of ω_Λ, and embeds* Top *in* SL-Top *as a concretely reflective full subcategory.*

Proof. $\Leftarrow$. Suppose $D\subseteq L$ is a directed set and $a\in L$. We need only show that $a\wedge\bigvee D\leq\bigvee a\wedge D$.

Without loss of generality, we can assume that D is also a lower set , i.e., $D=\downarrow D$.

Let $X=L$ and $\mathcal{T}$ denote the topology on X generated as a base by $\{\uparrow d\mid d\in D\}$. By assumption, the collection of functions $X\longrightarrow L$ of the form $\bigvee\limits_{t\in T} a_t\wedge U_t$ is an L-topology τ on X, where $a_t\in L, U_t\in\mathcal{T}$. Clearly $f=a\wedge\bigvee\limits_{d\in D}(d\wedge\uparrow d)\in\tau$. Therefore, there exist $\{a_s\mid s\in S\}\subseteq L$ and $\{U_s\mid s\in S\}\subseteq\mathcal{T}$ such that $f=\bigvee\limits_{s\in S} a_s\wedge U_s$.

For each $s\in S$, there is some $d\in D$ such that $\uparrow d\subseteq U_s$, thus

$$a_s\leq a_s\wedge U_s(d)\leq f(d)=a\wedge\bigvee\{y\in D\mid y\in D, y\leq d\}=a\wedge d.$$

Therefore

$$a\wedge\bigvee D=f\big(\bigvee D\big)\leq\bigvee_{s\in S}a_s\leq\bigvee_{d\in D}a\wedge d.$$

$\Rightarrow$. Let $(X,\mathcal{T})$ be a topological space and τ, the collection of functions $X\longrightarrow L$ of the form $\bigvee\limits_{t\in T}a_t\wedge U_t$, is an L-topology on X, where $a_t\in L, U_t\in\mathcal{T}$. We need only show that τ is closed under binary meets.

Suppose $f_1=\bigvee\limits_{s\in S}a_s\wedge U_s\in\tau$ and $f_2=\bigvee\limits_{t\in T}b_t\wedge V_t\in\tau$.

Let $S^{<\omega}=\{F\subseteq S\mid F\text{ is finite}\}$ and $T^{<\omega}=\{G\subseteq T\mid G\text{ is finite}\}$.

Since for all $F\in S^{<\omega}, G\in T^{<\omega}$, we have

$$\bigvee_{s\in F}a_s\wedge U_s=\bigvee_{K\subseteq F}\Big(\Big(\bigvee_{s\in K}a_s\Big)\wedge\bigcap_{s\in K}U_s\Big),\quad \bigvee_{t\in G}b_t\wedge V_t=\bigvee_{H\subseteq G}\Big(\Big(\bigvee_{t\in H}b_t\Big)\wedge\bigcap_{t\in H}V_t\Big),$$

and thus

$$\Big(\bigvee_{s\in F}a_s\wedge U_s\Big)\wedge\Big(\bigvee_{t\in G}b_t\wedge V_t\Big)=\bigvee_{K\subseteq F,H\subseteq G}\Big(\bigvee_{s\in K}a_s\wedge\bigvee_{t\in H}b_t\Big)\wedge\Big(\bigcap_{s\in K}U_s\cap\bigcap_{t\in H}V_t\Big).$$

Therefore, by meet continuity of L,

$$\begin{aligned} f_1 \wedge f_2 &= \Big(\bigvee_{F\in S^{<\omega}} \bigvee_{s\in F} a_s \wedge U_s\Big) \wedge \Big(\bigvee_{G\in T^{<\omega}} \bigvee_{t\in G} b_t \wedge V_t\Big) \\ &= \bigvee_{F\in S^{<\omega}, G\in T^{<\omega}} \Big(\bigvee_{s\in F} a_s \wedge \bigvee_{t\in G} b_t\Big) \wedge \Big(\bigcap_{s\in F} U_s \cap \bigcap_{t\in G} V_t\Big) \\ &\in \tau. \end{aligned}$$

The second half of the conclusion is straightforward.

The following two lemmas have been proved in [18] when L is a complete Heyting algebra. We include here the proofs, which are similar to those in [18], for the convenience of the reader.

Lemma 2.2. *Suppose L_1, L_2 are complete lattices and $\Delta \subseteq L_1^{L_2}$ is a stratified L_1-topology on L_2 such that $(\omega_\Delta, \iota_\Delta)$ is a Galois connection between* SL_1-Top *and* SL_2-Top. *Suppose $L \subseteq L_2$ is a subset closed with respect to finite meets and arbitrary joins. Let Δ^* be the restriction of Δ on L, i.e. (L, Δ^*) is a subspace of (L_2, Δ). Then, the pair of functors $(\omega_{\Delta^*}, \iota_{\Delta^*})$ is a Galois connection between* SL_1-Top *and* SL-Top.

Proof. Given a stratified L_1-topological space (X, τ), a function $f : (X, \tau) \longrightarrow (L, \Delta^*)$ is continuous if and only if f is continuous regarded as a function from (X, τ) to (L_2, Δ). Thus, the set $\{f : (X, \tau) \longrightarrow (L, \Delta^*) \mid f \text{ is continuous}\}$ is closed under finite meets and arbitrary joins since $(\omega_\Delta, \iota_\Delta)$ is a Galois connection. Hence, $(\omega_{\Delta^*}, \iota_{\Delta^*})$ is a Galois connection by Theorem 2.1.

Lemma 2.3. *If L_1 is a meet continuous lattice and $\Gamma \subseteq L_1 \times L_1$ is a stratified L_1-topology on 2 such that $(\omega_\Gamma, \iota_\Gamma)$ is a Galois connection between* SL_1-Top *and* Top, *then $\Gamma \subseteq \{(a, b) \mid a \leq b\}$.*

Proof. Suppose there is some $(a, b) \in \Gamma$ such that $a \not\leq b$. Let $(X, \mathcal{T})$ be a crisp topological space with a family of clopen sets $\{U_t \mid t \in T\}$ such that $F = \bigcap_{t\in T} U_t$ is not open.

Let $\tau = \pi_{L_1}(\mathcal{T})$, i.e., $\lambda \in \tau$ if and only if there exists a family $\{U_s \mid s \in S\}$ of open sets in $(X, \mathcal{T})$ and $\{a_s \mid s \in S\} \subseteq L_1$ such that $\lambda = \bigvee_{s\in S} a_s \wedge U_s$. By the above lemma, τ is a stratified L_1-topology on X.

For each $t \in T$, define $f_t : X \longrightarrow 2$ by

$$f_t(x) = \begin{cases} 0, & x \in U_t; \\ 1, & x \notin U_t. \end{cases}$$

Since U_t is clopen, $f_t : (X, \tau) \longrightarrow (2, \Gamma)$ is continuous. Indeed, for all $(c, d) \in \Gamma$, $f_t^{\leftarrow}(c, d) = (c \wedge U_t) \vee (d \wedge (X \backslash U_t))$. Hence f_t is continuous.

Let $f = \bigvee_{t\in T} f_t$. Then

$$f(x) = \begin{cases} 0, & x \in \bigcap_{t\in T} U_t; \\ 1, & x \notin \bigcap_{t\in T} U_t. \end{cases}$$

We say that $f : (X, \tau) \longrightarrow (2, \Gamma)$ is not continuous. To this end, we show that

$$f^{\leftarrow}(a,b) = \Big(a \wedge \bigcap_{t\in T} U_t\Big) \vee \Big(b \wedge \Big(X \backslash \bigcap_{t\in T} U_t\Big)\Big) = (a \wedge F) \vee (b \wedge (X \backslash F)) \notin \tau.$$

Suppose, on the contrary, that $f^{\leftarrow}(a,b) \in \tau$. Then there exists a family $\{U_s \mid s \in S\}$ of open sets in $(X, \mathcal{T})$ and $\{a_s \mid s \in S\} \subseteq L_1$ such that

$$(a \wedge F) \vee (b \wedge (X \backslash F)) = \bigvee_{s\in S} a_s \wedge U_s.$$

Thus, $F = \bigcup\{U_s \mid a_s \not\leq b\}$ is open, contradictory to the assumption that F is not open.

Therefore, $\{f : (X, \tau) \longrightarrow (2, \Gamma) \mid f$ is continuous$\}$ is not a 2-topology since it is not closed under joins. Consequently, $(\omega_\Gamma, \iota_\Gamma)$ cannot be a Galois connection between SL_1-Top and Top by Theorem 2.1, a contradiction.

Theorem 2.2. *Let L_1 be a meet continuous lattice and L_2 a complete lattice. If $\Delta \subseteq L_1^{L_2}$ is a stratified L_1-topology on L_2 such that $(\omega_\Delta, \iota_\Delta)$ is a Galois connection between* SL_1-Top *and* SL_2-Top, *then every element $\lambda \in \Delta$ is Scott continuous.*

Proof. (1) Every $\lambda \in \Delta$ is order-preserving. The proof of this conclusion is similar to Theorem 3.3 in [18], we repeat it here for the convenience of the reader.

Suppose there is some $\lambda \in \Delta$ and $\alpha, \beta \in L_2$ such that $\alpha < \beta$ but $\lambda(\alpha) \not\leq \lambda(\beta)$, where, α is allowed to be 0 and β to be 1.

Clearly, $L = \{\alpha, \beta\} \cup \{0, 1\}$ is a complete sublattice of L_2. Let Δ^* be the restriction of Δ on L. Then, by Lemma 2.1, the pair of functors $(\omega_{\Delta^*}, \iota_{\Delta^*})$ is a Galois connection. On the other hand, let τ be the L_1-topology on X as defined in Lemma 2.2 and replace 0 by α if $0 \neq \alpha$, and 1 by β if $1 \neq \beta$ in the definition of f_t's in Lemma 2.2. Then, it can be checked that $f_t : (X, \tau) \longrightarrow (L, \Delta^*)$ is continuous for each $t \in T$, but $f^{\leftarrow}(\lambda) \notin \tau$, where f is the join of f_t's. Thus, $(\omega_{\Delta^*}, \iota_{\Delta^*})$ cannot be a Galois connection by Theorem 2.1(1), a contradiction.

(2) λ is Scott continuous.

Suppose that λ is not Scott continuous. Then there is a directed subset $D \subseteq L_2$ such that $\lambda(\bigvee D) \neq \bigvee_{d\in D} \lambda(d)$. By (1), λ preserves order, thus, $\lambda(\bigvee D) \geq \bigvee_{d\in D} \lambda(d)$. Hence $\lambda(\bigvee D) \not\leq \bigvee_{d\in D} \lambda(d)$.

Since λ is order-preserving, we can assume that $D = \downarrow D$. Write $\mathcal{T}$ for the topology on L_2 generated as a base by $\{\uparrow d \mid d \in D\}$ and let τ be the L_1-topology on L_2 consisting of functions $L_2 \longrightarrow L_1$ of the form $\bigvee_{t\in T} a_t \wedge U_t$, where $a_t \in L_1, U_t \in \mathcal{T}$.

For each $d \in D$, let $f_d : (L_2, \tau) \longrightarrow (L_2, \Delta)$ be defined by $f_d = d \wedge \uparrow d$. Then f_d is continuous since for each $\mu \in \Delta$,

$$f_d^{\leftarrow}(\mu) = \mu \circ f_d = (\mu(0) \wedge (L_2 \backslash \uparrow d)) \vee (\mu(d) \wedge \uparrow d) = (\mu(0) \wedge L_2) \vee (\mu(d) \wedge \uparrow d) \in \tau.$$

Let $f = \bigvee_{d\in D} f_d$. Clearly f is the identity when restricted to $\downarrow(\bigvee D)$. We assert that f is not continuous. Otherwise, there exist a subset $\{a_s \mid s \in S\} \subset L_1$ and a family of open sets $\{U_s \mid s \in S\}$ in $(L_2, \mathcal{T})$ such that $f^{\leftarrow}(\lambda) = \bigvee_{s\in S} a_s \wedge U_s$.

For each $s \in S$, there is some $d_s \in U_s \cap D$ since $\{\uparrow d \mid d \in D\}$ is a base for $\mathcal{T}$. Thus $a_s = (a_s \wedge U_s)(d_s) \leq \bigvee\limits_{s\in S}(a_s \wedge U_s)(d_s) = f^{\leftarrow}(\lambda)(d_s) = \lambda \circ f(d_s) = \lambda(d_s)$. Therefore $\bigvee\limits_{s\in S} a_s \leq \bigvee\limits_{d\in D} \lambda(d)$. On the other hand,

$$\lambda\big(\bigvee D\big) = \lambda\big(f\big(\bigvee D\big)\big) = \Big(\bigvee_{s\in S} a_s \wedge U_s\Big)\big(\bigvee D\big) \leq \bigvee_{s\in S} a_s.$$

Hence $\lambda(\bigvee D) \leq \bigvee\limits_{d\in D} \lambda(d)$, a contradiction to $\lambda(\bigvee D) \not\leq \bigvee\limits_{d\in D} \lambda(d)$.

Therefore the collection of continuous functions from (L_2, τ) to (L_2, Δ) is not an L_2-topology, hence $(\omega_\Delta, \iota_\Delta)$ cannot be a Galois connection.

Corollary 2.2. *Let L_1 be a meet continuous lattice and L_2 a complete lattice. If $\Delta \subseteq L_1^{L_2}$ is a stratified L_1-topology on L_2 such that for every stratified L_1-topological space (X, τ), the collection of all the continuous functions $(X, \tau) \longrightarrow (L_2, \Delta)$ is closed under arbitrary joins, then every element $\lambda \in \Delta$ is Scott continuous.*

Note 2.2. The condition that L is meet continuous in Theorem 2.2 is indispensable. For example, suppose that L is a complete lattice, $a \in L$ and $D \subseteq L$ is a directed set with $a \wedge \bigvee D \neq \bigvee\limits_{d\in D} a \wedge d$. Let Δ be the stratified L-topology on L generated as a subbase by $\{\mathrm{id}_L\} \cup \{b \wedge L \mid b \in L\}$. Then $(\omega_\Delta, \iota_\Delta)$ is a Galois connection between SL-Top itself and $a \wedge \mathrm{id}_L \in \Delta$. Clearly, $a \wedge \mathrm{id}_L \in \Delta$ is not Scott continuous.

§ 3 . The Main Result

Before proving our main result, we present at first a characterization of continuous lattices.

Lemma 3.1. *Let L be a complete lattice. The following conditions are equivalent:*

(1) *L is continuous.*

(2) *For every topological space X, a function $f : X \longrightarrow (L, \sigma(L))$ is continuous if and only if there exists a family $\{U_t\}_{t\in T}$ of open sets in X and $\{a_t\}_{t\in T} \subseteq L$ such that $f = \bigvee\limits_{t\in T} a_t \wedge U_t$.*

(3) *There is a topology Γ on L such that for every topological space X, a function $f : X \longrightarrow (L, \Gamma)$ is continuous if and only if there exists a family $\{U_t\}_{t\in T}$ of open sets in X and $\{a_t\}_{t\in T} \subseteq L$ with $f = \bigvee\limits_{t\in T} a_t \wedge U_t$.*

Proof. (1)$\Rightarrow$(2). Suppose $f : X \longrightarrow (L, \sigma(L))$ is continuous. Then

$$f = \bigvee_{a\in L} a \wedge f^{-1}(\Uparrow a).$$

Conversely, suppose there exists a family $\{U_t\}_{t\in T}$ of open sets in X and $\{a_t\}_{t\in T} \subseteq L$ such that

$$f = \bigvee_{t\in T} a_t \wedge U_t.$$

If $f(x) \gg a$, then

$$\bigvee\{a_t \mid x \in U_t\} \gg a,$$

thus there exists some $a_i, i \leq n$, such that

$$\bigvee_{i\leq n} a_i \geq a \text{ and } x \in \bigcap_{i\leq n} U_i.$$

Hence

$$\bigcap_{i\leq n} U_i \subseteq \Uparrow a,$$

thus, f is continuous.

(2)$\Rightarrow$(3). Trivial.

(3)$\Rightarrow$(1). Let $L_1 = 2$ and $L_2 = L$ in Corollary 2.2. We obtain that Γ is coarser than the Scott topology on L. Since $\mathrm{id}_L : (L, \Gamma) \longrightarrow (L, \Gamma)$ is continuous, there exist a family $\{U_t\}_{t\in T}$ of open sets in Γ and $\{a_t\}_{t\in T} \subseteq L$ such that

$$\mathrm{id}_L = \bigvee_{t\in T} a_t \wedge U_t.$$

Hence for each $a \in L$,

$$a = \bigvee_{U\in\Gamma, a\in U} \bigwedge_{b\in U} b.$$

Thus, for each $a \in L$,

$$a = \bigvee_{U\in\sigma(L), a\in U} \bigwedge_{b\in U} b$$

since Γ is coarser than $\sigma(L)$. Therefore, L is continuous.

Now, we are in a position to prove the main theorem in this note.

Theorem 3.1. *Let L be a meet continuous lattice. The following conditions are equivalent:*

(1) *L is continuous.*

(2) Top *can be embedded in* SL-Top *as a concretely both reflective and coreflective full subcategory.*

Proof. We need only prove that (2) $\Rightarrow$ (1). Suppose F : Top$\longrightarrow$ SL-Top is an embedding which has a concrete left adjoint G and a concrete right adjoint H. Then both of the pairs (F, G) and (H, F) are Galois connections. Thus, there exists a topology Γ on L and an L-topology Δ on 2 such that

$$F = \omega_\Gamma = \iota_\Delta.$$

By Theorem 2.2, we obtain that $\Delta \subseteq \{(a, b) \mid a \leq b\}$ and Γ is coarser than the Scott topology on L. Hence, for each topological space $(X, \mathcal{T})$,

$$\pi_L(\mathcal{T}) \subseteq \omega_\Gamma(\mathcal{T}) = F(\mathcal{T}) = \iota_\Delta(\mathcal{T}) \subseteq \pi_L(\mathcal{T}).$$

Therefore, a function $f : (X, \mathcal{T}) \longrightarrow (L, \Gamma)$ is continuous if and only if there exists a family $\{U_t\}_{t\in T}$ of open sets in X and $\{a_t\}_{t\in T} \subseteq L$ such that

$$f = \bigvee_{t\in T} a_t \wedge U_t.$$

Thus, L is continuous by Lemma 3.1(3).

It can be seen from the above theorem that when L is a continuous lattice, there is exactly one way to embed Top in SL-Top as a concretely both reflective and coreflective full subcategory, the embedding functor is the Lowen functor ω_L (see [15]) which coincides with π_L. This explains, to some extent, the importance played by the Lowen functors in the development of fuzzy topology.

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