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FRACTIONAL INTEGRATION ASSOCIATED TO HIGHER ORDER ELLIPTIC OPERATORS****

DENG Donggao* XU Ming** YAN Lixin***

Abstract

The authors prove the Hardy-Littlewood-Sobolev theorems for generalized fractional integrals $L^{-\alpha/2}$ for $0 < \alpha < n/m$, where L is a complex elliptic operator of arbitrary order $2m$ on $\mathbb{R}^n$.

Keywords Fractional integrals, Higher order elliptic operator, Hardy-Littlewood-Sobolev theorem
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§ 1 . Introduction

Let L be a homogeneous elliptic operator on $L^2(\mathbb{R}^n)$ of order $2m$ in divergence form

$$L = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (a_{\alpha,\beta} \partial^\beta), \tag{1.1}$$

where we assume $a_{\alpha,\beta} \in L^\infty(\mathbb{R}^n;\mathbb{C})$ for all α, β. The operator L is associated to the following form $Q(f, g)$ defined on the Sobolev space $H^m(\mathbb{R}^n)$ by

$$Q(f,g) = \int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha,\beta}(x) \partial^\beta f(x) \overline{\partial^\alpha g(x)} dx.$$

We assume that

$$Q(f,g) \le \Lambda \| \triangledown^m f \| \| \triangledown^m g\| \quad \text{and} \quad \operatorname{Re} Q(f,f) \ge \lambda \| \triangledown^m f\|_2^2 \tag{1.2}$$

for some $\lambda > 0$ and $\Lambda < +\infty$ independent of $f, g \in H^m(\mathbb{R}^n)$. Here $\triangledown^m f = (\partial^\alpha f)_{|\alpha|=m}$ and $\| \triangledown^m f\|_2 = \Big(\sum\limits_{|\alpha|=m} \int_{\mathbb{R}^n} |\partial^\alpha f|^2 \Big)^{1/2}$.

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*Department of Mathematics, Zhongshan University, Guangzhou 510275, China.
E-mail: stsddg@zsulink.zsu.edu.cn
**Institute of Mathematics, Chinese Academy of Sciences, Beijing 100080, China.
E-mail: stxmin@163.com
***Department of Mathematics, Zhongshan University, Guangzhou 510275, China.
E-mail: mcsylx@zsu.edu.cn
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By the holomorphic functional calculus theory (see [11]), L has a unique fractional power $L^{-\alpha/2}$ for $0<\alpha<n/m$, defined by

$$L^{-\alpha/2}(f)(x)=\frac{1}{\Gamma(\alpha/2)}\int_0^\infty e^{-tL}(f)(x)\frac{dt}{t^{-\alpha/2+1}}. \tag{1.3}$$

When $n\le 2m$, we can follow the standard harmonic analysis as in [12, Chapter 5] to obtain the Hardy-Littlewood-Sobolev theorems that the operator $L^{-\alpha/2}, 0<\alpha<n/m$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1<p<n/m\alpha$ and $1/q=1/p-m\alpha/n$, since the semigroup e^{-tL} generated by L has a kernel $p_t(x,y)$ satisfying a Gaussian upper bound (see [2, pp.58–59]), that is,

$$|p_t(x,y)|\le \frac{c}{t^{n/2m}}\exp\Big\{-\beta\Big(\frac{|x-y|}{t^{1/2m}}\Big)^{\frac{2m}{2m-1}}\Big\}\qquad \text{for some}\quad \beta>0, \tag{1.4}$$

for all $t>0$, and all $x,y\in\mathbb{R}^n$. Unfortunately, if $n>2m$, the operator L as in (1.1) generally fails to have a heat kernel (1.4) (see [6]), hence the method as in [12, Chapter 5] does not work in this case. Recently, by using an approach of Blunck and Kunstmann the first and third authors proved that when L is a second order elliptic operator, the Hardy-Littlewood-Sobolev theorems for $L^{-\alpha/2}$ are still true for $n>2$. In this paper, we generalize the results in [8] to an arbitrary elliptic operator L of order $2m$ by using an idea of Hofmann and Martell. The method here is different from that in [8]. Precisely, we have

Theorem 1.1. *Let $n>2m$ and $0<\alpha<\dfrac{n}{m}$. We assume that $p_0=\dfrac{2n}{n+2m}$, $p_1=\Big(\dfrac{n-2m}{2n}+\dfrac{m\alpha}{n}\Big)^{-1}$ and $\dfrac{1}{q}=\dfrac{1}{p}-\dfrac{m\alpha}{n}$. Then*

(i) *$L^{-\alpha/2}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $p_0<p<p_1$;*

(ii) *When $p=p_0$ and $q_0=\Big(\dfrac{1}{p_0}-\dfrac{m\alpha}{n}\Big)^{-1}$, $L^{-\alpha/2}$ is of weak-type (p_0,q_0), that is,*

$$\big|\{x:|L^{-\alpha/2}f(x)|>\lambda\}\big|\le C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{q_0}$$

for all $\lambda>0$.

The paper is organized as follows. In Section 2 we prove some technical estimates that will be used in the sequel. The proof of Theorem 1.1 will be given in Section 3.

§ 2. Preliminaries

We are given an elliptic operator as in (1.1) with ellipticity constants λ and Λ in (1.2). The identity operator will be written as $\mathcal{I}$. For any two closed sets E and F of $\mathbb{R}^n$, we set $d=\operatorname{dist}(E,F)$ as the distance between E and F. In this section, we prove some technical estimates, which will be used in the next section.

Firstly, Theorem 1.1 is true for $p=2$.

Lemma 2.1. *Let L be as in (1.1). Assume that $n > 2m$ and $0 < \alpha \leq 1$. Then, there exists a positive constant C independent of f such that*

$$\|L^{-\alpha/2}f\|_{q_2} \leq C\|f\|_2, \qquad \frac{1}{q_2} = \frac{1}{2} - \frac{m\alpha}{n}.$$

Proof. This lemma is a direct result from [1, Theorem 1.5]. In fact, P. Auscher et al have proved $\|L^{-1/2}f\|_{\dot{H}^m(R^n)} = \| \bigtriangledown^m L^{-1/2}f\|_2 \leq C\|f\|_2$. Note that $\|L^0 f\|_2 = \|f\|_2$. A complex interpolation theorem in [10] implies the estimates $\|L^{-\alpha/2}\|_{\dot{H}^{m\alpha}(R^n)} \leq C\|f\|_2$ for any $0 < \alpha \leq 1$. By the classical embedding theorem $\dot{H}^{m\alpha}(\mathbb{R}^n) \hookrightarrow L^{q_2}(\mathbb{R}^n), 1/q_2 = 1/2 - m\alpha/n$, we have

$$\|L^{-\alpha/2}f\|_{q_2} \leq C\|L^{-\alpha/2}f\|_{\dot{H}^{m\alpha}(\mathbb{R}^n)} \leq C\|f\|_2,$$

where C is a positive constant independent of f.

Lemma 2.2. *Let E and F be two closed sets of $\mathbb{R}^n$, and,* $\mathrm{supp} f \subset E$. *Then*

(i) $\|e^{-tL}f\|_{L^2(F)} + \|tLe^{-tL}f\|_{L^2(F)} \leq C\|f\|_{L^2(E)}\exp\Big\{ - c\Big(\frac{d(E,F)}{t^{1/2m}}\Big)^{\frac{2m}{2m-1}}\Big\}$,

(ii) $\|\sqrt{t}\nabla^m e^{-tL}f\|_{L^2(F)} \leq C\|f\|_{L^2(E)}\exp\Big\{ - c\Big(\frac{d(E,F)}{t^{1/2m}}\Big)^{\frac{2m}{2m-1}}\Big\}$,

where $c > 0$ depends only on λ, Λ, and C on n, λ, Λ.

Proof. For its proof, we refer to [5, Theorem 8] for the details. See also [4, Theorem 1.2].

Lemma 2.3. *Let $\nu > \alpha$ be an integer. Let E and F be two closed sets of $\mathbb{R}^n$, and* $\mathrm{supp} f \subset E$. *Then*

$$\|\sqrt{t}\nabla^m(L^{-\frac{\alpha}{2}}(\mathcal{I} - e^{-tL})^\nu)^* f\|_{L^2(F)} \leq Ct^{\frac{\alpha}{2}}\Big(\frac{d(E,F)}{t^{1/2m}}\Big)^{-\frac{2m}{2m-1}(\nu+\frac{1}{2})}\|f\|_{L^2(E)},$$

where C depends only on λ, Λ.

The proof of Lemma 2.3 is based on the following lemma, whose proof is similar to that of [9, Lemma 2.3]. We omit the details here.

Lemma 2.4. *Let $\{A_t\}_{t>0}$ and $\{B_t\}_{t>0}$ be two families of operators. Assume that for all closed sets E, F, for all f such that,* $\mathrm{supp} f \subset E$ *and for all $t > 0$, we have the following estimate*

$$\|A_t f\|_{L^2(F)} + \|B_t f\|_{L^2(F)} \leq C\|f\|_{L^2(E)}\exp\Big\{ - c\Big(\frac{d(E,F)}{t^{1/2m}}\Big)^{\frac{2m}{2m-1}}\Big\}.$$

Then, for all $t, s > 0$, we have

$$\|A_t B_s f\|_{L^2(F)} \leq \|f\|_{L^2(E)}\exp\Big\{ - c\Big(\frac{d(E,F)}{\max\{t,s\}^{1/2m}}\Big)^{\frac{2m}{2m-1}}\Big\}.$$

Proof of Lemma 2.3. We follow an idea of [9] to prove this lemma. Note that $L^{-\alpha/2}$ as in (1.3) can be rewritten as

$$L^{-\alpha/2}(f)(x) = C_{\nu,\alpha}\int_0^\infty e^{-(\nu+2)sL}(f)(x)\frac{ds}{s^{-\alpha/2+1}}.$$

This gives

$$\begin{aligned}\nabla^m(L^{-\frac{\alpha}{2}}(\mathcal{I}-e^{-tL})^\nu)^*(f) &= C_{\nu,\alpha}\int_0^\infty \nabla^m\big(e^{-(\nu+2)sL}(\mathcal{I}-e^{-tL})^\nu\big)^*(f)\frac{ds}{s^{-\alpha/2+1}}\\ &= C_{\nu,\alpha}\Big(\int_0^t\cdots+\int_t^\infty\cdots\Big) = C_{\nu,\alpha}(\mathrm{I_t}+\mathrm{II_t}).\end{aligned}$$

For the first one, using the following formula

$$(\mathcal{I}-e^{-tL})^\nu = \sum_{k=0}^\nu C_\nu^k(-1)^k e^{-ktL} = \sum_{k=0}^\nu C_{k,\nu}e^{-ktL},$$

we obtain

$$\mathrm{I_t} = \sum_{k=0}^\nu C_{k,\nu}\int_0^t \nabla^m e^{-(\nu+2)sL^*}e^{-ktL^*}(f)\frac{ds}{s^{-\alpha/2+1}} = \sum_{k=0}^\nu C_{k,\nu}\mathrm{I}_{t,k}.$$

By Lemma 2.2 for L^*,

$$\begin{aligned}\|\mathrm{I}_{t,0}\|_{L^2(F)} &\le \int_0^t \|\nabla^m e^{-(\nu+2)sL^*}(f)\|_{L^2(E)}\frac{ds}{s^{-\alpha/2+1}}\\ &\le \int_0^t \|(s^{\frac12}\nabla^m e^{-sL^*})\circ(e^{-\nu sL^*})\circ(e^{-sL^*})(f)\|_{L^2(F)}s^{\frac{\alpha-1}{2}}\frac{ds}{s}\\ &\le C\|f\|_{L^2(E)}\int_0^t s^{\frac{\alpha-1}{2}}\exp\Big\{-c\Big(\frac{d(E,F)}{s^{1/2m}}\Big)^{\frac{2m}{2m-1}}\Big\}\frac{ds}{s}\\ &\le C\|f\|_{L^2(E)}t^{\frac{\alpha-1}{2}}\int_1^\infty s^{\frac{1-\alpha}{2}}\exp\Big\{-c\Big(\frac{d(E,F)}{t^{1/2m}}\Big)^{\frac{2m}{2m-1}}\cdot s^{\frac{1}{2m-1}}\Big\}\frac{ds}{s}\\ &\le Ct^{\frac{\alpha-1}{2}}\Big(\frac{d(E,F)}{t^{1/2m}}\Big)^{-\frac{2m}{2m-1}(\nu+\frac12)}\|f\|_{L^2(E)}.\end{aligned}$$

Now, fix $1\le k\le\nu$. Then by Lemmas 2.2 and 2.4,

$$\begin{aligned}\|\mathrm{I}_{t,k}\|_{L^2(F)} &\le \int_0^t \|\nabla^m e^{-(\nu+2)sL^*}e^{-ktL^*}(f)\|_{L^2(E)}\frac{ds}{s^{-\alpha/2+1}}\\ &\le \int_0^t \Big\|\Big(\sqrt{\frac{kt}{2}}\nabla^m e^{-\frac{kt}{2}L^*}\Big)\circ(e^{-(\nu+2)sL^*})\circ(e^{-\frac{kt}{2}L^*})(f)\Big\|_{L^2(F)}\sqrt{\frac{2}{kt}}s^{\frac{\alpha}{2}}\frac{ds}{s}\\ &\le Ct^{-\frac12}\|f\|_{L^2(E)}\int_0^t s^{\frac{\alpha}{2}}\exp\Big\{-c\Big(\frac{d(E,F)}{\max}\{kt,s\}^{1/2m}\Big)^{\frac{2m}{2m-1}}\Big\}\frac{ds}{s}\\ &\le Ct^{-\frac12}\|f\|_{L^2(E)}\exp\Big\{-c\Big(\frac{d(E,F)}{t^{1/2m}}\Big)^{\frac{2m}{2m-1}}\Big\}\int_0^t s^{\frac{\alpha}{2}}\frac{ds}{s}\\ &\le Ct^{\frac{\alpha-1}{2}}\Big(\frac{d(E,F)}{t^{1/2m}}\Big)^{-\frac{2m}{2m-1}(\nu+\frac12)}\|f\|_{L^2(E)}.\end{aligned}$$

Using the following inequality of [9]:

$$\Big\|\frac{s}{t}(e^{-sL^*}-e^{-(s+t)L^*})g\Big\|_{L^2(F)} \leq C\exp\Big\{-c\Big(\frac{d(E,F)}{t^{1/2m}}\Big)^{\frac{2m}{2m-1}}\Big\}\|g\|_{L^2(E)},$$

we obtain

$$\begin{aligned}
\|\mathrm{II}_t\|_{L^2(F)} &\leq C\int_t^\infty \|(\sqrt{s}\nabla^m e^{-sL^*})\circ(e^{-sL^*}-e^{-(s+t)L^*})^\nu\circ(e^{-sL^*})f\|_{L^2(F)} s^{\frac{\alpha-1}{2}}\frac{ds}{s}\\
&\leq C\|f\|_{L^2(E)}\int_t^\infty \exp\Big\{-c\Big(\frac{d(E,F)}{s^{1/2m}}\Big)^{\frac{2m}{2m-1}}\Big\}\Big(\frac{t}{s}\Big)^\nu s^{\frac{\alpha-1}{2}}\frac{ds}{s}\\
&\leq C\|f\|_{L^2(E)}t^{\frac{\alpha-1}{2}}\int_0^1 \exp\Big\{-c\Big(\frac{sd(E,F)}{t^{1/2m}}\Big)^{\frac{2m}{2m-1}}\Big\}s^{\nu-\alpha+\frac12}\frac{ds}{s}\\
&\leq C\|f\|_{L^2(E)}t^{\frac{\alpha-1}{2}}\Big(\frac{d(E,F)}{t^{1/2m}}\Big)^{-\frac{2m}{2m-1}(\nu+\frac12)}\int_0^\infty e^{-s}s^{\nu-\alpha+\frac12}\frac{ds}{s}\\
&\leq Ct^{\frac{\alpha-1}{2}}\Big(\frac{d(E,F)}{t^{1/2m}}\Big)^{-\frac{2m}{2m-1}(\nu+\frac12)}\|f\|_{L^2(E)},
\end{aligned}$$

which completes the proof of Lemma 2.3 by collecting the estimates of $\mathrm{I_t}$ and $\mathrm{II_t}$.

§ 3. Proof of Theorem 1.1

It suffices to prove Theorem 1.1 in the case $0<\alpha\leq 1$ since for $1<\alpha<n/m$, the proof of this theorem is obtained by using the formula $L^{-(\beta+\gamma)/2}=L^{-\beta/2}\cdot L^{-\gamma/2}$. Recall that

$$p_0=\frac{2n}{n+2m},\quad q_0=\Big(\frac{1}{p_0}-\frac{m\alpha}{n}\Big)^{-1},\quad q_2=\Big(\frac{1}{2}-\frac{m\alpha}{n}\Big)^{-1}.$$

We will prove that the operator $L^{-\alpha/2}$ is of weak type (p_0,q_0). Applying Lemma 2.1 and Marcinkiewicz interpolation theorem, we obtain that $L^{-\alpha/2}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $p_0<p<2$ and $1/q=1/p-m\alpha/n$. Then by a standard duality argument we see that $L^{-\alpha/2}$ maps $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ boundly for all $2<p<((n-2m)/2n+m\alpha/n)^{-1}$ and $1/q=1/p-m\alpha/n$.

We begin to prove that $L^{-\alpha/2}$ is of weak-type (p_0,q_0), that is,

$$|\{x:|L^{-\alpha/2}f(x)|>\lambda\}|\leq C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{q_0} \tag{3.1}$$

for all $\lambda>0$.

Let us write M for the Hardy-Littlewood maximal function. We use a version of Calderón-Zygmund decomposition for $f(x)^{p_0}$ at height β^{p_0}, where

$$\beta=\|f\|_{p_0}\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{-\frac{q_0}{p_0}}.$$

See [7, p.247]. Then, there exists a collection of pairwise disjoint cubes $\{Q_j\}$ such that

$$\{x\in\mathbb{R}^n: M(f^{p_0})^{\frac{1}{p_0}}>\beta\}=\bigcup_j Q_j$$

and they satisfy the following property

$$\beta \le \Big(\frac{1}{Q_j}\int_{Q_j}|f(x)|^{p_0}dx\Big)^{\frac{1}{p_0}} \le C\beta.$$

One writes $f = g + b = g + \sum_j b_j$, where

$$g(x) = f(x)\chi_{\mathbb{R}^n\setminus\bigcup_j Q_j} + \sum_j P_{Q_j}(f)(x)\chi_{Q_j}(x),$$

$$b_j(x) = (f(x) - P_{Q_j}(f)(x))\chi_{Q_j}(x),$$

where $P_{Q_j}(f)(x)$ is a polynomial with order $m-1$ with the properties

$$\int_{Q_j}(f(x) - P_{Q_j}(f)(x))x^\alpha dx = 0$$

for $0 \le |\alpha| \le m-1$, and for any $x \in Q_j$,

$$|P_{Q_j}(f)(x)| \le C\frac{1}{|Q_j|}\int_{Q_j}|f(y)|dy.$$

The standard arguments yield $0 \le g(x) \le c\beta$ for almost every $x \in \mathbb{R}^n$. Besides, for $0 \le |\alpha| \le m-1$,

$$\int_{Q_j} b_j(x)x^\alpha dx = 0 \quad \text{and} \quad \|b_j\|_{p_0} \le C\beta|Q_j|^{1/p_0}. \tag{3.2}$$

For each j, we write $t_j = l(Q_j)^{2m}$, where $l(Q_j)$ stands for the side length of the cube Q_j. We then decompose $\sum_j b_j = h_1 + h_2$, where

$$h_1 = \sum_j(\mathcal{I} - (\mathcal{I} - e^{-t_jL})^\nu)b_j \quad \text{and} \quad h_2 = \sum_j(\mathcal{I} - e^{-t_jL})^\nu b_j.$$

Here ν will be chosen later. One writes

$$|\{x:\ |L^{-\alpha/2}f(x)| > 3\lambda\}| \le |\{x:\ |L^{-\alpha/2}g(x)| > \lambda\}| + \sum_{k=1}^2|\{x:\ |L^{-\alpha/2}h_k(x)| > \lambda\}|.$$

Since the operator $L^{-\alpha/2}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^{q_2}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}|L^{-\alpha/2}(g)(y)|^{q_2}dy &\le C\Big(\int_{\mathbb{R}^n}|g(y)|^2dy\Big)^{q_2/2}\\ &\le C\Big(\int_{\bigcup_j Q_j}|g(y)|^2dy\Big)^{q_2/2} + C\Big(\int_{\mathbb{R}^n\setminus\bigcup_j Q_j}|g(y)|^2dy\Big)^{q_2/2}\\ &\le C\beta^{q_2}\Big(\sum_j|Q_j|\Big)^{q_2/2} + C\beta^{(2-p_0)q_2/2}\|f\|_{p_0}^{p_0q_2/2}\\ &\le C\|f\|_{p_0}^{q_2}\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{(q_0/2-q_0/p_0)q_2}.\end{aligned}$$

Noting that $1/q_0 = 1/p_0 - m\alpha/n$ and $1/q_2 = 1/2 - m\alpha/n$, we have $q_0 = [1+(1/2-1/p_0)q_0]q_2$. This leads to

$$\begin{aligned}|\{x:\ |L^{-\alpha/2}g(x)|>\lambda\}| &\le \lambda^{-q_2}\int_{\mathbb{R}^n}|L^{-\alpha/2}(g)(x)|^{q_2}\,dx\\ &\le C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{[1+(1/2-1/p_0)q_0]q_2}\\ &\le C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{q_0}.\end{aligned}$$

We now estimate the second term, i.e. the term involving $h_1 = \sum_j(\mathcal{I}-(\mathcal{I}-e^{-t_jL})^\nu)b_j$. We obtain

$$\begin{aligned}|\{x:\ |L^{-\alpha/2}h_1(x)|>\lambda\}| &\le \lambda^{-q_2}\int_{\mathbb{R}^n}|L^{-\alpha/2}(h_1)(x)|^{q_2}\,dx\\ &\le C\lambda^{-q_2}\Big(\int_{\mathbb{R}^n}\Big|\sum_j(\mathcal{I}-(\mathcal{I}-e^{-t_jL})^\nu)b_j\Big|^2dx\Big)^{q_2/2}\\ &\le C\lambda^{-q_2}\sum_{k=1}^{\nu}\Big(\int_{\mathbb{R}^n}\Big|\sum_j e^{-kt_jL}b_j\Big|^2dx\Big)^{q_2/2}.\end{aligned}$$

We fix $1\le k\le\nu$. Then

$$\Big\|\sum_j e^{-kt_jL}b_j\Big\|_2 = \sup_h\Big|\int_{\mathbb{R}^n}\sum_j e^{-kt_jL}b_j(x)h(x)dx\Big|,$$

where the supremum is taken over all functions $h\in L^2$ with $\|h\|_{L^2}=1$. For each j, we set

$$S(0,j)=2Q_j;\quad S(l,j)=2^{l+1}Q_j\backslash 2^lQ_j,\qquad l=1,2,\cdots,$$

and $h_{(l,j)}(x)=h(x)\chi_{S(l,j)}(x)$. In this way, by (3.2)

$$\begin{aligned}&\Big\|\sum_j e^{-kt_jL}b_j\Big\|_2\\ &=\sup_h\Big|\sum_j\sum_{l=1}^{\infty}\int_{\mathbb{R}^n}e^{-kt_jL}b_j(x)h_{(l,j)}(x)dx\Big|\\ &=\sup_h\Big|\sum_j\sum_{l=1}^{\infty}\int_{Q_j}b_j(x)((e^{-kt_jL})^*h_{(l,j)}(x)-P_{Q_j}((e^{-kt_jL})^*h_{(l,j)}))dx\Big|\\ &\le C\sup_h\beta\sum_j\sum_{l=1}^{\infty}|Q_j|^{\frac{1}{p_0}}\|(e^{-kt_jL})^*h_{(l,j)}(x)-P_{Q_j}((e^{-k(t_j)^mL})^*h_{(l,j)})\|_{L^{\frac{2n}{n-2m}}(Q_j)},\end{aligned}$$

where $P_Q(f)$ is a polynomial with order $m-1$ satisfying the Sobolev-Poincaré inequality.

See [13, Chapter 4]. Now we are going to use the following Sobolev-Poincaré inequality,

$$\begin{aligned}&\|((e^{-kt_jL})^*h_{(l,j)}(x)-P_{Q_j}((e^{-kt_jL})^*h_{(l,j)}))\|_{L^{\frac{2n}{n-2m}}(Q_j)}\\ &\le C\|\bigtriangledown^m(e^{-kt_jL})^*h_{(l,j)}\|_{L^2(Q_j)}\\ &\le Ct_j^{-1/2}\exp\Big\{-c\Big(\frac{\mathrm{dist}(S(l,j),Q_j)}{(kt_j)^{1/2m}}\Big)^{\frac{2m}{2m-1}}\Big\}\|h_{(l,j)}\|_{L^2(S(l,j))}.\end{aligned}$$

Note that for $l\ge 1$, we get $\mathrm{dist}(S(l,j),Q_j)\ge 2^{l-2}l(Q_j)$ and $\frac{1}{p_0}=\frac{1}{2}+\frac{m}{n}$. We have

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}\sum_j e^{-kt_jL}b_j(x)h(x)dx\Big|\\ &\le C\beta\sum_j\sum_{l=1}^{\infty}|Q_j|^{\frac{1}{p_0}}t_j^{-\frac12}\exp\{-c2^{\frac{2m}{2m-1}l}\}\|h\|_{L^2(S(l,j))}\\ &\le C\beta\sum_j\sum_{l=1}^{\infty}|Q_j|^{\frac{1}{p_0}}l(Q_j)^{-m}\exp\{-c2^{\frac{2m}{2m-1}l}\}|2^{l+1}Q_j|^{\frac12}\Big(\frac{1}{|2^{l+1}Q_j|}\int_{2^{l+1}Q_j}|h(y)|^2dy\Big)^{1/2}\\ &\le C\beta\sum_j|Q_j|\,\mathrm{ess}\inf_{y\in Q_j}M(|h|^2)(y)^{\frac12}\sum_{l=0}^{\infty}\exp\{-c2^{\frac{2m}{2m-1}l}\}2^{\frac{ln}{2}}\\ &\le C\beta\int_{\bigcup_j Q_j}M(|h|^2)(y)^{\frac12}dx\\ &\le C\beta\Big|\bigcup_j Q_j\Big|^{1/2},\end{aligned}$$

which yields $\|h_1\|_2\le C\beta\big(\|f\|_{p_0}/\lambda\big)^{q_0/2}$, and then

$$|\{x:\ |L^{-\alpha/2}h_1(x)|>\lambda\}|\le C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{[1+(1/2-1/p_0)q_0]q_2}\le C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{q_0}. \tag{3.3}$$

We turn to the estimation of the third term, i.e. the term involving h_2. Denote $Q_j^*=2Q_j$ and $E=\Big(\bigcup_j Q_j^*\Big)$. Let $D_j=L^{-\alpha/2}(\mathcal{I}-e^{-t_jL})^\nu b_j$. We have

$$\begin{aligned}|\{x:\ |L^{-\alpha/2}h_2(x)|>\lambda\}|&\le\sum_j|Q_j^*|+\lambda^{-2}\int_{\big(\bigcup_j Q_j^*\big)^c}|L^{-\alpha/2}(h_2)(x)|^2dx\\ &\le C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{q_0}+\lambda^{-2}\int_{\big(\bigcup_j Q_j^*\big)^c}|L^{-\alpha/2}(h_2)(x)|^2dx\\ &\le C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{q_0}+\lambda^{-2}\Big(\sup_h\Big|\int_{\mathbb{R}^n}\sum_j D_jb_j(y)h(y)dy\Big|\Big)^2,\end{aligned}$$

where the supremum is taken over all functions $h\in L^2(E^*)$ with $\|h\|_{L^2(E^*)}=1$. By (3.2), we have

$$\Big|\int_{\mathbb{R}^n}\sum_j D_jb_j(y)h(y)dy\Big|\le C\sum_j\sum_{l=1}^{\infty}\|b_j\|_{L^{p_0}(Q_j)}\|D_j^*h_{(l,j)}(y)-P_{Q_j}(D_j^*h_{(l,j)})\|_{L^{p_0'}(Q_j)},$$

where $P_Q(f)$ is a polynomial with order $m-1$ satisfying the Sobolev-Poincaré inequality. By Sobolev-Poincaré inequality again and Lemma 2.3,

$$\begin{aligned}
&\|D_j^* h_{(l,j)}(y) - P_{Q_j}(D_j^* h_{(l,j)})\|_{L^{p_0'}(Q_j)} \\
&\leq \| \bigtriangledown^m D_j^* h_{(l,j)}\|_{L^2(S(l,j))} \\
&\leq C t_j^{\frac{\alpha-1}{2}} \Big(\frac{\operatorname{dist}(S(l,j),Q_j)}{{t_j}^{1/2m}}\Big)^{-\frac{2m}{2m-1}(\nu+\frac{1}{2})} \|h_{(l,j)}\|_{L^2(S(l,j))} \\
&\leq C t_j^{\frac{\alpha-1}{2}} 2^{-l[\frac{2m}{2m-1}(\nu+\frac{1}{2})]} \|h_{(l,j)}\|_{L^2(S(l,j))}.
\end{aligned}$$

We choose ν sufficiently large such that

$$\Big[\frac{2m}{(2m-1)}\Big(\nu+\frac{1}{2}\Big) - \frac{n}{2}\Big] > 0,$$

and then

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^n} \sum_j D_j b_j(x) h(x) dx \Big| \\
&\leq C\beta \sum_j \sum_{l=1}^{\infty} |Q_j|^{\frac{1}{p_0}} t_j^{\frac{\alpha-1}{2}} 2^{-l[\frac{2m}{2m-1}(\nu+\frac{1}{2})]} \|h_{(l,j)}\|_{L^2(S(l,j))} \\
&\leq C\beta \sum_j \sum_{l=0}^{\infty} |Q_j|^{\frac{1}{2}+\frac{m\alpha}{n}} |2^{l+1}Q_j|^{\frac{1}{2}} \Big(\frac{1}{|2^{l+1}Q_j|}\int_{2^{l+1}Q_j} |h(y)|^2 dy\Big)^{1/2} 2^{-l[\frac{2m}{2m-1}(\nu+\frac{1}{2})]} \\
&\leq C\beta \sum_j |Q_j|^{\frac{m\alpha}{n}} |Q_j| \operatorname{ess}\inf_{y\in Q_j} M(|h|^2)(y)^{\frac{1}{2}} \sum_{l=1}^{\infty} 2^{-l[\frac{2m}{(2m-1)}(\nu+\frac{1}{2})-\frac{n}{2}]} \\
&\leq C\beta \Big(\sum_j |Q_j|\Big)^{\frac{m\alpha}{n}+\frac{1}{2}}.
\end{aligned}$$

The same arguments as in (3.3) give

$$\begin{aligned}
|\{x:\ |L^{-\alpha/2}h_2(x)| > \lambda\}| &\leq C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{q_0} + \lambda^{-2}\int_{\left(\bigcup_j Q_j^*\right)^c} |L^{-\alpha/2}(h_2)(x)|^2 dx \\
&\leq C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{q_0} + C\lambda^{-2}\Big[\beta\Big(\sum_j |Q_j|\Big)^{\frac{m\alpha}{n}+\frac{1}{2}}\Big]^2 \\
&\leq C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{q_0} + C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{[2+2q_0(\frac{m\alpha}{n}-\frac{1}{p_0})+q_0]} \\
&\leq C\Big(\frac{\|f\|_{p_0}}{\lambda}\Big)^{q_0}
\end{aligned}$$

since $q_0 = \big(\frac{1}{p_0} - \frac{m\alpha}{n}\big)^{-1}$.

Hence, we have obtained (3.1), and then the proof of Theorem 1.1.

Remark 3.1. Consider complex bounded measurable coefficients $a_{\alpha\beta}$ on $\mathbb{R}^n$ such that the form $Q(f,g)=\displaystyle\int_{\mathbb{R}^n}\sum_{|\alpha|,|\beta|\le m} a_{\alpha,\beta}(x)\partial^\beta f(x)\overline{\partial^\alpha g(x)}dx$ satisfies

$$Q(f,g)\le \Lambda\|\bigtriangledown^m f\|\|\bigtriangledown^m g\|+k'\|f\|_2\|g\|_2,$$

the Gårding inequality, and

$$\text{Re } Q(f,f)\ge \lambda\|\bigtriangledown^m f\|_2^2-k\|f\|_2^2$$

for some $\lambda>0$, k, $k'\ge 0$ and $\Lambda<+\infty$ independent of f, $g\in H^m(\mathbb{R}^n)$. We define an inhomogeneous elliptic operator on $L^2(\mathbb{R}^n)$ of order $2m$ in divergence form by

$$L=\sum_{|\alpha|,|\beta|\le m}(-1)^\alpha\partial^\alpha(a_{\alpha,\beta}\partial^\beta). \tag{3.4}$$

For the above operator L, we have similar results as in Lemma 2.2 (see [3, Remark 3.1]). So, Theorem 1.1 is true for the fractional integrals $L^{-\alpha/2}$ of the operator L as in (3.4). For the proof, we omit its details here.

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