Chin. Ann. Math.
26B:2(2005),253–274.

BIFURCATION OF PERIODIC ORBITS OF A THREE-DIMENSIONAL SYSTEM***

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Abstract

Consider a three-dimensional system having an invariant surface. By using bifurcation techniques and analyzing the solutions of bifurcation equations, the authors study the spacial bifurcation phenomena of a k multiple closed orbit in the invariant surface. The sufficient conditions of the existence of many closed orbits bifurcate from the k multiple closed orbit are obtained.

Keywords Bifurcation, Invariant surface, Three-dimensional system, Closed orbit
2000 MR Subject Classification 34C23, 37G15

§ 1. Introduction

There has been a general theory on bifurcation of periodic orbits for two or higher-dimensional systems, and the theory for two-dimensional systems is much richer than that for higher-dimensional systems (see [1–11]). We have a relatively complete understanding to bifurcation phenomenon of plane systems (see [1, 2, 5, 6]). For example, a k multiple limit cycle of two-dimensional systems can generate one, two, even k limit cycles, and at most k limit cycles under autonomous perturbations. For three-dimensional systems, the bifurcation of periodic orbits were also studied in some cases. For example, from [3, 5, 8] we know that a non-hyperbolic periodic orbit of three-dimensional systems can generate one, two or more periodic orbits under suitable perturbations. The bifurcation of periodic orbits near a homoclinic or heteroclinic loop in the space were investigated in [7–11] in some details. A special three-dimensional analytic system

$$\begin{aligned} \dot{x} &= f_1(x) + \varepsilon g_1(x,y) + \varepsilon^2 \tilde{g}_1(x,y,\varepsilon), \\ \dot{y} &= f_2(x,y) + \varepsilon g_2(x,y) + \varepsilon^2 \tilde{g}_2(x,y,\varepsilon), \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^2, y \in \mathbb{R}, f_2(x,0) = 0$, was studied in [4]. Suppose that for $\varepsilon = 0$ System (1.1) has an m multiple closed orbit Γ on invariant plane $y = 0$. Sufficient conditions for Γ to generate one, two and at most m periodic orbits are obtained respectively. In this paper,

Manuscript received September 30, 2003. Revised March 15, 2004.
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***Project supported by the Ministry of Education of China (No.20010248019, No.20020248010) and the National Natural Science Foundation of China (No.10371072).

we consider the bifurcation of periodic orbits for a C^∞ three-dimensional system of the following form

$$\begin{aligned}\dot{x} &= F(x, y, \varepsilon),\\ \dot{y} &= G(x, y, \varepsilon),\end{aligned} \tag{1.2}$$

where $x = (x_1, x_2) \in \mathbb{R}^2, y \in \mathbb{R}, \varepsilon \in \mathbb{R}$. Suppose that for $\varepsilon = 0$ System (1.2) has a C^∞ invariant surface of the form $y = M(x)$. Assume that a closed orbit of multiplicity k restricted on the invariant surface is known. Our goal is to study the bifurcations of periodic orbits near a neighborhood of the surface. Without loss of generality, we may assume $M(x) \equiv 0$. Then System (1.2) can be written as follows

$$\begin{aligned}\dot{x} &= f(x) + yf_1(x, y) + \varepsilon P(x, y, \varepsilon),\\ \dot{y} &= yg(x, y) + \varepsilon Q(x, y, \varepsilon),\end{aligned} \tag{1.3}$$

where

$$\begin{aligned}&f(x) = (f_{01}(x), f_{02}(x))^T, \qquad f_1(x, y) = (f_{11}(x, y), f_{12}(x, y))^T,\\ &P(x, y, \varepsilon) = (P_1(x, y, \varepsilon), P_2(x, y, \varepsilon))^T.\end{aligned}$$

It is obvious that the form of (1.3) is more general than (1.1). We will obtain some results of new types for the number of periodic orbits of (1.3). For example, we give conditions for the k multiple closed orbit to generate $1, \cdots, 4$ periodic orbits and at most k, $k+1$, $2k$, $2k+2$ periodic orbits under suitable conditions respectively.

§ 2. A Transformation of Variables and Bifurcation Equations

Suppose the two-dimensional system

$$\dot{x} = f(x) \tag{2.1}$$

has a k multiple closed orbit L with a parameter representation

$$x = u(t), \qquad 0 \leqslant t \leqslant T,$$

where $T > 0$ denotes the period of L. Let

$$\begin{aligned}v(\theta) &= (v_1(\theta), v_2(\theta))^T = f(u(\theta))/|f(u(\theta))|,\\ Z(\theta) &= \begin{pmatrix} -v_2(\theta) & 0\\ v_1(\theta) & 0\\ 0 & 1\end{pmatrix}.\end{aligned}$$

Consider a transformation of variables from (x, y) to (θ, h) defined by the relation

$$(x, y)^T = (u(\theta), 0)^T + Z(\theta)h, \tag{2.2}$$

where

$$0 \leqslant \theta \leqslant T, \qquad h = (h_1, h_2)^T \in \mathbb{R}^2.$$

Applying Theorem 1.4 of Chapter two of reference [5] or [3], the transformation (2.2) transforms System (1.3) into a system for θ, h as follows

$$\begin{aligned}\dot\theta &= 1 + \tilde f_1(\theta,h) + E(\theta,h)(P(\tilde u(\theta)+Z(\theta)h,\varepsilon), Q(\tilde u(\theta)+Z(\theta)h,\varepsilon))^T\varepsilon,\\ \dot h &= A(\theta)h + f_2(\theta,h) + B(\theta,h)(P(\tilde u(\theta)+Z(\theta)h,\varepsilon), Q(\tilde u(\theta)+Z(\theta)h,\varepsilon))^T\varepsilon,\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}\tilde u(\theta) &= (u(\theta),0)^T, \qquad \tilde v(\theta) = (v^T(\theta),0)^T, \qquad \tilde x = (x,y),\\ \tilde f(x,y) &= \begin{pmatrix} f(x)+yf_1(x,y)\\ yg(x,y)\end{pmatrix},\\ \tilde f_1(\theta,h) &= E(\theta,h)[\tilde f(\tilde u(\theta)+Z(\theta)h) - \tilde f(\tilde u(\theta)) - Z'(\theta)h],\\ E(\theta,h) &= [|u'(\theta)| + \tilde v^T(\theta)Z'(\theta)h]^{-1}\tilde v^T(\theta),\\ A(\theta) &= Z^T(\theta)[-Z'(\theta) + \tilde f_{\tilde x}(\tilde u(\theta))Z(\theta)],\\ f_2(\theta,h) &= -Z^T(\theta)Z'(\theta)h\tilde f_1(\theta,h) + Z^T(\theta)[\tilde f(\tilde u(\theta)+Z(\theta)h) - \tilde f(\tilde u(\theta)) - \tilde f_{\tilde x}(\tilde u(\theta))Z(\theta)h],\\ B(\theta,h) &= Z^T(\theta)[I - Z'(\theta)hE(\theta,h)].\end{aligned}$$

Since $Z^T(\theta)Z'(\theta) = 0$, we have

$$\begin{aligned}B(\theta,h) &= Z^T(\theta),\\ A(\theta) &= Z^T(\theta)\tilde f_{\tilde x}(\tilde u(\theta))Z(\theta) = \begin{pmatrix} a(\theta) & v^\perp(\theta)f_1(u(\theta),0)\\ 0 & g(u(\theta),0)\end{pmatrix}\\ f_2(\theta,h) &= Z^T(\theta)[\tilde f(\tilde u(\theta)+Z(\theta)h) - \tilde f(\tilde u(\theta)) - \tilde f_{\tilde x}(\tilde u(\theta))Z(\theta)h],\end{aligned}$$

where (see [5])

$$a(\theta) = v^\perp(\theta)f_x(u(\theta))(v^\perp(\theta))^T = \mathrm{tr} f_x(u(\theta)) - \frac{d}{d\theta}\ln|f(u(\theta))|.$$

By Taylor Theorem and direct computation, we obtain

$$\tilde f_1(\theta,h) = \frac{1}{|u'(\theta)|}(v^T(\theta)f_x(u(\theta))(v^\perp(\theta))^T - (v'(\theta))^\perp v(\theta), v^T(\theta)f_1(u(\theta),0))h + O(|h|^2).$$

Let

$$\begin{aligned}w_1(\theta) &= [v^T(\theta)f_x(u(\theta))(v^\perp(\theta))^T - (v'(\theta))^\perp v(\theta)]/|u'(\theta)|,\\ w_2(\theta) &= [v^T(\theta)f_1(u(\theta),0)]/|u'(\theta)|,\\ w_3(\theta) &= [v^T(\theta)P(u(\theta),0,0)]/|u'(\theta)|.\end{aligned}$$

Then

$$\dot\theta = 1 + w_1(\theta)h_1 + w_2(\theta)h_2 + w_3(\theta)\varepsilon + O(|h,\varepsilon|^2). \tag{2.4}$$

Noting

$$f_2(\theta,0)=0, \qquad \frac{\partial f_2(\theta,0)}{\partial h}=0,$$

it follows that

$$f_2(\theta,h)=\begin{pmatrix} \frac{1}{2}d_1(\theta)h_1^2+d_2(\theta)h_1h_2+\frac{1}{2}d_3(\theta)h_2^2+O(|h|^3) \\ g_x(u(\theta),0)(v^\perp(\theta))^T h_1h_2+g_y(u(\theta),0)h_2^2+O(|h_2||h|^2) \end{pmatrix},$$

where

$$d_1(\theta)=-v_2(\theta)v^\perp(\theta)\, f''_{01}(u(\theta))\,(v^\perp(\theta))^T+v_1(\theta)v^\perp(\theta)f''_{02}(u(\theta))(v^\perp(\theta))^T,$$

$$d_2(\theta)=-v_2(\theta)f_{11x}(u(\theta),0)(v^\perp(\theta))^T+v_1(\theta)f_{12x}(u(\theta),0)(v^\perp(\theta))^T,$$

$$d_3(\theta)=2[-v_2(\theta)f_{11y}(u(\theta),0)+v_1(\theta)f_{12y}(u(\theta),0)],$$

$$f''_{0i}(u(\theta))=\begin{pmatrix} \dfrac{\partial^2 f_{0i}(u(\theta))}{\partial x_1^2} & \dfrac{\partial^2 f_{0i}(u(\theta))}{\partial x_1\partial x_2} \\ \dfrac{\partial^2 f_{0i}(u(\theta))}{\partial x_2\partial x_1} & \dfrac{\partial^2 f_{0i}(u(\theta))}{\partial x_2^2} \end{pmatrix}, \qquad i=1,2.$$

Hence, we have from (2.3)

$$\begin{aligned}\dot h_1 &= a(\theta)h_1+a_1(\theta)h_2+a_2(\theta)\varepsilon+a_3(\theta)h_1\varepsilon+a_4(\theta)h_2\varepsilon+a_5(\theta)\varepsilon^2+\frac{1}{2}d_1(\theta)h_1^2+d_2(\theta)h_1h_2\\ &\quad+\frac{1}{2}d_3(\theta)h_2^2+O(|h,\varepsilon|^3),\\ \dot h_2 &= b_1(\theta)h_2+b_2(\theta)\varepsilon+b_3(\theta)h_1\varepsilon+b_4(\theta)h_2\varepsilon+b_5(\theta)h_1h_2+b_6(\theta)h_2^2+b_7(\theta)\varepsilon^2\\ &\quad+O(|h_2||h|^2+|\varepsilon||h,\varepsilon|^2),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned} a_1(\theta)&=v^\perp(\theta)f_1(u(\theta),0), & a_2(\theta)&=v^\perp(\theta)P(u(\theta),0,0),\\ a_3(\theta)&=v^\perp(\theta)P_x(u(\theta),0,0)(v^\perp(\theta))^T, & a_4(\theta)&=v^\perp(\theta)P_y(u(\theta),0,0)\\ a_5(\theta)&=v^\perp(\theta)P_\varepsilon(u(\theta),0,0), & b_1(\theta)&=g(u(\theta),0),\\ b_2(\theta)&=Q(u(\theta),0,0), & b_3(\theta)&=Q_x(u(\theta),0,0)(v^\perp(\theta))^T,\\ b_4(\theta)&=Q_y(u(\theta),0,0), & b_5(\theta)&=g_x(u(\theta),0)(v^\perp(\theta))^T,\\ b_6(\theta)&=g_y(u(\theta),0), & b_7(\theta)&=Q_\varepsilon(u(\theta),0,0).\end{aligned}$$

Therefore we have the following lemma.

Lemma. *The periodic transformation* (2.2) *transforms System* (1.3) *into Equations* (2.4) *and* (2.5).

Using Equations (2.4) and (2.5), we obtain

$$\begin{aligned}\frac{dh_1}{d\theta} &= a(\theta)h_1+a_1(\theta)h_2+a_2(\theta)\varepsilon+\tilde a_3(\theta)h_1\varepsilon+\tilde a_4(\theta)h_2\varepsilon+\tilde a_5(\theta)h_1^2+\tilde a_6(\theta)h_1h_2\\ &\quad+\tilde a_7(\theta)h_2^2+\tilde a_8(\theta)\varepsilon^2+O(|h_1,h_2,\varepsilon|^3),\\ \frac{dh_2}{d\theta} &= b_1(\theta)h_2+b_2(\theta)\varepsilon+\tilde b_3(\theta)h_1\varepsilon+\tilde b_4(\theta)h_2\varepsilon+\tilde b_5(\theta)h_1h_2+\tilde b_6(\theta)h_2^2+\tilde b_7(\theta)\varepsilon^2\\ &\quad+O(|h_1,h_2,\varepsilon|^3),\end{aligned} \tag{2.6}$$

where

$$\tilde{a}_3(\theta) = a_3(\theta) - a_2(\theta)w_1(\theta) - a(\theta)w_3(\theta), \qquad \tilde{a}_4(\theta) = a_4(\theta) - a_2(\theta)w_2(\theta) - a_1(\theta)w_3(\theta),$$

$$\tilde{a}_5(\theta) = \frac{1}{2}d_1(\theta) - a(\theta)w_1(\theta), \qquad \tilde{a}_6(\theta) = d_2(\theta) - a(\theta)w_2(\theta) - a_1(\theta)w_1(\theta),$$

$$\tilde{a}_7(\theta) = \frac{1}{2}d_3(\theta) - a_1(\theta)w_2(\theta), \qquad \tilde{a}_8(\theta) = -w_3(\theta)a_2(\theta) + a_5(\theta),$$

$$\tilde{b}_3(\theta) = b_3(\theta) - b_2(\theta)w_1(\theta), \qquad \tilde{b}_4(\theta) = b_4(\theta) - b_2(\theta)w_2(\theta) - b_1(\theta)w_3(\theta),$$

$$\tilde{b}_5(\theta) = b_5(\theta) - b_1(\theta)w_1(\theta), \qquad \tilde{b}_6(\theta) = b_6(\theta) - b_1(\theta)w_2(\theta),$$

$$\tilde{b}_7(\theta) = -w_3(\theta)b_2(\theta) + b_7(\theta).$$

Equation (2.6) can be rewritten as

$$\begin{aligned} \frac{dh}{d\theta} &= \begin{pmatrix} a(\theta) & a_1(\theta) \\ 0 & b_1(\theta) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} a_2(\theta) \\ b_2(\theta) \end{pmatrix} \varepsilon + O(|h,\varepsilon|^2) \\ &= A(\theta)h + Z^T(\theta)F_0(u(\theta))\varepsilon + O(|h,\varepsilon|^2), \end{aligned} \tag{2.7}$$

where

$$F_0(u(\theta)) = (P(u(\theta),0,0), Q(u(\theta),0,0))^T.$$

Let $Y(\theta)$ be the fundamental matrix solution of the following equation

$$\frac{dh}{d\theta} = A(\theta)h \tag{2.8}$$

with $Y(0) = I$ (the identity matrix).

The Poincaré map of (2.7) has the form (see [5])

$$\widetilde{P}(h_0,\varepsilon) = Y(T)h_0 + Y(T)\int_0^T Y^{-1}(\theta)Z^T(\theta)F_0(u(\theta))d\theta\varepsilon + O(|h_0,\varepsilon|^2). \tag{2.9}$$

Let

$$E = Y(T) - I, \qquad K = \int_0^T Y(T)Y^{-1}(\theta)Z^T(\theta)F_0(u(\theta))d\theta.$$

For small $\varepsilon \neq 0$, System (1.3) has a closed orbit near L if and only if there exists a point h_0 with $|h_0|$ small, such that

$$\widetilde{P}(h_0,\varepsilon) - h_0 = Eh_0 + K\varepsilon + O(|h_0,\varepsilon|^2) = 0. \tag{2.10}$$

By (2.8), we have

$$Y(\theta) = \begin{pmatrix} e^{\int_0^\theta a(s)ds} & e^{\int_0^\theta a(s)ds}\displaystyle\int_0^\theta b(s)ds \\ 0 & e^{\int_0^\theta g(u(s),0)ds} \end{pmatrix},$$

where

$$\begin{aligned} b(s) &= v^\perp(s)f_1(u(s),0)\exp\Big(\int_0^s [g(u(\xi),0) - a(\xi)]d\xi\Big) \\ &= \frac{1}{|f(u(0))|}[f(u(s)) \wedge f_1(u(s),0)]\exp\Big(\int_0^s [g(u(\xi),0) - \mathrm{tr} f_x(u(\xi))]d\xi\Big). \end{aligned}$$

Thus

$$Y(T)=\begin{pmatrix} e^{\sigma_1} & e^{\sigma_1}\displaystyle\int_0^T b(s)ds \\ 0 & e^{\sigma_2} \end{pmatrix},$$

$$Y^{-1}(\theta)=\begin{pmatrix} \exp\Big(-\displaystyle\int_0^\theta a(s)ds\Big) & -\exp\Big(-\displaystyle\int_0^\theta g(u(s),0)ds\Big)\displaystyle\int_0^\theta b(s)ds \\ 0 & \exp\Big(-\displaystyle\int_0^\theta g(u(s),0)ds\Big) \end{pmatrix},$$

where

$$\sigma_1=\oint_L \mathrm{tr} f_x dt, \qquad \sigma_2=\oint_L g dt,$$
$$\exp\Big(-\int_0^\theta a(s)ds\Big)=\frac{|f(u(\theta))|}{|f(u(0))|}\exp\Big(-\int_0^\theta \mathrm{tr} f_x(u(s))ds\Big).$$

Therefore

$$E=\begin{pmatrix} e^{\sigma_1}-1 & e^{\sigma_1}\displaystyle\int_0^T b(s)ds \\ 0 & e^{\sigma_2}-1 \end{pmatrix},$$

$$K=(e^{\sigma_1}K_1, e^{\sigma_2}K_2)^T,$$

where

$$\begin{aligned} K_1&=\oint_L\Big(e^{-\int_0^t \mathrm{tr} f_x dt} f\wedge P_0+e^{\int_0^t(g-\mathrm{tr} f_x)dt}\int_0^t e^{-\int_0^s g dt}Q_0 ds\, f\wedge f_1\Big)dt/|f(u(0))|,\\ K_2&=\oint_L e^{-\int_0^t g dt}Q_0 dt,\\ P_0&=P(x,y,0), \qquad Q_0=Q(x,y,0). \end{aligned} \tag{2.11}$$

Let $h_0=(\rho_1,\rho_2)^T$. We can rewrite (2.10) as

$$(e^{\sigma_1}-1)\rho_1+e^{\sigma_1}\int_0^T b(s)ds\rho_2+e^{\sigma_1}K_1\varepsilon+O(|\rho_1,\rho_2,\varepsilon|^2)=0, \tag{2.12}$$
$$(e^{\sigma_2}-1)\rho_2+e^{\sigma_2}K_2\varepsilon+O(|\rho_1,\rho_2,\varepsilon|^2)=0. \tag{2.13}$$

Since $y=0$ is an invariant plane of System (1.3) for $\varepsilon=0$, the second coordinate of $\widetilde{P}(h_0,\varepsilon)-h_0$ must be 0 as $\varepsilon=\rho_2=0$, namely, (2.13) always holds when $\varepsilon=\rho_2=0$. Hence we can write (2.13) as the following form

$$(e^{\sigma_2}-1)\rho_2+e^{\sigma_2}K_2\varepsilon+O(|\rho_1\rho_2|+\rho_2^2+|\rho_1\varepsilon|+|\rho_2\varepsilon|+\varepsilon^2)=0. \tag{2.14}$$

Equations (2.12) and (2.14) are called the bifurcation equations of System (1.3).

§ 3 . Bifurcation near the Closed Orbit L

Case I. $\sigma_2 \neq 0$

If $\sigma_2 \neq 0$ and ε is small enough, by the Implicit Function Theorem, Equation (2.14) has a unique solution

$$\rho_2 = \varepsilon\Big(- \frac{e^{\sigma_2} K_2}{e^{\sigma_2} - 1} + O(|\varepsilon| + |\rho_1|)\Big) \equiv \rho_2(\rho_1, \varepsilon). \tag{3.1}$$

Let $G(\rho_1)$ denote the left-hand side function of (2.12) with $\varepsilon = \rho_2 = 0$. Then $G(\rho_1)$ represents the succession function of System (2.1) in the neighborhood of closed orbit L, and (2.12) becomes

$$G(\rho_1) + e^{\sigma_1} \int_0^T b(s) ds \rho_2 + e^{\sigma_1} K_1 \varepsilon + O(|\rho_1 \rho_2| + \rho_2^2 + |\rho_1 \varepsilon| + |\rho_2 \varepsilon| + \varepsilon^2) = 0. \tag{3.2}$$

Substituting (3.1) into (3.2), we obtain

$$G(\rho_1) + e^{\sigma_1} K_3 \varepsilon + O(|\rho_1 \varepsilon| + \varepsilon^2) = 0. \tag{3.3}$$

where

$$K_3 = K_1 - \frac{e^{\sigma_2} K_2}{e^{\sigma_2} - 1} \int_0^T b(s) ds.$$

Since L is a k multiple limit cycle of System (2.1), $k \geqslant 1$, there exists $a_k \neq 0$ such that

$$G(\rho_1) = a_k \rho_1^k + O(\rho_1^{k+1}). \tag{3.4}$$

Applying the Rolle Theorem, we conclude that (3.3) has at most k solutions in ρ_1. Furthermore, if $\sigma_1 \neq 0$, then

$$G(\rho_1) = (e^{\sigma_1} - 1)\rho_1 + O(\rho_1^2).$$

It follows that Equations (2.12) and (2.14) have exactly one solution. Hence the following theorem holds.

Theorem 3.1. *Suppose that $\sigma_2 \neq 0$, for small $\varepsilon \neq 0$. We have*

(i) *If L is a k multiple closed orbit, $k \geq 1$, then System* (1.3) *has at most k closed orbits in the neighborhood of L. At least one closed orbit near L exists if k is odd.*

(ii) *If $\sigma_1 \neq 0$, then System* (1.3) *has exactly one closed orbit in the neighborhood of L.*

Let $K_3 \neq 0$. We have

Theorem 3.2. *Let* (1.3) *be an analytic system. Suppose that $\sigma_2 K_3 \neq 0$ and $\varepsilon \neq 0$ is small enough. Let* (3.4) *hold.*

(i) *If k is odd, then System* (1.3) *has exactly one closed orbit.*

(ii) *If k is even, then System* (1.3) *has exactly two closed orbits when $a_k K_3 \varepsilon < 0$ and no closed orbits when $a_k K_3 \varepsilon > 0$.*

Proof. Equation (3.3) can be written as

$$a_k \rho_1^k + e^{\sigma_1} K_3 \varepsilon + O(|\rho_1|^{k+1} + |\rho_1 \varepsilon| + \varepsilon^2) = 0. \tag{3.5}$$

By the Implicit Function Theorem, we can obtain from (3.5) that

$$\varepsilon = -\frac{a_k \rho_1^k}{e^{\sigma_1} K_3} + O(\rho_1^{k+1}) \equiv \varepsilon^*(\rho_1),$$

which is analytic. If k is odd, then function $\varepsilon^*(\rho_1)$ has a unique inverse function of the form

$$\rho_1 = \Big(-\frac{e^{\sigma_1} K_3 \varepsilon}{a_k} \Big)^{\frac{1}{k}} + o(|\varepsilon|^{\frac{1}{k}}) \equiv \rho_1^*(\varepsilon).$$

Let $\rho_2^*(\varepsilon) = \rho_2(\rho_1^*(\varepsilon), \varepsilon)$. Then (2.10) has exactly one solution $h_0 = (\rho_1^*(\varepsilon), \rho_2^*(\varepsilon))$. Thus System (1.3) has exactly one closed orbit near L. If k is even and $a_k K_3 \varepsilon < 0$, then the function $\varepsilon^*(\rho_1)$ has two inverse functions

$$\rho_1 = (-1)^i \Big(-\frac{e^{\sigma_1} K_3 \varepsilon}{a_k} \Big)^{\frac{1}{k}} + o(|\varepsilon|^{\frac{1}{k}}) \equiv \rho_{1i}^*(\varepsilon), \qquad i = 1, 2.$$

It follows that (2.10) has precisely two solutions

$$h_0 = (\rho_{1i}^*(\varepsilon), \rho_2(\rho_{1i}^*(\varepsilon), \varepsilon)) \equiv \rho_{0i}(\varepsilon), \qquad i = 1, 2,$$

which means that System (1.3) has exactly two closed orbits in the neighborhood of closed orbit L. If k is even and $a_k K_3 \varepsilon > 0$, then (3.5) has no solution and hence System (1.3) has no closed orbits in a neighborhood of the closed orbit L. This completes the proof.

In order to obtain further bifurcations, we suppose that

$$(h_1(\theta, \rho_1, \rho_2, \varepsilon), h_2(\theta, \rho_1, \rho_2, \varepsilon))$$

is a solution of System (2.6) satisfying

$$(h_1(0, \rho_1, \rho_2, \varepsilon), h_2(0, \rho_1, \rho_2, \varepsilon)) = (\rho_1, \rho_2),$$

which has an expansion of the following form

$$\begin{aligned} h_1(\theta, \rho_1, \rho_2, \varepsilon) &= A_1(\theta)\rho_1 + A_2(\theta)\rho_2 + A_3(\theta)\varepsilon + A_4(\theta)\rho_1^2 + A_5(\theta)\rho_1\rho_2 + A_6(\theta)\rho_2^2 \\ &\quad + A_7(\theta)\rho_1\varepsilon + A_8(\theta)\rho_2\varepsilon + A_9(\theta)\varepsilon^2 + O(|\rho_1, \rho_2, \varepsilon|^3), \\ h_2(\theta, \rho_1, \rho_2, \varepsilon) &= B_1(\theta)\rho_2 + B_2(\theta)\varepsilon + B_3(\theta)\rho_1\rho_2 + B_4(\theta)\rho_1\varepsilon + B_5(\theta)\rho_2^2 + B_6(\theta)\rho_2\varepsilon \\ &\quad + B_7(\theta)\varepsilon^2 + O(|\rho_2, \varepsilon||\rho_1, \rho_2, \varepsilon|^2), \end{aligned} \tag{3.6}$$

where

$$A_1(0) = 1, \quad B_1(0) = 1, \quad A_i(0) = 0, \quad B_j(0) = 0, \qquad i = 2, \cdots, 9, \ j = 2, \cdots, 7.$$

Substituting (3.6) into (2.6) and comparing the coefficients of both sides of the resulting equations, we obtain

$$
\begin{aligned}
B_1'(\theta) &= b_1(\theta)B_1(\theta),\\
B_2'(\theta) &= b_1(\theta)B_2(\theta) + b_2(\theta),\\
A_1'(\theta) &= a(\theta)A_1(\theta),\\
A_2'(\theta) &= a(\theta)A_2(\theta) + a_1(\theta)B_1(\theta),\\
A_3'(\theta) &= a(\theta)A_3(\theta) + a_1(\theta)B_2(\theta) + a_2(\theta),\\
B_3'(\theta) &= b_1(\theta)B_3(\theta) + \tilde{b}_5(\theta)B_1(\theta)A_1(\theta),\\
B_4'(\theta) &= b_1(\theta)B_4(\theta) + \tilde{b}_3(\theta)A_1(\theta) + \tilde{b}_5(\theta)B_2(\theta)A_1(\theta),\\
B_5'(\theta) &= b_1(\theta)B_5(\theta) + \tilde{b}_5(\theta)B_1(\theta)A_2(\theta) + \tilde{b}_6(\theta)B_1^2(\theta),\\
B_6'(\theta) &= b_1(\theta)B_6(\theta) + \tilde{b}_3(\theta)A_2(\theta) + \tilde{b}_4(\theta)B_1(\theta) + \tilde{b}_5(\theta)[B_1(\theta)A_3(\theta) + B_2(\theta)A_2(\theta)]\\
&\quad + 2\tilde{b}_6(\theta)B_1(\theta)B_2(\theta),\\
B_7'(\theta) &= b_1(\theta)B_7(\theta) + \tilde{b}_3(\theta)A_3(\theta) + \tilde{b}_4(\theta)B_2(\theta) + \tilde{b}_5(\theta)B_2(\theta)A_3(\theta) + \tilde{b}_6(\theta)B_2^2(\theta) + \tilde{b}_7(\theta),\\
A_4'(\theta) &= a(\theta)A_4(\theta) + \tilde{a}_5(\theta)A_1^2(\theta),\\
A_5'(\theta) &= a(\theta)A_5(\theta) + a_1(\theta)B_3(\theta) + 2\tilde{a}_5(\theta)A_1(\theta)A_2(\theta) + \tilde{a}_6(\theta)A_1(\theta)B_1(\theta),\\
A_6'(\theta) &= a(\theta)A_6(\theta) + a_1(\theta)B_5(\theta) + \tilde{a}_5(\theta)A_2^2(\theta) + \tilde{a}_6(\theta)A_2(\theta)B_1(\theta) + \tilde{a}_7(\theta)B_1^2(\theta),\\
A_7'(\theta) &= a(\theta)A_7(\theta) + a_1(\theta)B_4(\theta) + \tilde{a}_3(\theta)A_1(\theta) + 2\tilde{a}_5(\theta)A_1(\theta)A_3(\theta) + \tilde{a}_6(\theta)A_1(\theta)B_2(\theta),\\
A_8'(\theta) &= a(\theta)A_8(\theta) + a_1(\theta)B_6(\theta) + \tilde{a}_3(\theta)A_2(\theta) + \tilde{a}_4(\theta)B_1(\theta) + 2\tilde{a}_5(\theta)A_2(\theta)A_3(\theta)\\
&\quad + \tilde{a}_6(\theta)(A_3(\theta)B_1(\theta) + A_2(\theta)B_2(\theta)) + 2\tilde{a}_7(\theta)B_1(\theta)B_2(\theta),\\
A_9'(\theta) &= a(\theta)A_9(\theta) + a_1(\theta)B_7(\theta) + \tilde{a}_3(\theta)A_3(\theta) + \tilde{a}_4(\theta)B_2(\theta) + \tilde{a}_5(\theta)A_3^2(\theta)\\
&\quad + \tilde{a}_6(\theta)A_3(\theta)B_2(\theta) + \tilde{a}_7(\theta)B_2^2(\theta) + \tilde{a}_8(\theta).
\end{aligned}
\tag{3.7}
$$

It is easy to solve all the above equations and obtain

$$
\begin{aligned}
B_1(\theta) &= e^{\int_0^\theta b_1(\theta)d\theta} = e^{\int_0^\theta g(u(s),0)ds},\\
B_2(\theta) &= e^{\int_0^\theta b_1(\theta)d\theta}\int_0^\theta e^{-\int_0^\theta b_1(s)ds}b_2(\theta)d\theta\\
&= e^{\int_0^\theta g(u(s),0)ds}\int_0^\theta e^{-\int_0^\theta g(u(s),0)ds}Q(u(\theta),0,0)d\theta,\\
A_1(\theta) &= e^{\int_0^\theta a(s)ds} = \exp\Big(\int_0^\theta \mathrm{tr} f_x(u(s))ds\Big)|f(u(0))|/|f(u(\theta))|,\\
A_2(\theta) &= e^{\int_0^\theta a(s)ds}\int_0^\theta e^{-\int_0^\theta a(s)ds}a_1(\theta)B_1(\theta)d\theta\\
&= e^{\int_0^\theta \mathrm{tr} f_x(u(s))ds}\int_0^\theta e^{\int_0^\theta [g(u(s),0)-\mathrm{tr} f_x(u(s))]ds}f(u(\theta))\wedge f_1(u(\theta),0)d\theta/|f(u(\theta))|,
\end{aligned}
$$

$$
\begin{aligned}
A_3(\theta) &= e^{\int_0^\theta a(s)ds} \int_0^\theta e^{-\int_0^\theta a(s)ds}(a_1(\theta)B_2(\theta)+a_2(\theta))d\theta \\
&= e^{\int_0^\theta \mathrm{tr} f_x(u(s))ds}\Big[\int_0^\theta e^{-\int_0^\theta \mathrm{tr} f_x(u(s))ds} f(u(\theta)) \wedge P(u(\theta),0,0)d\theta \\
&\quad + \int_0^\theta e^{\int_0^\theta (g(u(s),0)-\mathrm{tr} f_x(u(s)))ds} f(u(\theta)) \wedge f_1(u(\theta),0) \\
&\quad \cdot \Big(\int_0^\theta e^{-\int_0^s g(u(s),0)ds} Q(u(s),0,0)ds\Big)d\theta\Big] \Big/ |f(u(\theta))|, \\
B_3(\theta) &= \exp\Big(\int_0^\theta g(u(s),0)ds\Big)\int_0^\theta \tilde{b}_5(\theta)A_1(\theta)d\theta, \\
B_4(\theta) &= \exp\Big(\int_0^\theta g(u(s),0)ds\Big)\int_0^\theta [\tilde{b}_3(\theta)A_1(\theta)+\tilde{b}_5(\theta)B_2(\theta)A_1(\theta)] \\
&\quad \cdot \exp\Big(-\int_0^\theta g(u(s),0)ds\Big)d\theta, \\
B_5(\theta) &= \exp\Big(\int_0^\theta g(u(s),0)ds\Big)\int_0^\theta [\tilde{b}_5(\theta)B_1(\theta)A_2(\theta)+\tilde{b}_6(\theta)B_1^2(\theta)] \\
&\quad \cdot \exp\Big(-\int_0^\theta g(u(s),0)ds\Big)d\theta, \\
B_6(\theta) &= \exp\Big(\int_0^\theta g(u(s),0)ds\Big)\int_0^\theta [\tilde{b}_3(\theta)A_2(\theta)+\tilde{b}_4(\theta)B_1(\theta)+\tilde{b}_5(\theta)(B_1(\theta)A_3(\theta) \\
&\quad + B_2(\theta)A_2(\theta))+2\tilde{b}_6(\theta)B_1(\theta)B_2(\theta)]\exp\Big(-\int_0^\theta g(u(s),0)ds\Big)d\theta, \\
B_7(\theta) &= \exp\Big(\int_0^\theta g(u(s),0)ds\Big)\int_0^\theta [\tilde{b}_3(\theta)A_3(\theta)+\tilde{b}_4(\theta)B_2(\theta)+\tilde{b}_5(\theta)B_2(\theta)A_3(\theta) \\
&\quad + \tilde{b}_6(\theta)B_2^2(\theta)+\tilde{b}_7(\theta)]\exp\Big(-\int_0^\theta g(u(s),0)ds\Big)d\theta, \\
A_4(T) &= e^{\sigma_1}\int_0^T \Big[\tilde{a}_5(\theta)\exp\Big(\int_0^\theta \mathrm{tr} f_x(u(s))ds\Big)|f(u(0))|/|f(u(\theta))|\Big]d\theta, \\
A_5(T) &= e^{\sigma_1}\int_0^T [a_1(\theta)B_3(\theta)+2\tilde{a}_5(\theta)A_1(\theta)A_2(\theta)+\tilde{a}_6(\theta)A_1(\theta)B_1(\theta)] \\
&\quad \cdot \exp\Big(-\int_0^\theta \mathrm{tr} f_x(u(s))ds\Big)|f(u(\theta))|/|f(u(0))|d\theta, \\
A_6(T) &= e^{\sigma_1}\int_0^T [a_1(\theta)B_5(\theta)+\tilde{a}_5(\theta)A_2^2(\theta)+\tilde{a}_6(\theta)A_2(\theta)B_1(\theta)+\tilde{a}_7(\theta)B_1^2(\theta)] \\
&\quad \cdot \exp\Big(-\int_0^\theta \mathrm{tr} f_x(u(s))ds\Big)|f(u(\theta))|/|f(u(0))|d\theta, \\
A_7(T) &= e^{\sigma_1}\int_0^T [a_1(\theta)B_4(\theta)+\tilde{a}_3(\theta)A_1(\theta)+2\tilde{a}_5(\theta)A_1(\theta)A_3(\theta) \\
&\quad + \tilde{a}_6(\theta)A_1(\theta)B_2(\theta)]\exp\Big(-\int_0^\theta \mathrm{tr} f_x(u(s))ds\Big)|f(u(\theta))|/|f(u(0))|d\theta, \\
A_8(T) &= e^{\sigma_1}\int_0^T [a_1(\theta)B_6(\theta)+\tilde{a}_3(\theta)A_2(\theta)+\tilde{a}_4(\theta)B_1(\theta)+2\tilde{a}_5(\theta)A_2(\theta)A_3(\theta)
\end{aligned}
$$

$$
\begin{aligned}
&+ \tilde{a}_6(\theta)(A_3(\theta)B_1(\theta) + A_2(\theta)B_2(\theta)) + 2\tilde{a}_7(\theta)B_1(\theta)B_2(\theta)] \\
&\cdot \exp\Big(- \int_0^\theta \mathrm{tr} f_x(u(s))ds\Big)|f(u(\theta))|/|f(u(0))|d\theta, \\
A_9(T) = e^{\sigma_1} \int_0^T &[a_1(\theta)B_7(\theta) + \tilde{a}_3(\theta)A_3(\theta) + \tilde{a}_4(\theta)B_2(\theta) + \tilde{a}_5(\theta)A_3^2(\theta) \\
&+ \tilde{a}_6(\theta)A_3(\theta)B_2(\theta) + \tilde{a}_7(\theta)B_2^2(\theta) + \tilde{a}_8(\theta)] \\
&\cdot \exp\Big(- \int_0^\theta \mathrm{tr} f_x(u(s))ds\Big)|f(u(\theta))|/|f(u(0))|d\theta.
\end{aligned}
$$

Then the succession function of (2.6) is

$$
\begin{aligned}
h_1^*(\rho_1, \rho_2, \varepsilon) &= h_1(T, \rho_1, \rho_2, \varepsilon) - \rho_1 \\
&= (A_1(T) - 1)\rho_1 + A_2(T)\rho_2 + A_3(T)\varepsilon + \tilde{c}_4\rho_1^2 \\
&\quad + \tilde{c}_5\rho_1\rho_2 + \tilde{c}_6\rho_2^2 + \tilde{c}_7\rho_1\varepsilon + \tilde{c}_8\rho_2\varepsilon + \tilde{c}_9\varepsilon^2 + O(|\rho_1, \rho_2, \varepsilon|^3), \\
h_2^*(\rho_1, \rho_2, \varepsilon) &= h_2(T, \rho_1, \rho_2, \varepsilon) - \rho_2 \\
&= (B_1(T) - 1)\rho_2 + B_2(T)\varepsilon + \tilde{d}_3\rho_1\rho_2 + \tilde{d}_4\rho_1\varepsilon + \tilde{d}_5\rho_2^2 + \tilde{d}_6\rho_2\varepsilon \\
&\quad + \tilde{d}_7\varepsilon^2 + O(|\rho_2, \varepsilon||\rho_1, \rho_2, \varepsilon|^2), \qquad (3.8)
\end{aligned}
$$

where

$$\tilde{c}_i = A_i(T), \quad \tilde{d}_j = B_j(\theta)|_{\theta=T}, \qquad i = 4, \cdots, 9, \ j = 3, \cdots, 7.$$

Similarly to (3.2), we obtain bifurcation equations as follows

$$
\begin{aligned}
h_1^*(\rho_1, \rho_2, \varepsilon) &= a_k\rho_1^k + e^{\sigma_1}\int_0^T b(s)ds\rho_2 + e^{\sigma_1}K_1\varepsilon + \tilde{c}_5\rho_1\rho_2 + \tilde{c}_6\rho_2^2 + \tilde{c}_7\rho_1\varepsilon \\
&\quad + \tilde{c}_8\rho_2\varepsilon + \tilde{c}_9\varepsilon^2 + O(|\rho_2, \varepsilon||\rho_1, \rho_2, \varepsilon|^2 + |\rho_1|^{k+1}) = 0, \qquad (3.9) \\
h_2^*(\rho_1, \rho_2, \varepsilon) &= (e^{\sigma_2} - 1)\rho_2 + e^{\sigma_2}K_2\varepsilon + \tilde{d}_3\rho_1\rho_2 + \tilde{d}_4\rho_1\varepsilon + \tilde{d}_5\rho_2^2 + \tilde{d}_6\rho_2\varepsilon \\
&\quad + \tilde{d}_7\varepsilon^2 + O(|\rho_2, \varepsilon||\rho_1, \rho_2, \varepsilon|^2) = 0. \qquad (3.10)
\end{aligned}
$$

If $\sigma_2 \neq 0$, we can solve (3.10) and obtain

$$\rho_2 = \varepsilon(\tilde{\rho}_{21} + \tilde{\rho}_{22}\,\rho_1 + \tilde{\rho}_{23}\varepsilon + O(|\rho_1, \varepsilon|^2)) \equiv \rho_2(\rho_1, \varepsilon), \qquad (3.11)$$

where

$$
\begin{aligned}
\tilde{\rho}_{21} &= \frac{e^{\sigma_2}K_2}{1 - e^{\sigma_2}}, \\
\tilde{\rho}_{22} &= \frac{e^{\sigma_2}\tilde{d}_3K_2 + (1 - e^{\sigma_2})\tilde{d}_4}{(1 - e^{\sigma_2})^2}, \\
\tilde{\rho}_{23} &= \frac{e^{2\sigma_2}\tilde{d}_5(K_2)^2}{(1 - e^{\sigma_2})^3} + \frac{e^{\sigma_2}\tilde{d}_6K_2}{(1 - e^{\sigma_2})^2} + \frac{\tilde{d}_7}{1 - e^{\sigma_2}}.
\end{aligned}
$$

Substituting (3.11) into (3.9), we have

$$H_1(\rho_1,\varepsilon)\equiv h_1^*(\rho_1,\rho_2(\rho_1,\varepsilon),\varepsilon)=a_k\rho_1^k+\alpha_1\rho_1\varepsilon+\alpha_2\varepsilon^2$$
$$+e^{\sigma_1}K_3\varepsilon+O(|\rho_1|^{k+1}+|\rho_1^2\varepsilon|+|\rho_1|\varepsilon^2+|\varepsilon|^3)=0, \tag{3.12}$$

where

$$\alpha_1=e^{\sigma_1}\int_0^T b(s)ds\,\tilde\rho_{22}+\tilde c_5\tilde\rho_{21}+\tilde c_7,$$
$$\alpha_2=e^{\sigma_1}\int_0^T b(s)ds\,\tilde\rho_{23}+\tilde c_6\tilde\rho_{21}^2+\tilde c_8\tilde\rho_{21}+\tilde c_9,$$

$H_1(\rho_1,\varepsilon)$ is C^∞ in (ρ_1,ε) for $|\rho_1|+|\varepsilon|$ small.

Theorem 3.3. *Suppose that $\sigma_2\neq 0$ and $K_3=0$. Let L be a closed orbit of System* (2.1) *of multiplicity k, $k\geqslant 2$. For small $\varepsilon\neq 0$, we have*

(i) *If $k=2$, then L generates exactly two hyperbolic closed orbits* (*resp. no closed orbits*) *of System* (1.3) *when $\Delta_1>0$* (*resp. $\Delta_1<0$*), *where $\Delta_1=\alpha_1^2-4a_2\alpha_2$.*

(ii) *If $k>2$ and $\alpha_1\alpha_2\neq 0$, then for odd number k, L generates exactly one closed orbit* (*resp. three closed orbits*) *of System* (1.3) *when $\varepsilon\alpha_1 a_k>0$* (*resp. $\varepsilon\alpha_1 a_k<0$*), *for even number k, L generates exactly two closed orbits of System* (1.3).

Proof. Since $K_3=0$, by (3.12), we have

$$H_2(\rho_1,\varepsilon)\equiv H_1(\rho_1,\varepsilon)|_{K_3=0}$$
$$=a_k\rho_1^k+\alpha_1\rho_1\varepsilon+\alpha_2\varepsilon^2+O(|\rho_1|^{k+1}+|\rho_1^2\varepsilon|+|\rho_1|\varepsilon^2+|\varepsilon|^3)=0. \tag{3.13}$$

(i) If $k=2$, we rewrite $H_2(\rho_1,\varepsilon)$ as

$$H_2(\rho_1,\varepsilon)=\tilde\alpha_0(\varepsilon)+\tilde\alpha_1(\varepsilon)\rho_1+\tilde\alpha_2(\varepsilon)\rho_1^2(1+o(1)),$$

where

$$\tilde\alpha_0(\varepsilon)=\alpha_2\varepsilon^2+O(\varepsilon^3),\qquad \tilde\alpha_1(\varepsilon)=\alpha_1\varepsilon+O(\varepsilon^2),\qquad \tilde\alpha_2(\varepsilon)=a_2+O(\varepsilon).$$

We can solve $\dfrac{\partial H_2(\rho_1,\varepsilon)}{\partial\rho_1}=0$ and obtain

$$\rho_1=-\frac{\alpha_1}{2a_2}\varepsilon+O(\varepsilon^2)\equiv\eta(\varepsilon).$$

Then by the Taylor Formula, we can obtain

$$H_2(\rho_1,\varepsilon)=H_2(\eta(\varepsilon),\varepsilon)+\frac12\frac{\partial^2H_2(\eta(\varepsilon),\varepsilon)}{\partial\rho_1^2}(\rho_1-\eta(\varepsilon))^2(1+o(1))$$
$$=\frac12\frac{\partial^2H_2(\eta(\varepsilon),\varepsilon)}{\partial\rho_1^2}(1+o(1))\Big((\rho_1-\eta(\varepsilon))^2-\frac{\Delta_1^*}{1+o(1)}\Big),$$

where

$$H_2(\eta(\varepsilon),\varepsilon)=-\frac{\Delta_1}{4a_2}\varepsilon^2+O(|\varepsilon|^3),\qquad \Delta_1^*=-\frac{2H_2(\eta(\varepsilon),\varepsilon)}{\frac{\partial^2H_2(\eta(\varepsilon),\varepsilon)}{\partial\rho_1^2}}=\frac{\Delta_1}{4a_2^2}\varepsilon^2+O(\varepsilon^3).$$

Thus (3.13) has two solutions when $\Delta_1 > 0$, which have the forms

$$\rho_1 = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 - 4a_2\alpha_2}}{2a_2}\varepsilon + O(\varepsilon^2), \tag{3.14}$$

and no solutions when $\Delta_1 < 0$. Therefore the conclusion (i) holds.

(ii) If $k > 2$ and $\alpha_1\alpha_2 \neq 0$, by using (3.13) and the Malgrange Preparation Theorem (cf. [5] or [1]), (3.13) has at most two solutions in ε. In order to obtain the solution of (3.13), let $\varepsilon = \rho_1 v_1$. Substituting it into (3.13), we know that v_1 satisfies

$$\Phi_1(\rho_1, v_1) \equiv \alpha_1 v_1 + \alpha_2 v_1^2 + a_k \rho_1^{k-2} + O(|\rho_1|^{k-1} + |\rho_1 v_1|) = 0. \tag{3.15}$$

Since $\Phi_1\big(0, -\frac{\alpha_1}{\alpha_2}\big) = 0$ and $\frac{\partial \Phi_1\left(0, -\frac{\alpha_1}{\alpha_2}\right)}{\partial v_1} = -\alpha_1 \neq 0$, by using the Implicit Function Theorem, (3.15) has a solution $v_1 = -\frac{\alpha_1}{\alpha_2} + O(|\rho_1|)$. Hence (3.13) has a solution

$$\varepsilon = \rho_1 v_1 = -\frac{\alpha_1}{\alpha_2}\rho_1 + O(|\rho_1|^2). \tag{3.16}$$

To obtain another solution of (3.13), let $\varepsilon = \rho_1^{k-1} v_2$. Substituting it into (3.13), we know that v_2 satisfies

$$\Phi_2(\rho_1, v_2) \equiv a_k + \alpha_1 v_2 + \alpha_2 \rho_1^{k-2} v_2^2 + O(|\rho_1|) = 0. \tag{3.17}$$

Since $\Phi_2\big(0, -\frac{a_k}{\alpha_1}\big) = 0$ and $\frac{\partial \Phi_2\left(0, -\frac{a_k}{\alpha_1}\right)}{\partial v_2} = \alpha_1 \neq 0$, by using the Implicit Function Theorem, (3.17) has a solution $v_2 = -\frac{a_k}{\alpha_1} + O(\rho_1)$. Hence (3.13) has a solution

$$\varepsilon = \rho_1^{k-1} v_2 = -\frac{a_k}{\alpha_1}\rho_1^{k-1} + O(|\rho_1|^k). \tag{3.18}$$

(3.16) and (3.18) are the two solutions of (3.13).

The function in (3.16) has an inverse function

$$\rho_1 = -\frac{\alpha_2}{\alpha_1}\varepsilon + O(\varepsilon^2) \equiv \bar{\rho}_1(\varepsilon). \tag{3.19}$$

If k is odd, then (3.18) has exactly two inverse functions

$$\rho_1 = (-1)^i \Big(-\frac{\alpha_1 \varepsilon}{a_k}\Big)^{\frac{1}{k-1}} + o(|\varepsilon|^{\frac{1}{k-1}}) \equiv \bar{\rho}_{1i}(\varepsilon), \qquad i = 1, 2 \tag{3.20}$$

when $\varepsilon\alpha_1 a_k < 0$ and has no inverse functions when $\varepsilon\alpha_1 a_k > 0$. If k is even, then (3.18) has exactly one inverse function

$$\rho_1 = \Big(-\frac{\alpha_1 \varepsilon}{a_k}\Big)^{\frac{1}{k-1}} + o(|\varepsilon|^{\frac{1}{k-1}}) \equiv \bar{\rho}_{13}(\varepsilon). \tag{3.21}$$

By (3.11), and (3.19)–(3.21), we see that if k is odd and $\varepsilon\alpha_1 a_k < 0$, then the bifurcation equations (3.9) and (3.10) have exactly three solutions

$$(\rho_1, \rho_2) = (\bar{\rho}_1(\varepsilon), \rho_2(\bar{\rho}_1(\varepsilon), \varepsilon)),$$

$$(\rho_1, \rho_2) = (\bar{\rho}_{11}(\varepsilon), \rho_2(\bar{\rho}_{11}(\varepsilon), \varepsilon))$$

$$(\rho_1, \rho_2) = (\bar{\rho}_{12}(\varepsilon), \rho_2(\bar{\rho}_{12}(\varepsilon), \varepsilon)).$$

If k is odd and $\varepsilon\alpha_1 a_k > 0$, then the bifurcation equations (3.9) and (3.10) have a unique solution

$$(\rho_1, \rho_2) = (\bar{\rho}_1(\varepsilon), \rho_2(\bar{\rho}_1(\varepsilon), \varepsilon)).$$

If k is even, then the bifurcation equations (3.9) and (3.10) have exactly two solutions

$$(\rho_1, \rho_2) = (\bar{\rho}_1(\varepsilon), \rho_2(\bar{\rho}_1(\varepsilon), \varepsilon))$$
$$(\rho_1, \rho_2) = (\bar{\rho}_{13}(\varepsilon), \rho_2(\bar{\rho}_{13}(\varepsilon), \varepsilon)).$$

This completes the proof.

Case II. $\sigma_2 = 0$

If $\sigma_2 = 0$ and $\int_0^T b(s)ds \neq 0$, by (3.9) and the Implicit Function Theorem, we have

$$\rho_2 = \hat{\alpha}_1\varepsilon + \hat{\alpha}_2\rho_1^k + O(|\rho_1|^{k+1} + |\varepsilon|^2 + |\rho_1\varepsilon|) \equiv \tilde{\rho}_2(\rho_1, \varepsilon)\,, \tag{3.22}$$

where

$$\hat{\alpha}_1 = -\frac{K_1}{\int_0^T b(s)ds}\,, \qquad \hat{\alpha}_2 = -\frac{a_k}{e^{\sigma_1}\int_0^T b(s)ds}\,.$$

Substituting (3.22) into (3.10), we obtain

$$\hat{\alpha}_2\beta_{1k}\rho_1^{k+1} + K_2\varepsilon + \gamma_2\rho_1\varepsilon + \gamma_3\varepsilon^2 + O(|\rho_1|^{k+2} + |\rho_1^2\varepsilon| + |\rho_1\varepsilon^2| + |\varepsilon|^3) = 0\,, \tag{3.23}$$

where

$$\beta_{1k} = \begin{cases} \tilde{d}_3, & k > 1, \\ \tilde{d}_3 - \dfrac{a_1\tilde{d}_5}{e^{\sigma_1}\int_0^T b(s)ds}, & k = 1, \end{cases}$$

$$\gamma_2 = \begin{cases} \hat{\alpha}_1\tilde{d}_3 + \tilde{d}_4 + 2\hat{\alpha}_1\hat{\alpha}_2\tilde{d}_5 + \hat{\alpha}_2\tilde{d}_6, & k = 1, \\ \hat{\alpha}_1\tilde{d}_3 + \tilde{d}_4, & k > 1, \end{cases}$$

$$\gamma_3 = \hat{\alpha}_1^2\tilde{d}_5 + \hat{\alpha}_1\tilde{d}_6 + \tilde{d}_7.$$

If $\beta_{1k} \neq 0$, then (3.23) has at most $k+1$ solutions in ρ_1 and at least one solution in ρ_1 when k is even.

Suppose $K_2 \neq 0$. If k is odd, then (3.23) has exactly two solutions

$$\rho_1 = (-1)^i\left(\frac{e^{\sigma_1}\int_0^T b(s)dsK_2}{a_k\beta_{1k}}\varepsilon\right)^{\frac{1}{k+1}} + o(|\varepsilon|^{\frac{1}{k+1}}) \equiv \rho_{1i}(\varepsilon), \qquad i = 1, 2,$$

when $K_2\varepsilon a_k\beta_{1k}\int_0^T b(s)ds > 0$, and no solutions when $K_2\varepsilon a_k\beta_{1k}\int_0^T b(s)ds < 0$. Hence Equations (3.9) and (3.10) have exactly two solutions

$$(\rho_1, \rho_2) = (\rho_{1i}(\varepsilon), \tilde{\rho}_2(\rho_{1i}(\varepsilon), \varepsilon)), \qquad i = 1, 2,$$

when $K_2\varepsilon a_k\beta_{1k}\int_0^T b(s)ds > 0$, and no solutions when $K_2\varepsilon a_k\beta_{1k}\int_0^T b(s)ds < 0$. If k is even, then (3.23) has a unique solution

$$\rho_1 = \Big(\frac{e^{\sigma_1}\int_0^T b(s)dsK_2}{a_k\beta_{1k}}\varepsilon\Big)^{\frac{1}{k+1}} + o(|\varepsilon|^{\frac{1}{k+1}}) \equiv \rho_{13}(\varepsilon).$$

Hence Equations (3.9) and (3.10) have exactly one solution

$$(\rho_1, \rho_2) = (\rho_{13}(\varepsilon), \tilde{\rho}_2(\rho_{13}(\varepsilon), \varepsilon)).$$

If $\sigma_2 = 0$ and $K_2 = 0$, then (3.23) becomes

$$\hat{\alpha}_2\beta_{1k}\rho_1^{k+1} + \gamma_2\rho_1\varepsilon + \gamma_3\varepsilon^2 + O(|\rho_1|^{k+2} + |\rho_1^2\varepsilon| + |\rho_1\varepsilon^2| + |\varepsilon|^3) = 0. \tag{3.24}$$

Similarly to the proof of Theorem 3.3 and the discussion above, we have

Theorem 3.4. *Suppose that L is a k multiple closed orbit of System* (2.1)*, $\sigma_2 = 0$ and $\beta_{1k}\displaystyle\int_0^T b(s)ds \neq 0$. For small $\varepsilon \neq 0$, we have*

(1) *L generates at most $k+1$ closed orbits of System* (1.3)*. At least one closed orbit near L exists if k is even.*

(2) *If $K_2 \neq 0$, then*

(i) *For odd number k, L generates exactly two closed orbits of System* (1.3) *when $K_2\varepsilon a_k \cdot\beta_{1k}\displaystyle\int_0^T b(s)ds > 0$*, and no closed orbits when $K_2\varepsilon a_k\beta_{1k}\displaystyle\int_0^T b(s)ds < 0$.

(ii) *For even number k, L generates exactly one closed orbit of System* (1.3).

(3) *For $K_2 = 0$, we have*

(i) *If $k = 1$, then L generates exactly two hyperbolic closed orbits* (*resp. no closed orbits*) *of System* (1.3) *when $\Delta_2 > 0$* (*resp. $\Delta_2 < 0$*)*, where $\Delta_2 = \gamma_2^2 - 4\hat{\alpha}_2\beta_{11}\gamma_3$.*

(ii) *If $k > 1$ and $\gamma_2\gamma_3 \neq 0$, for even number k, L generates exactly one closed orbit* (*resp. three closed orbits*) *of System* (1.3) *when $\varepsilon a_k\beta_{1k}\displaystyle\int_0^T b(s)ds\gamma_2 < 0$ $\Big($resp. $\varepsilon a_k\beta_{1k}\displaystyle\int_0^T b(s)ds\gamma_2 > 0\Big)$, for odd number k, L generates exactly two closed orbits of System* (1.3).

If $\sigma_2 = \int_0^T b(s)ds = 0$, then (3.9) and (3.10) become

$$\begin{aligned} &a_k\rho_1^k + e^{\sigma_1}K_1\varepsilon + \tilde{c}_5\rho_1\rho_2 + \tilde{c}_6\rho_2^2 + \tilde{c}_7\rho_1\varepsilon + \tilde{c}_8\rho_2\varepsilon + \tilde{c}_9\varepsilon^2 \\ &\quad + O(|\rho_2, \varepsilon||\rho_1, \rho_2, \varepsilon|^2 + |\rho_1|^{k+1}) = 0, \end{aligned} \tag{3.25}$$

$$\begin{aligned} &K_2\varepsilon + \tilde{d}_3\rho_1\rho_2 + \tilde{d}_4\rho_1\varepsilon + \tilde{d}_5\rho_2^2 + \tilde{d}_6\rho_2\varepsilon + \tilde{d}_7\varepsilon^2 \\ &\quad + O(|\rho_2, \varepsilon||\rho_1, \rho_2, \varepsilon|^2) = 0. \end{aligned} \tag{3.26}$$

If $K_2 \neq 0$, we have

Theorem 3.5. *Let* (1.3) *be an analytic system. Suppose that L is a k multiple closed orbit of System* (2.1)*, $\sigma_2 = \displaystyle\int_0^T b(s)ds = 0$ and $K_2 \neq 0$. For small $\varepsilon \neq 0$, we have*

(1) *L generates at most $2k$ closed orbits of System* (1.3).

(2) *For $k=1$, L generates exactly two closed orbits of System* (1.3) *when $K_2\tilde{d}_5\varepsilon<0$ and no closed orbits when $K_2\tilde{d}_5\varepsilon>0$.*

(3) *Let*

$$\alpha_{1i}=\tilde{d}_5+\frac{-\beta_1+(-1)^i\sqrt{\beta_1^2-4a_2\beta_2}}{2a_2}\tilde{d}_3,\qquad i=1,2.$$

For $k=2$, if $\beta_1^2-4a_2\beta_2<0$, then System (1.3) *has no closed orbits in the neighborhood of L. If $\beta_1^2-4a_2\beta_2>0$ and $\alpha_{11}\alpha_{12}<0$, then System* (1.3) *has exactly two closed orbits in the neighborhood of L. If $\beta_1^2-4a_2\beta_2>0$ and $\alpha_{11}\alpha_{12}>0$, then System* (1.3) *has no closed orbits in the neighborhood of L when $K_2\alpha_{11}\varepsilon>0$ and has exactly four closed orbits when $K_2\alpha_{11}\varepsilon<0$.*

(4) *For $k>2$, if $\beta_1\beta_2\neq 0$, then we have*

(i) *If k is odd, then L generates exactly one closed orbits of System* (1.3) *when $K_2\beta_1(\tilde{c}_6\tilde{d}_3-\tilde{c}_5\tilde{d}_5)\varepsilon<0$ and L generates exactly three closed orbits of System* (1.3) *when $K_2\beta_1(\tilde{c}_6\tilde{d}_3-\tilde{c}_5\tilde{d}_5)\varepsilon>0$.*

(ii) *If k is even and $a_k\tilde{d}_3(\tilde{c}_6\tilde{d}_3-\tilde{c}_5\tilde{d}_5)<0$, then L generates exactly two closed orbits of System* (1.3). *If k is even and $a_k\tilde{d}_3(\tilde{c}_6\tilde{d}_3-\tilde{c}_5\tilde{d}_5)>0$, then L generates exactly four closed orbits of System* (1.3) *when $a_k\tilde{d}_3K_2\beta_1\varepsilon>0$ and no closed orbits when $a_k\tilde{d}_3K_2\beta_1\varepsilon<0$.*

Proof. If $\sigma_2=\int_0^T b(s)ds=0$ and $K_2\neq 0$, by (3.26), we have

$$\varepsilon=-\frac{1}{K_2}(\tilde{d}_3\rho_1\rho_2+\tilde{d}_5\rho_2^2)+O(\rho_2|\rho_1,\rho_2|^2),\tag{3.27}$$

Substituting (3.27) into (3.25), we have

$$a_k\rho_1^k+\beta_1\rho_1\rho_2+\beta_2\rho_2^2+O(|\rho_2||\rho_1,\rho_2|^2+|\rho_1|^{k+1})=0,\tag{3.28}$$

where $\beta_1=\tilde{c}_5-\frac{K_1}{K_2}e^{\sigma_1}\tilde{d}_3,\ \beta_2=\tilde{c}_6-\frac{K_1}{K_2}e^{\sigma_1}\tilde{d}_5$.

(1) Let $\tilde{f}(\rho_1,\rho_2)=a_k\rho_1^k+\beta_1\rho_1\rho_2+\beta_2\rho_2^2+O(|\rho_2||\rho_1,\rho_2|^2+|\rho_1|^{k+1})$. By using the Weierstrass Preparation Theorem (cf. [6] or [1]), we see that Equation (3.28) has at most k solutions in ρ_1. If $\tilde{f}(0,\rho_2)\not\equiv 0$, then by Theorem 1.9 of [6] (or by [1]) we know that the solutions of Equation (3.28) (if they exist) have the forms

$$\rho_1=\rho_2^{\bar{\beta}_j}\bar{v}_j(\rho_2^{\bar{r}_j}),\qquad j=1,\cdots,l,\quad 1\le l\le k,\tag{3.29}$$

where $\bar{v}_j(0)\neq 0$, $\bar{v}_j(u)$ is analytic in u near $u=0$ and $\bar{\beta}_j,\bar{r}_j$ are positive rational numbers. Substituting (3.29) into (3.27), we see that there exist $\bar{c}_j$ and $\bar{\alpha}_j$ such that the resulting equation has the form

$$\varepsilon=\bar{c}_j\rho_2^{\bar{\alpha}_j}+o(|\rho_2|^{\bar{\alpha}_j}),\qquad j=1,\cdots,l,\tag{3.30}$$

where $\bar{c}_j\in\mathbb{R}$ and $\bar{\alpha}_j$ is a positive rational number. It is easy to see that (3.30) has at most two inverse functions for every j. Hence by (3.29) and (3.30), we know that Equations (3.25) and (3.26) have at most $2k$ solutions. If $\tilde{f}(0,\rho_2)\equiv 0$, then we can rewrite $\tilde{f}(\rho_1,\rho_2)$ as $\tilde{f}(\rho_1,\rho_2)=\rho_1\tilde{f}_1(\rho_1,\rho_2)$, where $\tilde{f}_1(\rho_1,\rho_2)$ is analytic in (ρ_1,ρ_2), which has the form

$$\tilde{f}_1(\rho_1,\rho_2)=a_k\rho_1^{k-1}+\beta_1\rho_2+O(|\rho_1\rho_2|+|\rho_2|^2+|\rho_1|^k).$$

Equation $\tilde{f}(\rho_1, \rho_2) = 0$ yields $\rho_1 = 0$ or $\tilde{f}_1(\rho_1, \rho_2) = 0$. If $\rho_1 = 0$, then by (3.27) we have $\varepsilon = O(\rho_2^2)$, which has at most two inverse functions. If $\tilde{f}_1(0, \rho_2) \not\equiv 0$, by an analysis similar to the above, we know $\tilde{f}_1(\rho_1, \rho_2) = 0$ and (3.27) have at most $2(k-1)$ solutions, hence Equations (3.25) and (3.26) have at most $2k$ solutions. If $\tilde{f}_1(0, \rho_2) \equiv 0$, let $\tilde{f}_1(\rho_1, \rho_2) = \rho_1 \tilde{f}_2(\rho_1, \rho_2)$, where $\tilde{f}_2(\rho_1, \rho_2)$ is analytic in (ρ_1, ρ_2), which has the form

$$\tilde{f}_2(\rho_1, \rho_2) = a_k \rho_1^{k-2} + O(|\rho_2| + |\rho_1|^{k-1}).$$

Repeat the same process as above. In general, there is a natural number m and an analytic function $\tilde{f}_m(\rho_1, \rho_2)$, such that

$$\tilde{f}(\rho_1, \rho_2) = \rho_1^m \tilde{f}_m(\rho_1, \rho_2),$$

where $m \le k - 1$ and $\tilde{f}_m(0, \rho_2) \not\equiv 0$ or $m = k$. If $m \le k - 1$ and $\tilde{f}_m(0, \rho_2) \not\equiv 0$, then Equations (3.25) and (3.26) have at most $2(k - m + 1)$ solutions. If $m = k$, then

$$\tilde{f}_k(\rho_1, \rho_2) = a_k + O(|\rho_1| + |\rho_2|) \neq 0. \tag{3.31}$$

In this case Equations (3.25) and (3.26) have at most two solutions. Summarizing the above, we conclude that (3.25) and (3.26) have at most $2k$ solutions. Hence L generates at most $2k$ closed orbits of System (1.3).

(2) For $k = 1$, by (3.28), we have

$$\rho_1 = -\frac{\beta_2}{a_1}\rho_2^2 + O(|\rho_2|^3) \equiv \rho_1(\rho_2). \tag{3.32}$$

Substituting (3.32) into (3.27), we obtain

$$\varepsilon = -\frac{\tilde{d}_5}{K_2}\rho_2^2 + O(|\rho_2|^3). \tag{3.33}$$

If $K_2\tilde{d}_5\varepsilon > 0$, then (3.33) has no inverse functions. If $K_2\tilde{d}_5\varepsilon < 0$, then (3.33) has two inverse functions

$$\rho_2 = (-1)^i\Big(-\frac{K_2}{\tilde{d}_5}\varepsilon\Big)^{\frac{1}{2}} + o(|\varepsilon|^{\frac{1}{2}}) \equiv \rho_{2i}(\varepsilon), \qquad i = 1, 2.$$

Hence (3.25), (3.26) have no solutions when $K_2\tilde{d}_5\varepsilon > 0$ and have two solutions

$$(\rho_1, \rho_2) = (\rho_1(\rho_{2i}(\varepsilon)), \rho_{2i}(\varepsilon)), \qquad i = 1, 2,$$

when $K_2\tilde{d}_5\varepsilon < 0$.

(3) For $k = 2$, similarly to the proof of Theorem 3.3(1), we conclude that (3.28) has no solutions when $\beta_1^2 - 4a_2\beta_2 < 0$. If $\beta_1^2 - 4a_2\beta_2 > 0$, then (3.28) has two solutions

$$\rho_1 = \frac{-\beta_1 + (-1)^i\sqrt{\beta_1^2 - 4a_2\beta_2}}{2a_2}\rho_2 + O(\rho_2^2) \equiv \tilde{\rho}_{1i}(\rho_2), \qquad i = 1, 2. \tag{3.34}$$

Substituting (3.34) into (3.27), we have

$$\varepsilon = -\frac{\alpha_{1i}}{K_2}\rho_2^2 + O(|\rho_2|^3), \qquad i = 1, 2. \tag{3.35}$$

If $K_2\alpha_{1i}\varepsilon > 0$, $i = 1, 2$, then (3.35) has no inverse functions for $i = 1, 2$ respectively. If $K_2\alpha_{11}\varepsilon < 0$, then (3.35) for $i = 1$ has two inverse functions

$$\rho_2 = (-1)^l\Big(-\frac{K_2}{\alpha_{11}}\varepsilon\Big)^{\frac{1}{2}} + o(|\varepsilon|^{\frac{1}{2}}) \equiv \rho_{21l}(\varepsilon), \qquad l = 1, 2.$$

If $K_2\alpha_{12}\varepsilon < 0$, then (3.35) for $i = 2$ has two inverse functions

$$\rho_2 = (-1)^j\Big(-\frac{K_2}{\alpha_{12}}\varepsilon\Big)^{\frac{1}{2}} + o(|\varepsilon|^{\frac{1}{2}}) \equiv \rho_{22j}(\varepsilon), \qquad j = 1, 2.$$

Therefore for $\beta_1^2 - 4a_2\beta_2 > 0$, if $\alpha_{11}\alpha_{12} < 0$, then (3.35) has only two inverse functions $\rho_2 = \rho_{21l}(\varepsilon)$ $(l = 1, 2)$ or $\rho_2 = \rho_{22j}(\varepsilon)$ $(j = 1, 2)$. Thus (3.25), (3.26) have exactly two solutions

$$(\rho_1, \rho_2) = (\tilde{\rho}_{11}(\rho_{21l}(\varepsilon)),\ \rho_{21l}(\varepsilon)), \qquad l = 1, 2,$$

or

$$(\rho_1, \rho_2) = (\tilde{\rho}_{12}(\rho_{22j}(\varepsilon)),\ \rho_{22j}(\varepsilon)), \qquad j = 1, 2.$$

If $\alpha_{11}\alpha_{12} > 0$, then (3.35) has no inverse functions for $i = 1$ and $i = 2$ when $K_2\alpha_{11}\varepsilon > 0$ and has four inverse functions $\rho_2 = \rho_{211}(\varepsilon)$, $\rho_2 = \rho_{212}(\varepsilon)$, $\rho_2 = \rho_{221}(\varepsilon)$ and $\rho_2 = \rho_{222}(\varepsilon)$ when $K_2\alpha_{11}\varepsilon < 0$. Thus (3.25) and (3.26) have no solutions when $K_2\alpha_{11}\varepsilon > 0$ and have four solutions

$$(\rho_1, \rho_2) = (\tilde{\rho}_{11}(\rho_{21l}(\varepsilon)), \rho_{21l}(\varepsilon)), \qquad l = 1, 2,$$
$$(\rho_1, \rho_2) = (\tilde{\rho}_{12}(\rho_{22j}(\varepsilon)), \rho_{22j}(\varepsilon)), \qquad j = 1, 2,$$

when $K_2\alpha_{11}\varepsilon < 0$.

(4) For $k > 2$, suppose $\beta_1\beta_2 \neq 0$. Similarly to the proof of Theorem 3.3(2), we know that (3.28) has two solutions

$$\rho_1 = -\frac{\beta_2}{\beta_1}\rho_2 + O(|\rho_2|^2) \equiv \tilde{\tilde{\rho}}_1(\rho_2), \tag{3.36}$$

$$\rho_2 = -\frac{a_k}{\beta_1}\rho_1^{k-1} + O(|\rho_1|^k) \equiv \tilde{\tilde{\rho}}_2(\rho_1). \tag{3.37}$$

Substituting (3.36) into (3.27), we have

$$\varepsilon = \frac{1}{K_2\beta_1}(\beta_2\tilde{d}_3 - \beta_1\tilde{d}_5)\rho_2^2 + O(|\rho_2|^3) = \frac{1}{K_2\beta_1}(\tilde{c}_6\tilde{d}_3 - \tilde{c}_5\tilde{d}_5)\rho_2^2 + O(|\rho_2|^3). \tag{3.38}$$

If $K_2\beta_1(\tilde{c}_6\tilde{d}_3 - \tilde{c}_5\tilde{d}_5)\varepsilon < 0$, then (3.38) has no inverse functions. If $K_2\beta_1(\tilde{c}_6\tilde{d}_3 - \tilde{c}_5\tilde{d}_5)\varepsilon > 0$, then (3.38) has two inverse functions

$$\rho_2 = (-1)^i\Big(\frac{K_2\beta_1\varepsilon}{\tilde{c}_6\tilde{d}_3 - \tilde{c}_5\tilde{d}_5}\Big)^{\frac{1}{2}} + o(|\varepsilon|^{\frac{1}{2}}) \equiv \tilde{\rho}_{2i}(\varepsilon), \qquad i = 1, 2.$$

Substituting (3.37) into (3.27), we have

$$\varepsilon = \frac{a_k\tilde{d}_3}{K_2\beta_1}\rho_1^k + O(|\rho_1|^{k+1}). \tag{3.39}$$

If k is even, then (3.39) has no inverse functions when $a_k\tilde{d}_3K_2\beta_1\varepsilon < 0$ and has two inverse functions

$$\rho_1 = (-1)^i\Big(\frac{K_2\beta_1}{a_k\tilde{d}_3}\varepsilon\Big)^{\frac{1}{k}} + o(|\varepsilon|^{\frac{1}{k}}) \equiv \tilde{\tilde{\rho}}_{1i}(\varepsilon), \qquad i = 1, 2$$

when $a_k\tilde{d}_3K_2\beta_1\varepsilon > 0$. If k is odd, then (3.39) has a unique inverse function

$$\rho_1 = \Big(\frac{K_2\beta_1}{a_k\tilde{d}_3}\varepsilon\Big)^{\frac{1}{k}} + o(|\varepsilon|^{\frac{1}{k}}) \equiv \tilde{\tilde{\rho}}_{13}(\varepsilon).$$

Therefore if k is odd, then (3.25) and (3.26) have exactly one solution

$$(\rho_1, \rho_2) = (\tilde{\tilde{\rho}}_{13}(\varepsilon), \tilde{\tilde{\rho}}_2(\tilde{\tilde{\rho}}_{13}(\varepsilon))),$$

when $K_2\beta_1(\tilde{c}_6\tilde{d}_3 - \tilde{c}_5\tilde{d}_5)\varepsilon < 0$ and have three solutions

$$(\rho_1, \rho_2) = (\tilde{\tilde{\rho}}_1(\tilde{\rho}_{2i}(\varepsilon)), \tilde{\rho}_{2i}(\varepsilon)), \qquad i = 1, 2,$$

and

$$(\rho_1, \rho_2) = (\tilde{\tilde{\rho}}_{13}(\varepsilon), \tilde{\tilde{\rho}}_2(\tilde{\tilde{\rho}}_{13}(\varepsilon)))$$

when $K_2\beta_1(\tilde{c}_6\tilde{d}_3 - \tilde{c}_5\tilde{d}_5)\varepsilon > 0$.

If k is even and $a_k\tilde{d}_3(\tilde{c}_6\tilde{d}_3 - \tilde{c}_5\tilde{d}_5) < 0$, then (3.25) and (3.26) have two solutions

$$(\rho_1, \rho_2) = (\tilde{\tilde{\rho}}_1(\tilde{\rho}_{2i}(\varepsilon)), \tilde{\rho}_{2i}(\varepsilon)), \qquad i = 1, 2,$$

or

$$(\rho_1, \rho_2) = (\tilde{\tilde{\rho}}_{1i}(\varepsilon), \tilde{\tilde{\rho}}_2(\tilde{\tilde{\rho}}_{1i}(\varepsilon))), \qquad i = 1, 2.$$

If k is even and $a_k\tilde{d}_3(\tilde{c}_6\tilde{d}_3 - \tilde{c}_5\tilde{d}_5) > 0$, then (3.25) and (3.26) have four solutions

$$(\rho_1, \rho_2) = (\tilde{\tilde{\rho}}_1(\tilde{\rho}_{2i}(\varepsilon)), \tilde{\rho}_{2i}(\varepsilon)), \qquad i = 1, 2,$$

and

$$(\rho_1, \rho_2) = (\tilde{\tilde{\rho}}_{1i}(\varepsilon), \tilde{\tilde{\rho}}_2(\tilde{\tilde{\rho}}_{1i}(\varepsilon))), \qquad i = 1, 2.$$

when $a_k\tilde{d}_3K_2\beta_1\varepsilon > 0$ and have no solutions when $a_k\tilde{d}_3K_2\beta_1\varepsilon < 0$. Hence Theorem 3.5(4) follows from the above analysis. This completes the proof.

If $K_2 = 0$, we have

Theorem 3.6. *Let* (1.3) *be an analytic system. Suppose that L is a k multiple closed orbit of System* (2.1) *and* $\sigma_2 = \int_0^T b(s)ds = K_2 = 0$. *For small $\varepsilon \neq 0$, we have*

(1) *For $k = 1$, assume $\tilde{d}_5 \neq 0$. Then L generates at most two closed orbits of System* (1.3). *Furthermore, L generates exactly two closed orbits of System* (1.3) *when $\widetilde{\Delta} > 0$ and generates no closed orbits of System* (1.3) *when $\widetilde{\Delta} < 0$, where*

$$\tilde{\Delta} = \Big(\tilde{d}_6 - \frac{1}{a_1}\tilde{d}_3e^{\sigma_1}K_1\Big)^2 - 4\tilde{d}_5\Big(\tilde{d}_7 - \frac{1}{a_1}\tilde{d}_4e^{\sigma_1}K_1\Big).$$

(2) *If $k > 1$ and $K_1\tilde{d}_4 \neq 0$, then L generates at most $2k+2$ closed orbits of System* (1.3).

(3) *For $k > 1$, if $K_1\tilde{d}_3\tilde{d}_4\tilde{d}_5 \neq 0$, assume*

$$\tilde{\gamma}_1 = \begin{cases} \tilde{c}_6 - \dfrac{\tilde{c}_5\tilde{d}_5}{\tilde{d}_3} + \dfrac{a_2\tilde{d}_5^2}{\tilde{d}_3^2}, & k = 2, \\ \tilde{c}_6 - \dfrac{\tilde{c}_5\tilde{d}_5}{\tilde{d}_3}, & k > 2. \end{cases}$$

Then

(i) *For odd k, L generates exactly one closed orbits of System* (1.3) *when $K_1\tilde{\gamma}_1\varepsilon > 0$ and generates exactly three closed orbits of System* (1.3) *when $K_1\tilde{\gamma}_1\varepsilon < 0$.*

(ii) *For even k, if $a_k\tilde{\gamma}_1 < 0$, then L generates exactly two closed orbits of System* (1.3)*; if $a_k\tilde{\gamma}_1 > 0$, then L generates exactly four closed orbits of System* (1.3) *when $K_1\tilde{\gamma}_1\varepsilon < 0$ and no closed orbits of System* (1.3) *when $K_1\tilde{\gamma}_1\varepsilon > 0$.*

Proof. If $\sigma_2 = \int_0^T b(s)ds = K_2 = 0$, then (3.25) and (3.26) become

$$\begin{aligned} &a_k\rho_1^k + e^{\sigma_1}K_1\varepsilon + \tilde{c}_5\rho_1\rho_2 + \tilde{c}_6\rho_2^2 + \tilde{c}_7\rho_1\varepsilon + \tilde{c}_8\rho_2\varepsilon + \tilde{c}_9\varepsilon^2 \\ &+ O(|\rho_2, \varepsilon||\rho_1, \rho_2, \varepsilon|^2 + |\rho_1|^{k+1}) = 0, \end{aligned} \tag{3.40}$$

$$\begin{aligned} &\tilde{d}_3\rho_1\rho_2 + \tilde{d}_4\rho_1\varepsilon + \tilde{d}_5\rho_2^2 + \tilde{d}_6\rho_2\varepsilon + \tilde{d}_7\varepsilon^2 \\ &+ O(|\rho_2, \varepsilon||\rho_1, \rho_2, \varepsilon|^2) = 0. \end{aligned} \tag{3.41}$$

(1) If $k = 1$, by the Implicit Function Theorem, (3.40) has a unique solution

$$\rho_1 = -\frac{1}{a_1}(e^{\sigma_1}K_1\varepsilon + \tilde{c}_6\rho_2^2) + O(|\rho_2|^3 + |\varepsilon|^2 + |\rho_2\varepsilon|). \tag{3.42}$$

Substituting (3.42) into (3.41), we have

$$\tilde{d}_5\rho_2^2 + \Big(\tilde{d}_6 - \frac{1}{a_1}\tilde{d}_3e^{\sigma_1}K_1\Big)\rho_2\varepsilon + \Big(\tilde{d}_7 - \frac{1}{a_1}\tilde{d}_4e^{\sigma_1}K_1\Big)\varepsilon^2 + O(|\rho_2, \varepsilon|^3) = 0. \tag{3.43}$$

Suppose that $\tilde{d}_5 \neq 0$. It is easy to see that (3.43) has at most two solutions in ρ_2. Hence (3.40) and (3.41) have at most two solutions. Furthermore, similar to the proof of Theorem 3.3(1), we obtain that (3.43) has no solutions when $\tilde{\Delta} < 0$ and has two solutions of the forms

$$\rho_2 = \frac{-\tilde{d}_6 + \frac{1}{a_1}\tilde{d}_3e^{\sigma_1}K_1 \pm \sqrt{\tilde{\Delta}}}{2\tilde{d}_5}\varepsilon + O(\varepsilon^2)$$

when $\widetilde{\Delta} > 0$. Therefore Theorem 3.6(1) holds.

(2) If $k > 1$ and $K_1 \neq 0$, then by (3.40) we have

$$\varepsilon = -\frac{1}{e^{\sigma_1}K_1}(a_k\rho_1^k + \tilde{c}_5\rho_1\rho_2 + \tilde{c}_6\rho_2^2) + O(|\rho_1|^{k+1} + |\rho_1^2\rho_2| + |\rho_1\rho_2^2| + |\rho_2|^3). \tag{3.44}$$

Substituting (3.44) into (3.41), we have

$$-\frac{\tilde{d}_4a_k}{e^{\sigma_1}K_1}\rho_1^{k+1} + \tilde{d}_3\rho_1\rho_2 + \tilde{d}_5\rho_2^2 + O(|\rho_1, \rho_2|^3) = 0. \tag{3.45}$$

If $\tilde{d}_4 \neq 0$, using (3.44) and (3.45), by a process similar to the proof of Theorem 3.5(1), we have that the conclusion of Theorem 3.6(2) holds.

(3) For $k > 1$, if $K_1\tilde{d}_3\tilde{d}_4\tilde{d}_5 \neq 0$, in a way similar to the analysis of Theorem 3(2), we obtain two solutions of Equation (3.45) as follows

$$\rho_1 = -\frac{\tilde{d}_5}{\tilde{d}_3}\rho_2 + O(\rho_2^2) \equiv \bar{\bar{\rho}}_1(\rho_2), \tag{3.46}$$

$$\rho_2 = \frac{\tilde{d}_4 a_k}{e^{\sigma_1}K_1\tilde{d}_3}\rho_1^k + O(|\rho_1|^{k+1}) \equiv \bar{\bar{\rho}}_2(\rho_1). \tag{3.47}$$

Substituting (3.46) into (3.44), we have

$$\varepsilon = -\frac{\tilde{\gamma}_1}{e^{\sigma_1}K_1}\rho_2^2 + O(|\rho_2|^3). \tag{3.48}$$

Hence (3.48) has no inverse functions when $K_1\tilde{\gamma}_1\varepsilon > 0$ and has two inverse functions

$$\rho_2 = (-1)^i\Big(-\frac{e^{\sigma_1}K_1}{\tilde{\gamma}_1}\varepsilon\Big)^{\frac{1}{2}} + o(|\varepsilon|^{\frac{1}{2}}) \equiv \bar{\bar{\rho}}_{2i}(\varepsilon), \qquad i = 1, 2,$$

when $K_1\tilde{\gamma}_1\varepsilon < 0$.

Substituting (3.47) into (3.44), we have

$$\varepsilon = -\frac{a_k}{e^{\sigma_1}K_1}\rho_1^k + O(|\rho_1|^{k+1}). \tag{3.49}$$

For even k, if $K_1a_k\varepsilon > 0$, then (3.49) has no inverse solutions. If $K_1a_k\varepsilon < 0$, then (3.49) has two inverse solutions

$$\rho_1 = (-1)^i\Big(-\frac{e^{\sigma_1}K_1}{a_k}\varepsilon\Big)^{\frac{1}{k}} + o(|\varepsilon|^{\frac{1}{k}}) \equiv \bar{\bar{\rho}}_{1i}(\varepsilon), \qquad i = 1, 2.$$

For odd k, (3.49) has a unique inverse function

$$\rho_1 = \Big(-\frac{e^{\sigma_1}K_1}{a_k}\varepsilon\Big)^{\frac{1}{k}} + o(|\varepsilon|^{\frac{1}{k}}) \equiv \bar{\bar{\rho}}_{13}(\varepsilon).$$

Therefore for odd k, (3.40) and (3.41) have exactly one solution

$$(\rho_1, \rho_2) = (\bar{\bar{\rho}}_{13}(\varepsilon), \bar{\bar{\rho}}_2(\bar{\bar{\rho}}_{13}(\varepsilon)))$$

when $K_1\tilde{\gamma}_1\varepsilon > 0$ and have exactly three solutions

$$(\rho_1, \rho_2) = (\bar{\bar{\rho}}_1(\bar{\bar{\rho}}_{2i}(\varepsilon)), \bar{\bar{\rho}}_{2i}(\varepsilon)), \qquad i = 1, 2$$

and

$$(\rho_1, \rho_2) = (\bar{\bar{\rho}}_{13}(\varepsilon), \bar{\bar{\rho}}_2(\bar{\bar{\rho}}_{13}(\varepsilon)))$$

when $K_1\tilde{\gamma}_1\varepsilon < 0$.

For even k, if $a_k\tilde{\gamma}_1 < 0$, then (3.40) and (3.41) have two solutions

$$(\rho_1, \rho_2) = (\bar{\bar{\rho}}_1(\bar{\bar{\rho}}_{2i}(\varepsilon)), \bar{\bar{\rho}}_{2i}(\varepsilon)), \qquad i = 1, 2,$$

or

$$(\rho_1, \rho_2) = (\bar{\bar{\rho}}_{1i}(\varepsilon), \bar{\bar{\rho}}_2(\bar{\bar{\rho}}_{1i}(\varepsilon))), \qquad i = 1, 2.$$

If $a_k \tilde{\gamma}_1 > 0$, then (3.40) and (3.41) have four solutions

$$(\rho_1, \rho_2) = (\bar{\bar{\rho}}_1(\bar{\bar{\rho}}_{2i}(\varepsilon)), \bar{\bar{\rho}}_{2i}(\varepsilon)), \qquad i = 1, 2$$

and

$$(\rho_1, \rho_2) = (\bar{\bar{\rho}}_{1i}(\varepsilon), \bar{\bar{\rho}}_2(\bar{\bar{\rho}}_{1i}(\varepsilon))), \qquad i = 1, 2,$$

when $K_1 \tilde{\gamma}_1 \varepsilon < 0$ and have no solutions when $K_1 \tilde{\gamma}_1 \varepsilon > 0$.

From the analysis above, we conclude that Theorem 3.6(3) holds. This completes the proof.

References

[1] Chow, S. N. & Hale, J. K., Methods of Bifurcation Theory, Springer-Verlag, New York, 1982.

[2] Guckenheimer, J. & Holmes, P., Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields, Springer-Verlag, New York, 1983.

[3] Hale, J. K., Ordinary Differential Equations, Robert E. Krieger Publishing Co., Florida, 1980.

[4] Han, M., Bifurcation of periodic orbits for three dimensional system (in Chinese), *Acta. Math. Appl. Sin.*, **18:**4(1995), 528–537.

[5] Han, M., Theory of Periodic Solutions and Bifurcations of Dynamical Systems (in Chinese), Science Publishing House, Beijing, 2002.

[6] Han, M. & Zhu, D., Bifurcation Theory in Differential Equations (in Chinese), Beijing, Coal Industry Publishing House, Beijing, 1994.

[7] Jin, Y. & Zhu, D., Bifurcations of rough heteroclinic loop with two saddle points, *Science in China, Series A,* **16:**4(2003), 459–468.

[8] Kuznetsov, Y., Elements of Applied Bifurcation Theory, Springer-Verlag, New York, 1995.

[9] Luo, D., Wang, X., Zhu, D. & Han, M., Bifurcation Theory and Methods of Dynamical Systems, World Scientific, Singapore, 1997.

[10] Zhu, D. & Han, M., Bifurcation of homoclinic orbits in fast variable space (in Chinese), *Chin. Ann. Math.,* **23A:**4(2002), 429–438.

[11] Zhu, D. & Xia, Z., Bifurcations of heteroclinic loops, *Science in China, Series A,* **41:**8(1998), 837–848.