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NON-DEGENERATE INVARIANT BILINEAR FORMS ON NONASSOCIATIVE TRIPLE SYSTEMS

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Abstract

A bilinear form f on a nonassociative triple system $\mathcal{T}$ is said to be invariant if and only if $f(\langle abc\rangle, d) = f(a, \langle dcb\rangle) = f(c, \langle bad\rangle)$ for all $a, b, c, d \in \mathcal{T}$. $(\mathcal{T}, f)$ is called a pseudo-metric triple system if f is non-degenerate and invariant. A decomposition theory for triple systems and pseudo-metric triple systems is established. Moreover, the finite-dimensional metric Lie triple systems are characterized in terms of the structure of the non-degenerate, invariant and symmetric bilinear forms on them.

Keywords Triple system, Lie triple system, Bilinear form, Lie algebra
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§ 1. Introduction

Lie algebras admitting non-degenerate and invariant bilinear forms (i.e. self-dual Lie algebras or pseudo-metric Lie algebras) has been a hot topic in the study of Lie theory. The motivation for studying these algebras comes from the fact that metric Lie or associative algebras have been appearing repeatedly in several areas of mathematics and physics (see, for example, [1, 2]). Lie triple systems play an important part in the study of the theory of Lie algebras and Lie groups. It is well known that a Lie algebra becomes a Lie triple system in a natural way whereas a Lie triple system can be imbedded into a Lie algebra. As Lie algebras carrying a non-degenerate invariant bilinear form are of particular importance in studying Lie theory and other related fields, it should be worthwhile to look into the effect of the action of non-degenerate and invariant bilinear forms on a Lie triple system. In [3] the authors looked into the properties of Lie triple systems admitting non-degenerate invariant bilinear forms when discussing the relationship between the symmetric invariant bilinear forms on a Lie triple system and the ones on its standard embedding Lie algebra. In the present paper, we investigate the decomposition theory both for a triple system $\mathcal{T}$ and for a metric triple system $(\mathcal{T},f)$. We also characterize the finite-dimensional metric Lie triple systems in terms of the structure of the non-degenerate invariant and symmetric bilinear

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forms on them. Jacobi identity is not needed in our discussion before Section 4, so we do not confine our attention to triple systems of Lie type until then.

Section 2 contains some basic concepts and preliminary results. We prove that a pseudo-metric triple system is metrizable when it is either commutative or anti-commutative (see Lemma 2.4). As a result, a pseudo-metric Lie triple system over a field $\mathcal{F}$ is a metric Lie triple system provided Char$\mathcal{F} \neq 2$.

In Section 3, we develop the decomposition theory for triple systems in the following four steps. Firstly, we establish a theorem for decomposing a finite-dimensional triple system $\mathcal{T}$ into a direct sum of indecomposable ideals, which may be called the general decomposition theorem for triple systems. Secondly, we give the notion of f-decomposition of a metric triple system and show that the actual difference between the so-called general decomposition of a metric Lie triple system and its f-decomposition is that they are the same up to annihilating Abelian ideals (see Theorem 3.2). Thirdly, in what we may call an f-decomposition theorem (see Theorem 3.3), we solve the problem of decomposing any finite-dimensional pseudo-metric triple system $(\mathcal{T}, f)$ into an orthogonal direct sum of f-indecomposable ideals. Finally, we show that, giving a decomposition of $(\mathcal{T}, f)$ into a direct sum of indecomposable ideals, a bilinear form g can be found such that the decomposition is just an orthogonal direct sum of g-indecomposable ideals.

In Section 4, we advance the results on metric simple Lie triple system to the semi-simple and reductive Lie triple system. Any Lie triple system $\mathcal{T}$ admitting a unique, up to a constant, quadratic structure over a field $\mathcal{F}$ of characteristic zero is necessarily a simple Lie triple system. If the field $\mathcal{F}$ is algebraically closed, such a condition is also sufficient. In addition, we characterize all the semi-simple and reductive Lie triple systems with the non-degenerate, invariant and symmetric bilinear forms on them.

§ 2 . Preliminaries

Let T be a nonassociative (i.e. not necessarily associative) triple system over a field $\mathcal{F}$ and f a bilinear form on $\mathcal{T}$. In this section, we first introduce the concept of a pseudo-metric triple system, a triple system admitting a non-degenerate invariant bilinear form, and give the preliminary results on them. And then we prove that a pseudo-metric Lie triple system is metrizable. For basic concepts not specified in this section, the reader is referred to [4–8].

Let $\mathcal{R}$ be a commutative ring with 1. A triple system is a unital $\mathcal{R}$-module $\mathcal{T}$ together with a trilinear map $\mathcal{T} \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$, $(x, y, z) \longmapsto \langle xyz \rangle$. $\mathcal{T}$ is called a Lie triple system, if, for all $u, v, x, y, z \in \mathcal{T}$,

$$\langle xxz \rangle = 0, \tag{2.1}$$

$$\langle xyz \rangle + \langle yzx \rangle + \langle zxy \rangle = 0, \tag{2.2}$$

$$\langle uv\langle xyz \rangle\rangle = \langle\langle uvx \rangle yz\rangle + \langle x\langle uvy \rangle z\rangle + \langle xy\langle uvz \rangle\rangle. \tag{2.3}$$

Define $L(x, y), R(x, y), P(x, y) \in \mathrm{End}_{\mathcal{R}}(T)$ by

$$\langle xyz \rangle = P(x, z)y = L(x, y)z = R(z, y)x.$$

Then (2.1), (2.2), (2.3) are equivalent to, respectively,

$$L(x,x) = 0, \tag{2.1$'$}$$

$$L(x,y) = R(x,y) - R(y,x), \tag{2.2$'$}$$

$$[L(x,y), L(u,v)] = L(\langle xyu\rangle, v) + L(u, \langle xyv\rangle). \tag{2.3$'$}$$

A submodule $\mathcal{S}$ of a triple system $\mathcal{T}$ is called a subsystem if $\langle\mathcal{SSS}\rangle \subseteq \mathcal{S}$. $\mathcal{S}$ is called an ideal if $\langle\mathcal{STT}\rangle + \langle\mathcal{TST}\rangle + \langle\mathcal{TTS}\rangle \subseteq \mathcal{S}$. In the Lie triple system case, $\mathcal{S}$ is an ideal of $\mathcal{T}$ if and only if $\langle\mathcal{STT}\rangle \subseteq \mathcal{S}$.

A homomorphism from one triple system $\mathcal{T}$ to another, $\mathcal{T}'$, is a $\mathcal{R}$-linear map $f : \mathcal{T} \longrightarrow \mathcal{T}'$, satisfying $f(\langle xyz\rangle) = \langle f(x)f(y)f(z)\rangle$ for all $x, y, z \in \mathcal{T}$.

The image of any homomorphism is a subsystem of $\mathcal{T}'$ whereas the kernel is an ideal of $\mathcal{T}$. Conversely, if $\mathcal{S}$ is an ideal of a triple system $\mathcal{T}$, then $\overline{\mathcal{T}} = \mathcal{T}/\mathcal{S}$, together with $\langle(x+\mathcal{S})(y+\mathcal{S})(z+\mathcal{S})\rangle := \langle xyz\rangle + \mathcal{S}$, is again a triple system. $\mathcal{T}/\mathcal{S}$, is called the factor triple system of $\mathcal{T}$ mod $\mathcal{S}$.

Definition 2.1. *A bilinear form $f : \mathcal{T}\times\mathcal{T} \longrightarrow \mathcal{F}$ is called invariant, if*

$$f(\langle abc\rangle, d) = f(a, \langle dcb\rangle) = f(c, \langle bad\rangle), \qquad \forall a, b, c, d \in \mathcal{T}.$$

For any subspace $\mathcal{V}$ of $\mathcal{T}$, let $\mathcal{V}^{\perp}$ (resp. ${}^{\perp}\mathcal{V}$) denote the right orthogonal space (resp. the left orthogonal space) of $\mathcal{V}$, i.e. $\mathcal{V}^{\perp} := \{t \in \mathcal{T} \mid f(v,t) = 0, \forall v \in \mathcal{V}\}$; ${}^{\perp}\mathcal{V} := \{t \in \mathcal{T} \mid f(t,v) = 0, \forall v \in \mathcal{V}\}$. f is called non-degenerate if $\mathcal{T}^{\perp} = 0 = {}^{\perp}\mathcal{T}$.

We see that the equation in Definition 2.1 is equivalent to $f(R(c,b)a, d) = f(a, R(b,c)d)$ and $f(L(a,b)c, d) = f(c, L(b,a)d)$, which corresponds to right- and left-invariant, respectively.

If f is invariant and symmetric, then $f(\langle abc\rangle, d) = f(b, \langle cda\rangle)$. Therefore, a symmetric bilinear form f on a Lie triple system $\mathcal{T}$ is invariant if and only if f is left invariant (see [3, Lemma 4.1]).

Definition 2.2. *A pseudo-metric triple system $(\mathcal{T}, f)$ is a triple system admitting a non-degenerate invariant bilinear form f. In this case, $\mathcal{T}$ is pseudo-metrizable. If in addition f is symmetric, we call $(\mathcal{T}, f)$ a* **metric triple system**. *$\mathcal{T}$ is then called metrizable.*

Definition 2.3. *The annihilator of a triple system $\mathcal{T}$ is the set*

$$\mathcal{Z}(\mathcal{T}) := \{x \in \mathcal{T} \mid \langle x\mathcal{TT}\rangle = \langle\mathcal{T}x\mathcal{T}\rangle = \langle\mathcal{TT}x\rangle = 0\}. \tag{2.4}$$

More generally, for any submodule $\mathcal{V}$ of $\mathcal{T}$, the annihilator $\mathcal{Z}_{\mathcal{T}(\mathcal{V})}$ of $\mathcal{V}$ in $\mathcal{T}$ is defined as the set of all x in $\mathcal{T}$ such that for any permutation of x, $\mathcal{T}$ and $\mathcal{V}$, the triple composition of the three is zero.

$\mathcal{Z}_{\mathcal{T}}(\mathcal{V})$ is a submodule, but not a subsystem, even if $\mathcal{V}$ is an ideal. For a Lie triple system $\mathcal{T}$, $\mathcal{Z}(\mathcal{T})$ is also called the center of $\mathcal{T}$.

Definition 2.4. *A triple system $\mathcal{T}$ is decomposable if it is either zero or equal to a direct sum of two non-zero ideals. $\mathcal{T}$ is called reductive if* $\mathrm{Rad}\,(\mathcal{T}) = \mathcal{Z}(\mathcal{T})$.

Any Abelian or semi-simple triple system is a reductive Lie triple system.

Lemma 2.1. *For any subspaces $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{T}$, the following basic duality relations hold:*

$$\mathcal{V} \subseteq \mathcal{W} \quad \textit{implies} \quad \mathcal{V}^{\perp} \supseteq \mathcal{W}^{\perp} \quad \textit{and} \quad {}^{\perp}\mathcal{V} \supseteq {}^{\perp}\mathcal{W}, \tag{2.5}$$

$$(\mathcal{V}+\mathcal{W})^{\perp} = \mathcal{V}^{\perp} \cap \mathcal{W}^{\perp} \quad \textit{and} \quad {}^{\perp}(\mathcal{V}+\mathcal{W}) = {}^{\perp}\mathcal{V} \cap {}^{\perp}\mathcal{W}, \tag{2.6}$$

$$(\mathcal{V}\cap\mathcal{W})^{\perp} \supseteq \mathcal{V}^{\perp} + \mathcal{W}^{\perp} \quad \textit{and} \quad {}^{\perp}(\mathcal{V}\cap\mathcal{W}) \supseteq {}^{\perp}\mathcal{V} + {}^{\perp}\mathcal{W}. \tag{2.7}$$

Furthermore, when $\mathcal{T}$ is in the finite case, the following formulae hold:

$${}^{\perp}(\mathcal{V}^{\perp}) = \mathcal{V} + {}^{\perp}\mathcal{T} \quad \textit{and} \quad ({}^{\perp}\mathcal{V})^{\perp} = \mathcal{V} + \mathcal{T}^{\perp}, \tag{2.8}$$

$$\dim \mathcal{V}^{\perp} = \dim \mathcal{T} - \dim \mathcal{V} + \dim(\mathcal{V} \cap {}^{\perp}\mathcal{T}), \tag{2.9}$$

$$\dim {}^{\perp}\mathcal{V} = \dim \mathcal{T} - \dim \mathcal{V} + \dim(\mathcal{V} \cap \mathcal{T}^{\perp}), \tag{2.10}$$

$$\dim(\mathcal{V} \cap \mathcal{V}^{\perp}) = \dim(\mathcal{V} \cap {}^{\perp}\mathcal{V}). \tag{2.11}$$

For the special case that f is a symmetric non-degenerate bilinear form on $\mathcal{T}$, we have $({}^{\perp}\mathcal{V})^{\perp} = \mathcal{V}$ and $\dim \mathcal{V}^{\perp} = \dim {}^{\perp}\mathcal{V} = \dim \mathcal{T} - \dim \mathcal{V}$.

Let f (resp. g) be a bilinear form on a vector space $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) over a field $\mathcal{F}$. Then there is a canonical bilinear form $f \perp g$ on the direct sum $\mathcal{T}_1 \dot{+} \mathcal{T}_2$ of $\mathcal{T}_1$ and $\mathcal{T}_2$ defined by

$$f \perp g(t_1 + t_2, t_1' + t_2') := f(t_1, t_1') + g(t_2, t_2')$$

for all $t_1, t_1' \in \mathcal{T}_1$, $t_2, t_2' \in \mathcal{T}_2$. Moreover, $f \perp g$ is non-degenerate if and only if both f and g are non-degenerate.

A subspace $\mathcal{V}$ of $\mathcal{T}$ is called non-degenerate if $\mathcal{V} \cap \mathcal{V}^{\perp} = 0$ and $\mathcal{V} \cap {}^{\perp}\mathcal{V} = 0$. For a finite dimensional triple system $\mathcal{T}$, the two conditions are equivalent according to (2.11) in Lemma 2.1. Moreover, Lemma 2.1 tells us $\mathcal{T} = \mathcal{V} \oplus \mathcal{V}^{\perp}$ and $\mathcal{T} = \mathcal{V} \oplus {}^{\perp}\mathcal{V}$ for each non-degenerate $\mathcal{V}$ and each finite dimensional $\mathcal{T}$. Hence, it is readily checked that the bilinear form f on $\mathcal{T}$ is non-degenerate if and only if both the restrictions of f to $\mathcal{V} \times \mathcal{V}$ and to $\mathcal{V}^{\perp} \times \mathcal{V}^{\perp}$ are non-degenerate.

Definition 2.5. *The intersection of ${}^{\perp}\mathcal{T}$ and $\mathcal{T}^{\perp}$ is called the kernel of f and is denoted by $\mathcal{N}_f$. If $\mathcal{T}'$ is another vector space over the field $\mathcal{F}$ and $m : \mathcal{T}' \longrightarrow \mathcal{T}$ is a linear map, then the pull back m^*f is defined to be the bilinear form $(a, a') \longmapsto f(ma, ma')$ for all $a, a' \in \mathcal{T}'$.*

The kernel $\mathcal{N}_{m^*f}$ of m^*f contains the kernel of m. It is readily checked that for a surjective m, $\mathcal{N}_{m^*f} = m^{-1}\mathcal{N}_f$.

Suppose g is a bilinear form on $\mathcal{T}'$, m is surjective and $\ker m \subseteq \mathcal{N}_g$. Then the projective g^m of g given by $g^m(ma_1', ma_2') := g(a_1', a_2')$ is a well-defined bilinear form on $\mathcal{T}$, with the kernel $\mathcal{N}_{g^m} \cong \mathcal{N}_g / \ker m$.

Let $\mathcal{T}$ be a Lie triple system and $\mathcal{H}$ the submodule of derivation algebra $V(\mathcal{T})$ of $\mathcal{T}$ ($V(\mathcal{T})$ is the $\mathcal{F}$-module of all derivations of $\mathcal{T}$ together with the map $(D_1, D_2) \longmapsto [D_1, D_2]$. $V(\mathcal{T})$ is also an algebra). If $\mathcal{H}$ is generated by all $L(x, y)$, $\forall x, y \in \mathcal{T}$, then $\mathcal{H}$ is an ideal of $V(\mathcal{T})$ (see [5] for the definition of $L(x, y)$).

Lemma 2.2. *Let $(\mathcal{T}, f)$ be a pseudo-metric triple system over a field $\mathcal{F}$ and $\mathcal{V}$ an arbitrary vector subspace of $\mathcal{T}$.*

(1) *If* $\mathcal{I}$ *is an ideal of* $\mathcal{T}$, ${}^{\perp}\mathcal{I}$ *and* $\mathcal{I}^{\perp}$ *are both ideals and* $\langle \mathcal{I}\mathcal{T}\mathcal{I}^{\perp}\rangle + \langle \mathcal{I}^{\perp}\mathcal{T}\mathcal{I}\rangle = 0$. *Moreover, if* f *is symmetric, we have* $\mathcal{I}^{\perp} \subseteq \mathcal{Z}_{\mathcal{T}(\mathcal{I})}$ *and* ${}^{\perp}\mathcal{I} \subseteq \mathcal{Z}_{\mathcal{T}(\mathcal{I})}$.

(2) $\mathcal{Z}_{\mathcal{T}(\mathcal{V})} = (\langle \mathcal{V}\mathcal{T}\mathcal{T}\rangle)^{\perp} \cap (\langle \mathcal{T}\mathcal{V}\mathcal{T}\rangle)^{\perp} \cap (\langle \mathcal{T}\mathcal{T}\mathcal{V}\rangle)^{\perp} \cap^{\perp} (\langle \mathcal{V}\mathcal{T}\mathcal{T}\rangle) \cap^{\perp} (\langle \mathcal{T}\mathcal{V}\mathcal{T}\rangle) \cap^{\perp} (\langle \mathcal{T}\mathcal{T}\mathcal{V}\rangle)$.

In particular, if f *is symmetric or antisymmetric, or* $\mathcal{T}$ *is commutative or anti-commuta- tive (i.e.* $\langle xyz\rangle = \langle yxz\rangle$ *or* $\langle xyz\rangle = -\langle yxz\rangle$, $\forall x, y, z \in \mathcal{T}$*), then*

$$\mathcal{Z}_{\mathcal{T}(\mathcal{V})} =^{\perp} (\langle \mathcal{T}\mathcal{T}\mathcal{V}\rangle + \langle \mathcal{T}\mathcal{V}\mathcal{T}\rangle + \langle \mathcal{V}\mathcal{T}\mathcal{T}\rangle) = (\langle \mathcal{T}\mathcal{T}\mathcal{V}\rangle + \langle \mathcal{T}\mathcal{V}\mathcal{T}\rangle + \langle \mathcal{V}\mathcal{T}\mathcal{T}\rangle)^{\perp}.$$

And $\mathcal{Z}_{\mathcal{T}(\mathcal{V})}$ *is an ideal if* $\mathcal{V}$ *is.*

(3) $\mathcal{Z}(\mathcal{T}) =^{\perp} (\mathcal{T}^{(1)}) = (\mathcal{T}^{(1)})^{\perp}$, *where* $\mathcal{T}^{(1)} := \langle \mathcal{T}\mathcal{T}\mathcal{T}\rangle$.

Remark 2.1. Any solvable nonzero finite dimensional pseudo-metrizable triple system has a nonzero annihilator. Because $\mathcal{T}^{(1)} \subset \mathcal{T}$, $\mathcal{T}^{(1)} \neq \mathcal{T}$ and $\dim \mathcal{T}^{(1)} + \dim(\mathcal{T}^{(1)})^{\perp} = \dim \mathcal{T}$, we have $\dim(\mathcal{T}^{(1)})^{\perp} = \dim \mathcal{Z}(\mathcal{T}) > 0$.

Two dimensional non-Abelian Lie triple systems cannot be pseudo-metrizable because it has nonzero annihilator.

The next lemma is a basic result about the transfer of invariant bilinear forms from one triple system to another.

Lemma 2.3. *Let* $\mathcal{T}$ *and* $\mathcal{T}'$ *be two triple systems over a field* $\mathcal{F}$ *and* f *(resp.* g*) an invariant bilinear form on* $\mathcal{T}$ *(resp. on* $\mathcal{T}'$*). If* $m : \mathcal{T} \longrightarrow \mathcal{T}'$ *be a homomorphism of triple systems, then*

(1) *The pull back* m^*g *of* g *is an invariant bilinear form on* $\mathcal{T}$.

(2) *When* m *is surjective and* $\operatorname{Ker} m$ *is contained in the kernel of* f, *the projection* f^m *of* f *is an invariant bilinear form on* $\mathcal{T}'$.

(3) *When* $\mathcal{H}$ *is a subsystem of* $\mathcal{T}$ *and* $\mathcal{H} \cap \mathcal{H}^{\perp} = \mathcal{H} \cap^{\perp} \mathcal{H}$, $\mathcal{H} \cap \mathcal{H}^{\perp}$ *is an ideal of* $\mathcal{H}$. *Given that* $p : \mathcal{H} \longrightarrow \mathcal{H}/(\mathcal{H} \cap \mathcal{H}^{\perp})$ *is the canonical projection and* $f_{\mathcal{H}}$ *the restriction of* f *to* $\mathcal{H} \times \mathcal{H}$, *then the projection* $(f_{\mathcal{H}})^p$ *is a non-degenerate invariant bilinear form on the factor system* $\mathcal{H}/(\mathcal{H} \cap \mathcal{H}^{\perp})$.

(4) *The bilinear form* $f \perp g$ *is invariant on the direct sum* $\mathcal{T} \oplus \mathcal{T}'$. *Moreover,* $f \perp g$ *is non-degenerate if and only if* f *and* g *are non-degenerate.*

Definition 2.6. *Let* $(\mathcal{T}, f)$ *and* $(\mathcal{T}', g)$ *be two pseudo-metric triple systems. A linear map* $\phi : \mathcal{T} \longrightarrow \mathcal{T}'$ *is an isometry or isomorphism of pseudo-metric triple systems if* ϕ *is an isomorphism of the two triple systems and* $f = \phi^* g$.

Lemma 2.4. *Let* $\mathcal{T}$ *be a triple system over a field* $\mathcal{F}$, *satisfying* $\langle xyz\rangle = \varepsilon\langle yxz\rangle$ *for* $\varepsilon = \pm 1$ *and all* x, y, z *in* $\mathcal{T}$. *If* f *is an invariant bilinear form on* $\mathcal{T}$, *then*

(1) $f(\langle xyz\rangle, w) = f(w, \langle xyz\rangle)$, $\forall x, y, z, w \in \mathcal{T}$.

(2) *If* $\operatorname{Char} \mathcal{F} \neq 2$ *and* $(\mathcal{T}, f)$ *is a pseudo-metric triple system,* $\mathcal{T}$ *is metrizable.*

Proof. (1) As $\langle xyz\rangle = \varepsilon\langle yxz\rangle$ for $\varepsilon^2 = 1$, we have

$$\begin{aligned} f(\langle xyz\rangle, w) &= f(x, \langle wzy\rangle) = \epsilon f(x, \langle zwy\rangle = \varepsilon f(\langle xyw\rangle, z) \\ &= \varepsilon f(w, \langle yxz\rangle) = \varepsilon^2 f(w, \langle xyz\rangle) = f(w, \langle xyz\rangle). \end{aligned}$$

(2) Let $f^t(x,y)=f(y,x), \forall x,y\in\mathcal{T}$. Clearly, f^t is a non-degenerate bilinear form on $\mathcal{T}$. We point out that f^t is invariant. In fact, $\forall x,y,z,w\in\mathcal{T}$,

$$\begin{aligned} f^t(\langle xyz\rangle,w) &= \varepsilon f^t(\langle yxz\rangle,w)=\varepsilon f(w,\langle yxz\rangle)=\varepsilon f(\langle xyw\rangle,z)\\ &=\varepsilon f^t(z,\langle xyw\rangle)=f^t(z,\langle yxw\rangle). \end{aligned}$$

Since $f_s(a,b):=\frac{1}{2}(f(a,b)+f^t(a,b))$ (resp. $f_{as}(a,b):=\frac{1}{2}(f(a,b)-f^t(a,b))$) is a symmetric (resp. anti-symmetric) invariant bilinear form on $\mathcal{T}$, $f=f_s+f_{as}$ is a sum with one part symmetric and the other anti-symmetric.

By (1) and the definition of invariance, we get the equalities

$$f_{as}(\mathcal{T},\mathcal{T}^{(1)})=f_{as}(\mathcal{T}^{(1)},\mathcal{T})=0 \quad \text{and} \quad f(\mathcal{T},\mathcal{T}^{(1)})=f_s(\mathcal{T},\mathcal{T}^{(1)}). \tag{2.12}$$

Let $\mathcal{N}$ be the kernel of f_s and $\mathcal{T}^{\perp'}$ (resp. ${}^{\perp'}\mathcal{T}$) be the left (resp. right) orthogonal space of $\mathcal{T}$ with respect to f_s. Since $\mathcal{N}=\mathcal{T}^{\perp'}={}^{\perp'}\mathcal{T}$ by the symmetry of f_s, $\mathcal{N}$ is an ideal of $\mathcal{T}$ by Lemma 2.3(3). As a particular case of (2.12), we get

$$f(\mathcal{T},\mathcal{N}\cap\mathcal{T}^{(1)})=f_s(\mathcal{T},\mathcal{N}\cap\mathcal{T}^{(1)})\subset f_s(\mathcal{T},\mathcal{N})=0,$$

meaning $\mathcal{N}\cap\mathcal{T}^{(1)}=0$. Since f is non-degenerate, it is easy to deduce $\mathcal{N}\subseteq\mathcal{Z}(\mathcal{T})$ from

$$\langle\mathcal{N}\mathcal{T}\mathcal{T}\rangle+\langle\mathcal{T}\mathcal{N}\mathcal{T}\rangle+\langle\mathcal{T}\mathcal{T}\mathcal{N}\rangle\subset\mathcal{N}\cap\mathcal{T}^{(1)}=0.$$

Now take any vector subspace $\mathcal{V}$ of $\mathcal{T}$ such that $\mathcal{T}=\mathcal{V}\oplus(\mathcal{T}^{(1)}\oplus\mathcal{N})$. Let $\mathcal{B}:=\mathcal{V}\oplus\mathcal{T}^{(1)}$ be an ideal. Since $\mathcal{N}\subseteq\mathcal{Z}(\mathcal{T})$, the restriction of the canonical projection $p:\mathcal{T}\longrightarrow\mathcal{T}/\mathcal{N}$ to $\mathcal{B}$ is an isomorphism of triple systems. Again by Lemma 2.3(3) we see that f_s restricting to $\mathcal{B}$ is non-degenerate. Choose a vector space base (e_i) of $\mathcal{N}$ and define $g(e_i,e_j):=\delta_{ij}$. Then g is a non-degenerate symmetric invariant bilinear form on triple system $\mathcal{N}$.

It is clear from Lemma 2.3(4) and $\mathcal{T}=\mathcal{N}\oplus\mathcal{B}$ that $g\perp f_s$ is a non-degenerate symmetric invariant bilinear form on $\mathcal{T}$. Hence $\mathcal{T}$ is metrizable.

Corollary 2.1. *A finite dimensional pseudo-metric Lie triple system $\mathcal{T}$ over a field $\mathcal{F}$ with $Char\mathcal{F}\neq 2$ is a metric Lie triple system.*

§ 3 . Decomposition Theory

In this section, we give a decomposition theorem for a finite dimensional triple system $\mathcal{T}$ and an f-decomposition theorem for a finite dimensional pseudo-metric triple system $(\mathcal{T},f)$. We also investigate the connection between the two different kinds of decompositions.

Theorem 3.1. *Let $\mathcal{T}$ be a finite-dimensional triple system over a field $\mathcal{F}$. If there are two decompositions of $\mathcal{T}$ into direct sums of indecomposable ideals*

$$\mathcal{T}=\mathcal{I}_1\oplus\cdots\oplus\mathcal{I}_k\oplus\cdots\oplus\mathcal{I}_K \quad \textit{and} \quad \mathcal{T}=\mathcal{J}_1\oplus\cdots\oplus\mathcal{J}_m\oplus\cdots\oplus\mathcal{J}_M,$$

where k,K,m,M are integers with $0\le k\le K$, $0\le m\le M$ and the ideal $\mathcal{I}_i$ (resp. $\mathcal{J}_j$) is non-Abelian for $1\le i\le k$ (resp. $1\le j\le m$) and Abelian (i.e. $\langle\mathcal{I}_i\mathcal{I}_i\mathcal{I}_i\rangle=0$) otherwise, then

(1) $K=M$ and $k=m$. And there is a permutation of the set $\{1,2,\cdots,M\}$ leaving invariant the set $\{1,2,\cdots,m\}$ such that the restriction of the canonical projection $p_{j'}$:

$\mathcal{T} \longrightarrow \mathcal{I}_{j'}$ to the ideal $\mathcal{J}_j$ is an isomorphism of the triple systems. The induced permutation of $\{1, 2, \cdots, m\}$ is uniquely defined by the condition $\mathcal{J}_j \cap \mathcal{I}_{j'} \neq 0$.

(2) *All the indecomposable Abelian ideals in the above decomposition are one dimensional and belong to the annihilator $\mathcal{Z}(\mathcal{T})$ of $\mathcal{T}$.*

(3) *$\mathcal{J}_j + \mathcal{Z}(\mathcal{T}) = \mathcal{I}_{j'} + \mathcal{Z}(\mathcal{T})$ and ${\mathcal{J}_j}^{(1)} = {\mathcal{I}_{j'}}^{(1)}$ for all $1 \leq j \leq M$.*

(4) *If $\mathcal{T}$ is perfect, i.e. $\mathcal{T}$ has a vanishing annihilator, then $m = M$ and the above decomposition is unique up to permutations.*

Proof. Let LRP$(\mathcal{T})$ denote the associative subalgebra of the space of all $\mathcal{F}$-linear maps $\mathcal{T} \longrightarrow \mathcal{T}$ generated by $L(x, y), R(x, y)$ and $P(x, y)$, with the multiplication of the subalgebra being the composition of maps, i.e. for any f_1, f_2 in LRP$(\mathcal{T})$ and x in $\mathcal{T}$,

$$(f_1 f_2)(x) = (f_1 \cdot f_2)(x) = f_1(f_2(x)).$$

Then $(f_1(f_2 f_3)) = ((f_1 f_2) f_3))$. So LRP$(\mathcal{T})$ is associative.

Next, without loss of generality, we make a standard construction to imbed LRP$(\mathcal{T})$ into the algebra LRP$(\mathcal{T}, 1)$ with unit element. Consider the $\mathcal{F}$-module

$$\mathrm{LRP}(\mathcal{T}, 1) := \mathcal{F} \cdot 1 \oplus f = \{(\alpha, f) \mid \forall \alpha \in \mathcal{F}, f \in \mathrm{LRP}(\mathcal{T}, 1)\}.$$

LRP$(\mathcal{T}, 1)$ is an algebra with the multiplication defined by the formula

$$(\alpha, f_1)(\beta, f_2) = (\alpha\beta, \alpha f_2 + \beta f_1 + f_1 f_2).$$

It is evident that LRP$(\mathcal{T}, 1)$ has a unit element $(1, 0)$ and $f \longmapsto (0, f)$ defines a homomorphism from LRP$(\mathcal{T})$ to LRP$(\mathcal{T}, 1)$. So LRP$(\mathcal{T}, 1)$ is a unital associative algebra since LRP$(\mathcal{T})$ is associative.

As a unital associative algebra, LRP$(\mathcal{T}, 1)$ is both Artinian and Noetherian. In fact, as $\mathcal{T}$ is a finite-dimensional module over $\mathcal{F}$, so is the space $\mathrm{End}_\phi(\mathcal{T})$ of $\mathcal{F}$-linear maps. Then LRP$(\mathcal{T}, 1)$ is finite dimensional as a subspace of $\mathrm{End}_\phi(\mathcal{T})$ and is both Artinian and Noetherian, for a finite-dimensional ring is both Artinian and Noetherian.

Next, we observe that $\mathcal{T}$ is a finite-dimensional triple system and a module over the ring LRP$(\mathcal{T}, 1)$. Let $\mathrm{LRP}(\mathcal{T}, 1) \times \mathcal{T} \longrightarrow \mathcal{T}$ be defined by

$$(f, x) \longmapsto f(x), \qquad \forall f \in \mathrm{LRP}(\mathcal{T}, 1),\ \forall x \in \mathcal{T}.$$

Obviously $\mathcal{T}$ is a LRP$(\mathcal{T}, 1)$-module. Moreover, since the map $\mathcal{F} \cdot 1 \longrightarrow \mathrm{LRP}(\mathcal{T}, 1)$ is a standard imbedding map and $\mathcal{F} \cdot 1$ is isomorphic to $\mathcal{F}$ and $\mathcal{T}$ is finite-dimensional as a $\mathcal{F}$-module, $\mathcal{T}$ is a finite LRP$(\mathcal{T}, 1)$-module. So $\mathcal{T}$ is both Noetherian and Artinian because LRP$(\mathcal{T}, 1)$ is both Noetherian and Artinian.

An LRP$(\mathcal{T}, 1)$-submodule $\mathcal{T}_1$ of module $\mathcal{T}$ is an ideal of the triple system $\mathcal{T}$ and vice versa because

$$\langle \mathcal{T}_1 \mathcal{T} \mathcal{T} \rangle + \langle \mathcal{T} \mathcal{T}_1 \mathcal{T} \rangle + \langle \mathcal{T} \mathcal{T} \mathcal{T}_1 \rangle \subseteq \mathcal{T}_1 \quad \text{iff} \quad L(\mathcal{T}\mathcal{T})\mathcal{T}_1 + R(\mathcal{T}\mathcal{T})\mathcal{T}_1 + P(\mathcal{T}\mathcal{T})\mathcal{T}_1 \subseteq \mathcal{T}_1.$$

Moreover, the LRP$(\mathcal{T}, 1)$-submodule $\mathcal{T}_1$ of $\mathcal{T}$ is an indecomposable submodule if and only if $\mathcal{T}_1$ is indecomposable as an ideal of the triple system $\mathcal{T}$. Use the first part of Krull-Schmidt theorem (Let $\mathcal{T}$ be a module that is both Artinian and Noetherian and let

$$\mathcal{T} = \mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_K \quad \text{and} \quad \mathcal{T} = \mathcal{J}_1 \oplus \cdots \oplus \mathcal{J}_M,$$

where the submodule $\mathcal{I}_i$ and $\mathcal{J}_j$ are indecomposable. Then $K = M$) (see [9]), we see that the first statement of the theorem has been proved, i.e. $K = M$. Moreover, it is seen from the proof of Krull-Schmidt theorem (see [9, pp.110-115]) that there is a permutation of $\{1, 2, \cdots, M\}$ such that $\mathcal{T} = \mathcal{J}_j \oplus \mathcal{I}^{(j')}$, where $\mathcal{I}^{(j')}$ denotes the ideal $\mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_{j'-1} \oplus \mathcal{I}_{j'+1} \oplus \cdots \oplus \mathcal{I}_M$. Hence $\mathcal{J}_j$ is isomorphic to the factor module $\mathcal{T}/\mathcal{I}^{(j')}$ which is obviously isomorphic to $\mathcal{I}_{j'}$. In order to prove that the above module isomorphism is also a triple system isomorphism it now suffices to show that the canonical projection $p_{j'} : \mathcal{T} \longrightarrow \mathcal{I}_{j'}$ is a triple system homomorphism. In fact, for any $x = \bigoplus_{i=1}^{m} x_i$, $y = \bigoplus_{i=1}^{m} y_i$, $z = \bigoplus_{i=1}^{m} z_i \in \mathcal{T}$ and $x_i, y_i, z_i \in \mathcal{I}_i$,

$$\begin{aligned} p_{j'}(\langle xyz\rangle) &= p_{j'}(L(x,y)z) = L(x,y)p_{j'}(z) = \langle xyz_{j'}\rangle \\ &= \Big\langle \bigoplus_{i=1}^{m} x_i, \bigoplus_{i=1}^{m} y_i, z_{j'} \Big\rangle = \langle x_{j'}y_{j'}z_{j'}\rangle = \langle p_{j'}(x)p_{j'}(y)p_{j'}(z)\rangle. \end{aligned}$$

Therefore, the restriction of $p_{j'}$ to $\mathcal{J}_j$ is indeed a triple system isomorphism.

The annihilator $\mathcal{Z}_{\mathcal{T}}(\mathcal{I}^{(j')})$ of $\mathcal{I}^{(j')}$ in $\mathcal{T}$ contains the ideals $\mathcal{I}_{j'}, \mathcal{J}_j$ and $\mathcal{Z}(\mathcal{T})$. Hence

$$\mathcal{I}_{j'} \oplus (\mathcal{Z}_{\mathcal{T}}(\mathcal{I}^{(j')}) \cap \mathcal{I}^{(j')}) = \mathcal{Z}_{\mathcal{T}}(\mathcal{I}^{(j')}) = \mathcal{J}_j \oplus (\mathcal{Z}_{\mathcal{T}}(\mathcal{I}^{(j')}) \cap \mathcal{I}^{(j')}).$$

On the other hand, $\mathcal{Z}_{\mathcal{T}}(\mathcal{I}^{(j')}) \cap \mathcal{I}^{(j')}$ is contained in $\mathcal{Z}(\mathcal{T})$. Therefore

$$\mathcal{I}_{j'} + \mathcal{Z}(\mathcal{T}) = \mathcal{J}_j + \mathcal{Z}(\mathcal{T}).$$

Cubing both sides of this equation gives $\mathcal{I}_{j'}^{(1)} = \mathcal{J}_j^{(1)}$. Clearly, $\mathcal{I}_{j'}$ is non-Abelian if and only if $\mathcal{J}_j$ is. And in this case, $\mathcal{I}_{j'} \cap \mathcal{J}_j \neq 0$, which determines a 1–1 correspondence between the non-Abelian ideals $\mathcal{I}'_j$ and the non-Abelian ideals $\mathcal{J}_j$. Therefore $k = m$. With this we come to the end of the proof of (1) and (3).

(2) is true since every Abelian ideal $\mathcal{I}$ of $\mathcal{T}$ (i.e. $\langle\mathcal{I}\mathcal{I}\mathcal{I}\rangle$=0) whose dimension is greater than one can be decomposed into a direct sum of one-dimensional Abelian ideals and every Abelian indecomposable ideals in the theorem belongs to the annihilator $\mathcal{Z}(\mathcal{T})$ of $\mathcal{T}$.

Next we prove (4). If $\mathcal{T}$ is perfect, $\mathcal{T}^{(1)} = \langle\mathcal{T}\mathcal{T}\mathcal{T}\rangle = \mathcal{T}$ by definition. Then

$$\mathcal{T} = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_{\mathcal{M}} = \mathcal{T}_1^{(1)} \oplus \cdots \oplus \mathcal{T}_m^{(1)} = \mathcal{T}^{(1)}.$$

So $\mathcal{T}_i^{(1)} = \mathcal{T}_i$ for all $i \leq m$ and $\mathcal{T}_i = 0$ for all $i > m$. Because $\mathcal{J}_j^{(1)} = \mathcal{I}_{j'}^{(1)}, \mathcal{J}_j = \mathcal{J}_j^{(1)} = \mathcal{I}_{j'}^{(1)} = \mathcal{I}_{j'}$, it follows that $m = M$ and the decomposition in the theorem is unique. If $\mathcal{T}$ has a vanishing annihilator, we have the same conclusion because

$$\mathcal{J}_j + \mathcal{Z}(\mathcal{T}) = \mathcal{J}_j = \mathcal{I}_{j'} = \mathcal{I}_{j'} + \mathcal{Z}(\mathcal{T}).$$

We now consider the decomposition problem for a finite-dimensional pseudo-metric triple system $(\mathcal{T}, f)$.

Definition 3.1. *An ideal $\mathcal{I}$ of a finite-dimensional pseudo-metric triple system $(\mathcal{T}, f)$ is f-non-degenerate if $\mathcal{I} \cap \mathcal{I}^{\perp} = 0$.*

By the dimension formulae in Lemma 2.1, it is easy to see that, for an ideal $\mathcal{I}$ in a pseudo-metric triple system $(\mathcal{T}, f)$,

$$\mathcal{I} \cap \mathcal{I}^{\perp} = 0 \quad \text{iff} \quad \mathcal{I} \cap^{\perp} \mathcal{I} = 0 \quad \text{iff} \quad \mathcal{T} = \mathcal{I} \oplus \mathcal{I}^{\perp} \quad \text{iff} \quad \mathcal{T} = \mathcal{I} \oplus^{\perp} \mathcal{I}.$$

So the following four statements are equivalent:

(1) $\mathcal{I}$ is f-non-degenerate;

(2) $\mathcal{I}^{\perp}$ is f-non-degenerate;

(3) the restriction of f to $\mathcal{I} \times \mathcal{I}$ is non-degenerate;

(4) the restriction of f to $\mathcal{I}^{\perp} \times \mathcal{I}^{\perp}$ is non-degenerate.

Definition 3.2. *A pseudo-metric triple system $(\mathcal{T}, f)$ is called f-decomposable if $\mathcal{T} = 0$ or $\mathcal{T}$ contains a nonzero f-non-degenerate ideal $\mathcal{I} \neq \mathcal{T}$. Otherwise, $(\mathcal{T}, f)$ is called f-indecomposable.*

Lemma 3.1. *A finite-dimensional pseudo-metric triple system has a decomposition into a direct sum of f-indecomposable ideals.*

Proof. Let $(\mathcal{T}, f)$ be a finite-dimensional pseudo-metric triple system. If $(\mathcal{T}, f)$ is f-indecomposable or zero, the proof is self-evident.

Suppose that $(\mathcal{T}, f)$ is f-decomposable and that $\mathcal{I}$ is an f-non-degenerate ideal of $\mathcal{T}$. Then $\mathcal{T} = \mathcal{I} \oplus \mathcal{I}^{\perp}$. Since the restriction of f to $\mathcal{I} \times \mathcal{I}$ is non-degenerate, there is a nontrivial f-non-degenerate ideal $\mathcal{J}$ of $\mathcal{I}$. Let $\mathcal{J}^{\perp'}$ denote the right orthogonal space of $\mathcal{J}$ in $\mathcal{I}$. Then $\mathcal{J}^{\perp'}$ is f-non-degenerate as $\mathcal{I}(\mathcal{I}^{\perp}) = (\mathcal{I}^{\perp})\mathcal{I} = 0$. Hence $\mathcal{T}$ can be decomposed into a direct sum $\mathcal{J} \oplus \mathcal{J}^{\perp'} \oplus \mathcal{I}^{\perp}$. Proceeding in this way, we end up with a decomposition of $\mathcal{T}$ into a direct sum of finite f-indecomposable f-non-degenerate ideals.

Before discussing the relationship between any two f-decompositions of $(\mathcal{T}, f)$ into orthogonal f-indecomposable ideals, we first look into the connection between the f-indecomposability and the indecomposability in the general sense of a pseudo-metric triple system.

Theorem 3.2. *Let $(\mathcal{T}, f)$ be a finite f-indecomposable pseudo-metric triple system over a field $\mathcal{F}$.*

(1) *If $\mathcal{T}$ is non-Abelian, $\mathcal{T}$ is indecomposable.*

(2) *If $\mathcal{T}^{(1)} = 0$, then either $\mathcal{T}$ is one dimensional and hence indecomposable or $\mathcal{T}$ is two-dimensional with f being anti-symmetric.*

Proof. First, if $\mathcal{T}$ is such that $\mathcal{T} = \mathcal{I} + \mathcal{J}$, where $\mathcal{I}$ and $\mathcal{J}$ are ideals in $\mathcal{T}$, and for any permutation of $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{J}$, the multiplication of the three is zero, then $\mathcal{T}^{(1)} = \mathcal{I}^{(1)} \oplus \mathcal{J}^{(1)}$.

In fact, from $\mathcal{I} \subseteq \mathcal{Z}_{\mathcal{T}(\mathcal{J})}$ and $\mathcal{J} \subseteq \mathcal{Z}_{\mathcal{T}(\mathcal{I})}$, we have

$$\begin{aligned} \langle \mathcal{I}\mathcal{T}\mathcal{T} \rangle &= \langle \mathcal{I}(\mathcal{I}+\mathcal{J})(\mathcal{I}+\mathcal{J}) \rangle = \mathcal{I}^{(1)} = \langle \mathcal{T}\mathcal{I}\mathcal{T} \rangle = \langle \mathcal{T}\mathcal{T}\mathcal{I} \rangle, \\ \langle \mathcal{J}\mathcal{T}\mathcal{T} \rangle &= \langle \mathcal{T}\mathcal{J}\mathcal{T} \rangle = \langle \mathcal{T}\mathcal{T}\mathcal{J} \rangle = \mathcal{J}^{(1)}. \end{aligned}$$

So $\mathcal{T}^{(1)} = \mathcal{I}^{(1)} + \mathcal{J}^{(1)}$. Since $\mathcal{I}^{(1)}$ and $\mathcal{J}^{(1)}$ are ideals of $\mathcal{T}$, we get by Lemma 2.1(2.6),

$$\begin{aligned} {}^{\perp}(\mathcal{I}^{(1)}) &= {}^{\perp}(\langle \mathcal{I}\mathcal{T}\mathcal{T} \rangle + \langle \mathcal{T}\mathcal{I}\mathcal{T} \rangle + \langle \mathcal{T}\mathcal{T}\mathcal{I} \rangle) \supseteq \mathcal{Z}_{\mathcal{T}(\mathcal{I})}, \\ (\langle \mathcal{I}\mathcal{T}\mathcal{T} \rangle + \langle \mathcal{T}\mathcal{I}\mathcal{T} \rangle + \langle \mathcal{T}\mathcal{T}\mathcal{I} \rangle)^{\perp} &= (\mathcal{I}^{(1)})^{\perp} \supseteq \mathcal{Z}_{\mathcal{T}(\mathcal{I})}. \end{aligned}$$

Hence $\mathcal{J} \subseteq \mathcal{Z}_{\mathcal{T}(\mathcal{I})} \subseteq {}^{\perp}(\mathcal{I}^{(1)})$ and $\mathcal{J} \subseteq (\mathcal{I}^{(1)})^{\perp}$. It follows that

$$\mathcal{I}^{(1)} \subseteq {}^{\perp}\mathcal{J} \cap \mathcal{J}^{\perp}.$$

Similarly, $\mathcal{J}^{(1)} \subseteq {}^{\perp}\mathcal{I} \cap \mathcal{I}^{\perp}$ since

$$\mathcal{I}^{\perp} \cap \mathcal{J}^{\perp} = (\mathcal{I} + \mathcal{J})^{\perp} = \mathcal{T}^{\perp} = 0 = {}^{\perp}\mathcal{T} = 0 = {}^{\perp}\mathcal{I} \cap {}^{\perp}\mathcal{J}.$$

Therefore $\mathcal{I}^{(1)} \cap \mathcal{J}^{(1)} = 0$, which then gives $\mathcal{T} = \mathcal{I}^{(1)} \oplus \mathcal{J}^{(1)}$.

(1) Let $(\mathcal{T}, f)$ be f-indecomposable and $\mathcal{T}^{(1)} \neq 0$. We prove in three steps that either $\mathcal{I} = \mathcal{T}$ and $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{T}) \subseteq \mathcal{I}^{(1)}$ or $\mathcal{J} = \mathcal{T}$ and $\mathcal{I} \subseteq \mathcal{Z}(\mathcal{T}) \subseteq \mathcal{I}^{(1)}$, thus reaching the conclusion that $\mathcal{T}$ is indecomposable.

First, we show $\mathcal{Z}(\mathcal{T}) \subseteq \mathcal{T}^{(1)}$. Let $\mathcal{Z}_0$ be a subspace of $\mathcal{Z}(\mathcal{T})$ such that $\mathcal{Z}(\mathcal{T}) = \mathcal{Z}_0 \oplus (\mathcal{Z}(\mathcal{T}) \cap \mathcal{T}^{(1)})$. Using Lemma 2.1(2.7), we get

$$\mathcal{T}^{(1)} = \mathcal{Z}(\mathcal{T})^{\perp} = \mathcal{Z}_0^{(1)} \cap (\mathcal{T}^{(1)} + \mathcal{Z}(\mathcal{T})),$$

the last equality being from

$$(\mathcal{Z}(\mathcal{T}) \cap \mathcal{T}^{(1)})^{\perp} = ({}^{\perp}(\mathcal{T}^{(1)}) \cap^{\perp} \mathcal{Z}(\mathcal{T}))^{\perp} = ({}^{\perp}(\mathcal{T}^{(1)} + \mathcal{Z}(\mathcal{T})))^{\perp} = \mathcal{T}^{(1)} + \mathcal{Z}(\mathcal{T}).$$

Consequently

$$0 = \mathcal{Z}_0 \cap \mathcal{T}^{(1)} = \mathcal{T}_0 \cap \mathcal{Z}_0^{\perp} \cap (\mathcal{T}^{(1)} + \mathcal{Z}(\mathcal{T})) = \mathcal{Z}_0 \cap \mathcal{Z}_0^{\perp}.$$

Hence, as a subspace of $\mathcal{Z}(\mathcal{T})$, $\mathcal{Z}_0$ is an f-non-degenerate ideal of $\mathcal{T}$. By the f-indecom -posability of $(\mathcal{T}, f)$, we get $\mathcal{Z}_0 = 0$. Therefore, $\mathcal{Z}(\mathcal{T}) \subseteq \mathcal{T}^{(1)}$.

Next, without loss of generality, suppose $\mathcal{I}^{(1)} \neq 0$. We shall prove $\mathcal{I} \cap^{\perp} \mathcal{I} = \mathcal{I} \cap \mathcal{I}^{\perp}$, from which it is clear that $\mathcal{I} \cap \mathcal{I}^{\perp}$ is an ideal of $\mathcal{T}$ by Lemma 2.3(3). Since $\mathcal{I}^{(1)} \subseteq \mathcal{I}$, it follows that

$${}^{\perp}\mathcal{I} \subseteq^{\perp} (\mathcal{I}^{(1)}) = \mathcal{Z}_{\mathcal{T}(\mathcal{I})} = (\mathcal{I}^{(1)})^{\perp} \supseteq \mathcal{I}^{\perp}.$$

So

$$\mathcal{I} \cap \mathcal{I}^{\perp} \subseteq \mathcal{Z}_{\mathcal{T}(\mathcal{I})} \quad \text{and} \quad \mathcal{I} \cap^{\perp} \mathcal{I} \subseteq \mathcal{Z}_{\mathcal{T}(\mathcal{I})}.$$

Thus

$$\mathcal{I} \cap \mathcal{I}^{\perp} \subseteq \mathcal{Z}(\mathcal{T}) \quad \text{and} \quad \mathcal{I} \cap^{\perp} \mathcal{I} \subseteq \mathcal{Z}(\mathcal{T})$$

because $\mathcal{I} \cap \mathcal{I}^{\perp} \subseteq \mathcal{I} \subset \mathcal{Z}_{\mathcal{T}(\mathcal{J})}$. Furthermore

$$\mathcal{I} \cap \mathcal{I}_{\perp} \subseteq \mathcal{Z}(\mathcal{T}) \subseteq \mathcal{T}^{(1)} = \mathcal{I}^{(1)} \oplus \mathcal{J}^{(1)}.$$

Since $\mathcal{I}^{(1)} \cap \mathcal{I}^{\perp} \subseteq \mathcal{J}^{\perp} \cap \mathcal{I}^{\perp} = \mathcal{T}^{\perp} = 0$, we have $\mathcal{I} \cap \mathcal{I}^{\perp} \subseteq \mathcal{J}^{(1)}$. Thus $\mathcal{I} \cap \mathcal{I}^{\perp} \subseteq \mathcal{J}^{(1)} \cap \mathcal{I}$. On the other hand, $\mathcal{J}^{(1)} \cap \mathcal{I} \subseteq \mathcal{I} \cap \mathcal{I}^{\perp}$ as $\mathcal{J}^{(1)} \subseteq \mathcal{I}^{\perp}$. Therefore

$$\mathcal{I} \cap \mathcal{I}^{\perp} = \mathcal{I} \cap^{\perp} \mathcal{I} = \mathcal{J}^{(1)} \cap \mathcal{I}.$$

Since $0 = \mathcal{T}^{\perp} = \mathcal{I}^{\perp} \cap \mathcal{J}^{\perp} \supseteq \mathcal{I}^{\perp} \cap \mathcal{I}^{(1)} = \mathcal{I} \cap \mathcal{I}^{\perp} \cap \mathcal{I}^{(1)}$,

$$(\mathcal{I} \cap \mathcal{I}^{\perp}) + \mathcal{I}^{(1)} = (\mathcal{I} \cap \mathcal{I}^{\perp}) \oplus \mathcal{I}^{(1)}.$$

Choose a vector subspace $\mathcal{V}$ of $\mathcal{I}$ such that $\mathcal{I} = \mathcal{V} \oplus \mathcal{I}^{(1)} \oplus (\mathcal{I} \cap \mathcal{I}^{\perp})$. Then $\mathcal{I}' := \mathcal{V} \oplus \mathcal{I}^{(1)}$ is an ideal of $\mathcal{I}$.

Finally, we show that $\mathcal{I}'$ is an f-non-degenerate ideal of $\mathcal{T}$, which will lead to $\mathcal{I}' = \mathcal{T}$ and $\mathcal{I} = \mathcal{T}$. Indeed, let $x \in \mathcal{I}'$ be such that $f(x, \mathcal{I}') = 0$. Obviously, $f(x, \mathcal{I} \cap \mathcal{I}^{\perp}) = 0$. And hence $f(x, \mathcal{I}' \oplus (\mathcal{I} \cap \mathcal{I}^{\perp}) = 0$. So $\mathcal{I} = \mathcal{T}$ and $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{T}) \subseteq \mathcal{I}^{(1)} = \mathcal{T}^{(1)}$.

(2) In the case where $\mathcal{T}$ is Abelian, every f-non-degenerate one-dimensional subspace of $\mathcal{T}$ is an f-non-degenerate ideal. Therefore, either there is a one-dimensional f-non-degenerate subspace, meaning $\mathcal{T}$ is one-dimensional by the f-non-degeneracy of $\mathcal{T}$ or every one dimensional subspace is f-degenerate, which implies that f is anti-symmetric by Theorem 3.1(2). In the latter case, for every nonzero vector a in $\mathcal{T}$, there is another nonzero vector $b \in \mathcal{T}$, such that a and b are independent. Then $f(a, b) \neq 0$ by the f-non-degeneracy of $\mathcal{T}$. Since $f(b, a) = -f(a, b)$, the restriction of f to the two-dimensional ideal $\mathcal{B}$ of $\mathcal{T}$ spanned by a and b is non-degenerate, which leads to $\mathcal{T} = \mathcal{B}$. With this the proof of the theorem is completed.

The next theorem tells us the connection between f-decomposition and f'-decompo -sition of the same pseudo-metric triple system. It could be regarded as an f-decomposition theorem for finite-dimensional pseudo-metric triple systems.

Theorem 3.3. *Let $(\mathcal{T}, f)$ be a finite-dimensional pseudo-metric triple system over a field $\mathcal{F}$ and f' another non-degenerate invariant bilinear form on $\mathcal{T}$. If there is a decomposition $\mathcal{T} = \mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_k \oplus \cdots \oplus \mathcal{I}_K$ (resp. $\mathcal{T} = \mathcal{J}_1 \oplus \cdots \oplus \mathcal{J}_m \oplus \cdots \oplus \mathcal{J}_M$) of $\mathcal{T}$ into a direct sum of f-indecomposable (resp. f'-indecomposable) ideals, where k, K, m, M are integers with $0 \leq k \leq K$ and $0 \leq m \leq M$ and the ideals $\mathcal{I}_i$ (resp. $\mathcal{J}_j$) are non-Abelian for $1 \leq k \leq K$ (resp. for $1 \leq m \leq M$) and Abelian otherwise, then*

(1) $k = m$ and there is a permutation of $\{1, 2, \cdots, m\}$ such that the restriction of the canonical projection $p_{j'} : \mathcal{T} \longrightarrow \mathcal{I}_{j'}$ to the ideal $\mathcal{J}_j$ is an isomorphism of the triple systems, with the permutation being uniquely defined by $\mathcal{J}_j \cap \mathcal{I}_{j'} \neq 0$.

(2) $\mathcal{J}_j + Z(\mathcal{T}) = \mathcal{I}_{j'} + Z(\mathcal{T})$, $\mathcal{J}_j^{(1)} = \mathcal{I}_{j'}^{(1)}$ for all $1 \leq j \leq m$.

(3) if $\mathcal{T}$ is perfect or has a vanishing annihilator, then $m = M$ and the above decomposition is unique up to permutations.

(4) If f and f' are symmetric and $\mathrm{Char}\,\mathcal{F} \neq 2$, *then $K = M$ and all f-indecomposable (resp. f'-indecomposable) Abelian ideals are one-dimensional.*

Proof. Every non-Abelian f-indecomposable ideal of $\mathcal{T}$ is indecomposable by Theorem 3.2. So (1)–(3) of the theorem follows from the decomposition Theorem 3.1. Moreover, if f and f' are symmetric bilinear forms, then they are not antisymmetric because $\mathrm{Char}\,\mathcal{F} \neq 2$ and f (resp. f') is not zero. Since every f-indecomposable Abelian ideal is one-dimensional by Theorem 3.1(2), every f-indecomposable ideal is indecomposable.

Corollary 3.1. *Let $(\mathcal{T}, f)$ be a finite-dimensional pseudo-metric Lie triple system over a field $\mathcal{F}$ with* $\mathrm{Char}\,\mathcal{F} \neq 2$. *Let f' be another non-degenerate invariant bilinear form on $\mathcal{T}$. If $\mathcal{T} = \mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_K$ (resp. $\mathcal{T} = \mathcal{J}_1 \oplus \cdots \oplus \mathcal{J}_M$) is a decomposition of $\mathcal{T}$ into a direct sum of f-indecomposable (resp. f'-indecomposable) ideals, then $K = M$ and all f-indecomposable (resp. f'-indecomposable) Abelian ideals are one-dimensional.*

Theorem 3.4. *Let $(\mathcal{T}, f)$ be a pseudo-metric triple system over a field $\mathcal{F}$ and $\mathcal{T} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_N$ a decomposition of $\mathcal{T}$ into indecomposable ideals $\mathcal{G}_r$, where N is a positive integer with $1 \leq r \leq N$. Then there is a non-degenerate invariant bilinear form g on $\mathcal{T}$ such that* *each ideal $\mathcal{G}_r$ of $\mathcal{T}$ is g-non-degenerate.*

Proof. Rearrange the index of $\mathcal{G}_i$ such that, for an integer n $(0 \leq n \leq N)$, the first k ideals $\mathcal{G}_r$ are non-Abelian ideals and Abelian otherwise in the decomposition. Then we have

$$\mathcal{T} = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_k \oplus \cdots \oplus \mathcal{T}_K$$

as a decomposition of $\mathcal{T}$ into f-indecomposable ideals where the ideals $\mathcal{T}_i$ are non-Abelian for $1 \leq i \leq k$ and Abelian otherwise. Let (z_i), $(k+1 \leq i \leq K' = \dim \mathcal{Z}(\mathcal{T}) + k)$ be a vector space basis for the direct sum $Z_0 = \mathcal{T}_{k+1} \oplus \cdots \oplus \mathcal{T}_K$ of the Abelian ideals. Define $f_0 : Z_0 \times Z_0 \longrightarrow \mathcal{F}$ to be the bilinear form $f_0(z_i, z_j) := \delta_{ij}$. Then f_0 is a non-degenerate invariant bilinear form on Z_0 with the one-dimensional ideals $\mathcal{F}z_i$ being f_0-indecomposable and hence indecomposable. Let f_1 be the restriction of f to the direct sum $\mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_k$ of the non-Abelian ideals. Then the orthogonal sum $h := f_0 \perp f_1$ is a non-degenerate invariant bilinear form on $\mathcal{T}$. By Theorem 3.2 and Theorem 3.1, we get $k = n$ and $K = N$ because the h-indecomposable Abelian ideals $\mathcal{F}z_{k+1}, \cdots \mathcal{F}z_{K'}$ are one-dimensional and so are the indecomposable Abelian ideals $\mathcal{G}_{n+1}, \cdots, \mathcal{G}_N$. Denote by h_i $(1 \leq i \leq k')$ the restriction of h to the ideal $\mathcal{I}_i$. Then h_i is a non-degenerate invariant bilinear form on $\mathcal{T}_i$. Since $\mathcal{T} = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_k \oplus \mathcal{F}z_{k+1} \oplus \cdots \oplus \mathcal{F}z_{K'}$ is a decomposition of $\mathcal{T}$ into indecomposable ideals, the restriction of the canonical projection $p_{j'} : \mathcal{T} \longrightarrow \mathcal{T}_{j'}$ to the ideal $\mathcal{G}_j$ (i.e. $p^j_{j'} : \mathcal{G}_j \longrightarrow \mathcal{T}_{j'}$) is an isomorphism of the triple systems by Theorem 3.1. Let $g_j := p^*_{j'}(h_{j'})$ be the pull back bilinear form on $\mathcal{G}_j$. Then g_j is an invariant bilinear form on $\mathcal{G}_j$. g_j is non-degenerate since $p^j_{j'}$ is an isomorphism by Lemma 2.3(1) and (3). The orthogonal sum $g := g_1 \perp g_2 \perp \cdots \perp g_N$ is then a non-degenerate invariant bilinear form on $\mathcal{T}$, with each $\mathcal{G}_r$ being g-non-degenerate.

Remark 3.1. For any finite-dimensional metric triple system $(\mathcal{T}, f)$ over a field $\mathcal{F}$ and $\operatorname{Char}\mathcal{F} \neq 2$, there is always a decomposition of $\mathcal{T}$ into a direct sum of f-indecomposable ideals such that the number of summand components is constant and independent of the choice of non-degenerate invariant bilinear form f.

Corollary 3.2. *Let $(\mathcal{T}, f)$ be a pseudo-metric triple system. If $\mathcal{I}^{(1)} \neq 0$ for every nonzero ideal $\mathcal{I}$ of $\mathcal{T}$, then $\mathcal{T}$ is a direct sum of simple ideals of $\mathcal{T}$.*

Proof. Let $\mathcal{I}$ be a minimal ideal of $\mathcal{T}$, then $\mathcal{I}^\perp$ is an ideal of $\mathcal{T}$ by Lemma 2.1 and hence $\mathcal{I} \cap \mathcal{I}^\perp = \mathcal{I}$ or $\mathcal{I} \cap \mathcal{I}^\perp = 0$. Suppose the first case occurs. Then we have

$$0 = f(\langle yxx' \rangle, x'') = f(y, \langle x''x'x \rangle), \qquad \forall\, x, x', x'' \in \mathcal{I},\ y \in \mathcal{T}$$

because f is invariant. Since f is non-degenerate, $\langle x''x'x \rangle = 0$ and $\mathcal{I}^{(1)} = 0$, which is contrary to the assumption. Hence $\mathcal{I} \cap \mathcal{I}^\perp = 0$ and $\mathcal{T} = \mathcal{I} \oplus \mathcal{I}^\perp$ by the dimension formula in Lemma 2.1. Any ideal of $\mathcal{I}$ is an ideal of $\mathcal{T}$. By the minimality of $\mathcal{I}$, it has no proper ideals. As $\mathcal{I}^{(1)} \neq 0$ by assumption, we see that $\mathcal{I}$ is simple. Since the assumption also holds in $\mathcal{I}^\perp$, we get a decomposition of $\mathcal{T}$ into a direct sum of simple ideals by an recursive argument.

§ 4 . Metric Lie Triple System

From now on, we confine ourself to finite dimensional Lie triple systems, on a field $\mathcal{F}$ of characteristic zero, with an invariant non-degenerate bilinear form. A pseudo-metric

Lie triple system is a metric Lie triple system by Lemma 2.3(2) when $\mathrm{Char}\,\mathcal{F} \neq 2$. In this section, we first find a sufficient condition for a symmetric invariant bilinear form on $\mathcal{T}$ to be non-degenerate and then discuss how to characterize a Lie triple system by the bilinear forms on it.

Let $\mathcal{T}$ be a finite-dimensional triple system over a field $\mathcal{F}$ of characteristic zero and $\mathcal{A}$ be an algebra respectively. Let us denote by $\mathcal{F}(\mathcal{T})$ (resp. $\mathcal{F}(\mathcal{A})$) the linear space of all symmetric invariant bilinear form on $\mathcal{T}$ (resp. on $\mathcal{A}$) and by $\mathcal{B}(\mathcal{T})$ (resp. $\mathcal{B}(\mathcal{A})$) the subspace of $\mathcal{F}(\mathcal{T})$ (resp. $\mathcal{F}(\mathcal{A})$) spanned by the set of invariant symmetric and non-degenerate bilinear forms. We shall say that $\mathcal{T}$ (resp. $\mathcal{A}$) admits a unique (up to constant) quadratic structure if $\mathcal{B}(\mathcal{T})$ (resp. $\mathcal{B}(\mathcal{A})$) is one dimensional.

I. Bajo and S. Benayadi characterized the Lie algebra with a unique quadratic structure (see [10]). Their result is as follows.

Lemma 4.1. *Let $\mathcal{G}$ be a Lie algebra over $\mathcal{F}$ such that* $\dim\mathcal{G} > 1$.

(1) *If* $\dim\mathcal{B}(\mathcal{G}) = 1$, *then $\mathcal{G}$ is a simple Lie algebra;*

(2) *If $\mathcal{F}$ is algebraically closed, then* $\dim\mathcal{B}(\mathcal{G}) = 1$ *if and only if $\mathcal{G}$ is a simple Lie algebra.*

Readers can refer to [10]–[17] for more conclusions on self-dual Lie algebra and Lie superalgebra. We will soon find that a metric Lie triple system $\mathcal{T}$ can be characterized by $\dim\mathcal{B}(\mathcal{T})$.

Lemma 4.2. *Let $\mathcal{F}$ be an algebraically closed field of characteristic zero and $\mathcal{T}$ a nonabelian L.t.s. over $\mathcal{F}$. Then $\mathcal{T}$ is simple iff* $\dim\mathcal{P}(\mathcal{T}) = 1$.

Proof. See [3].

Example 4.1. Let $\mathcal{T}$ be the four-dimensional Lie triple system with the basis $h, e_\pm, g$ and the Lie triple product

$$[h, e_+, e_-] = g, \qquad [h, e_-, e_+] = g, \qquad [e_\pm, h, h] = e_\pm.$$

The triple system is solvable but not nilpotent. For any scalars $\alpha \neq 0$ and β, the bilinear form f:

$$f(e_+, e_-) = \alpha, \qquad f(h, h) = \beta \qquad \text{and} \qquad f(h, g) = \alpha,$$

is invariant and non-degenerate. Hence it is a metric Lie triple system. It is readily verified that $(\mathcal{T}, f)$ is f-indecomposable but $\mathcal{T}$ is not a simple triple system.

Lemma 4.3. *Let $(\mathcal{T}, f)$ be a metric Lie triple system and $\mathcal{I} \subset \mathcal{T}$ be a minimal ideal.*

(1) *If $\mathcal{I}$ is f-non-degenerate, then $\mathcal{I}$ is a factor and hence simple or one-dimensional;*

(2) *If $\mathcal{I}$ is f-degenerate, then it is isotropic (i.e. $\mathcal{I} \subseteq \mathcal{I}^\perp$) and Abelian;*

(3) *$\mathcal{I}^\perp$ is a maximal ideal.*

Proof. If $\mathcal{I} \subset T$ is an ideal, so are $\mathcal{I}^\perp$ and $\mathcal{I} \cap \mathcal{I}^\perp \subset \mathcal{I}$ since the intersection of two ideals is an ideal. Since $\mathcal{I}$ is minimal, $\mathcal{I} \cap \mathcal{I}^\perp$ is either 0 or $\mathcal{I}$.

(1) Let us consider the first possibility, $\mathcal{I} \cap \mathcal{I}^\perp = 0$. Then $\mathcal{I}$ is f-non-degenerate by the definition and the symmetry of f. Since $\mathcal{I}$ and $\mathcal{I}^\perp$ are ideals of $\mathcal{T}$, $\mathcal{T} = \mathcal{I} \oplus \mathcal{I}^\perp$ means that $\mathcal{I}$ is a factor of $\mathcal{T}$. Any ideal of $\mathcal{I}$ is automatically an ideal of $\mathcal{T}$. So $\mathcal{I}$ is either simple or one-dimensional because $\mathcal{I}$ is a minimal ideal.

(2) The second possibility, $\mathcal{I} \cap \mathcal{I}^{\perp} = \mathcal{I}$, which means that $\mathcal{I}$ is f-degenerate. In this case, $\mathcal{I} \subseteq \mathcal{I}^{\perp}$ is isotropic by definition. And by Lemma 2.1, $\mathcal{I} \subseteq \mathcal{I}^{\perp} \subseteq \mathcal{Z}_{\mathcal{T}}(\mathcal{I})$, where $\mathcal{I}$ is Abelian.

(3) Finally, suppose that there exists a proper ideal $\mathcal{J}$ such that $\mathcal{I}^{\perp} \subset \mathcal{J}$. Then we find $\mathcal{J}^{\perp} \subset \mathcal{I}^{\perp\perp} = \mathcal{I}$, which violates the minimality. Hence $\mathcal{I}^{\perp}$ is maximal.

The following preposition is an immediate consequence of Lemma 4.3.

Lemma 4.4. *Let $(\mathcal{T}, f)$ be f-indecomposable Lie triple system. Then exactly one of the following cases holds:*

(1) *$\mathcal{T}$ is a simple Lie triple system;*

(2) *$\mathcal{T}$ is a one-dimensional Lie triple system;*

(3) *$\mathcal{T}$ is not simple, $\dim \mathcal{T} > 1$ and every proper ideals of $\mathcal{T}$ is f-degenerate.*

Let $(\mathcal{T}, f)$ be a metric triple system over $\mathcal{F}$ and $\mathcal{T} = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_n$ be the f-decomposition of $\mathcal{T}$ into a direct sum of f-indecomposable ideals. Denote by $\mathcal{F}(\mathcal{T}_i)$ (resp. $\mathcal{B}(\mathcal{T}_i)$) a subspace of $\mathcal{F}(\mathcal{T})$ (resp. $\mathcal{B}(\mathcal{T})$) by extending any $r_i \in \mathcal{F}(\mathcal{T}_i)$ (resp. $\mathcal{B}(\mathcal{T}_i)$) by zero in a natural way. Since f is symmetric, n is an invariant in $\mathcal{F}$ by Theorem 3.3. We denote it by $n(\mathcal{T})$. Then $\mathcal{F}(\mathcal{T})$ contains the direct sum $\mathcal{F}(\mathcal{T}_1) \oplus \cdots \oplus \mathcal{F}(\mathcal{T}_{n(\mathcal{T})})$.

Lemma 4.5. *Let $(\mathcal{T}, f)$ be a metric triple system and $\mathcal{T} = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_{n(\mathcal{T})}$ be the f-decomposition of $\mathcal{T}$ into f-indecomposable ideals. If $\mathcal{T}$ is perfect, then*

$$\mathcal{F}(\mathcal{T}) = \mathcal{F}(\mathcal{T}_1) \oplus \cdots \oplus \mathcal{F}(\mathcal{T}_{n(\mathcal{T})}).$$

Proof. It is obvious that $\mathcal{F}(\mathcal{T}) \supseteq \mathcal{F}(\mathcal{T}_1) \oplus \cdots \oplus \mathcal{F}(\mathcal{T}_{n(\mathcal{T})})$ from the statement made before Lemma 4.5. Given f_0 in $\mathcal{F}(\mathcal{T})$, we have, for all $0 \le i, j \le n(\mathcal{T}), i \ne j$,

$$f_0(\mathcal{T}_i, \mathcal{T}_j) = f_0(\mathcal{T}_i, \langle \mathcal{T}_j\mathcal{T}_j\mathcal{T}_j\rangle) = f_0(\langle \mathcal{T}_i\mathcal{T}_j\mathcal{T}_j\rangle, \mathcal{T}_j) = f_0(0, \mathcal{T}_j) = 0.$$

Because $\mathcal{T}$ is perfect, we have

$$f_0 \in \mathcal{F}(\mathcal{T}_1) \oplus \cdots \oplus \mathcal{F}(\mathcal{T}_{n(\mathcal{T})}),$$

which completes the proof.

Theorem 4.1. *Let $(\mathcal{T}, f)$ be a finite-dimensional metric Lie triple system.*

(1) *If $\dim \mathcal{F}(\mathcal{T}) = n(\mathcal{T})$ and every one-dimensional ideal is f-degenerate, then $\mathcal{T}$ is semi-simple.*

(2) *If $\mathcal{F}$ is algebraically closed, then the above condition is also necessary.*

Proof. Let $\mathcal{T} = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_{n(\mathcal{T})}$ be an orthogonal direct sum of f-non-degenerate f-indecomposable ideals. Then $n(\mathcal{T})$ is an invariant number and independent of the choice of the non-degenerate symmetric and invariant bilinear form f.

(1) Since $\mathcal{F}(\mathcal{T}) \supseteq \mathcal{F}(\mathcal{T}_1) \oplus \cdots \oplus \mathcal{F}(\mathcal{T}_{n(\mathcal{T})})$ and

$$n(\mathcal{T}) = \dim \mathcal{F}(\mathcal{T}) \ge \sum_{i=1}^{n(\mathcal{T})} \dim \mathcal{F}(\mathcal{T}_i) \ge n(\mathcal{T}),$$

we have $\sum_{i=1}^{n(\mathcal{T})} \dim \mathcal{F}(\mathcal{T}_i) = n(\mathcal{T})$ and hence $\dim \mathcal{F}(\mathcal{T}_i) = 1$ for any f-non-degenerate f-indecomposable $\mathcal{T}_i$. For every $i \leq n(\mathcal{T})$, one gets

$$\dim \mathcal{B}(\mathcal{T}_i) = \dim \mathcal{F}(\mathcal{T}_i) = 1 \quad \text{and} \quad \dim(\mathcal{T}_i) > 1$$

since every one-dimensional ideal is f-degenerate. Then $\mathcal{T}_i$ is a simple Lie triple system by Theorem 3.4. And thus $\mathcal{T}$ is a direct sum of simple ideals.

(2) If $\mathcal{T}$ is a semi-simple metric Lie triple system over an algebraically closed field $\mathcal{F}$, then

$$\mathcal{F}(\mathcal{T}) = \mathcal{F}(\mathcal{T}_1) \oplus \cdots \oplus \mathcal{F}(\mathcal{T}_n(\mathcal{T})).$$

What remains to be proved is that either $\dim \mathcal{F}(\mathcal{T}_i) = 1$ or $\mathcal{T}_i$ is a simple ideal. In fact, $Z(\mathcal{T}) = 0$ since $\mathcal{T}$ is a semi-simple triple system. And hence $\mathcal{T} = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_{n(\mathcal{T})}$ as an f-decomposition is also a decomposition of $\mathcal{T}$ into a direct sum of indecomposable ideals in the general sense. This decomposition is unique up to permutations by Theorem 3.1 for the perfect triple system $\mathcal{T}$. At the same time, $\mathcal{T}$ is a direct sum of simple ideals, so every $\mathcal{T}_i$ $(1 \leq i \leq n(\mathcal{T}))$ is a simple ideal of $\mathcal{T}$ and $\dim \mathcal{B}(\mathcal{T}_i) = \dim \mathcal{F}(\mathcal{T}_i) = 1$. Thus $\dim \mathcal{F}(\mathcal{T}) = \sum_{i=1}^{n(\mathcal{T})} \dim \mathcal{F}(\mathcal{T}_i) = n(\mathcal{T})$ is a natural result.

Theorem 4.2. *Let $(\mathcal{T}, f)$ be a metric Lie triple system over $\mathcal{F}$.*

(1) *If $\dim \mathcal{F}(\mathcal{T}) = n(\mathcal{T})$, $\mathcal{T}$ is a reductive Lie triple system with $\dim \mathcal{Z}(\mathcal{T}) \leq 1$.*

(2) *If $\mathcal{F}$ is algebraically closed, then $\dim \mathcal{F}(\mathcal{T}) = n(\mathcal{T})$ if and only if $\mathcal{T}$ is a reductive Lie triple system with $\dim \mathcal{Z}(\mathcal{T}) \leq 1$.*

Proof. (1) If $\mathcal{T}$ has no one-dimensional f-non-degenerate Abelian ideals, then Theorem 4.1 is applicable and thereby $\mathcal{T}$ is semi-simple. Hence $\mathcal{T}$ is reductive with $\mathcal{Z}(\mathcal{T}) = 0$. Let us suppose that $\mathcal{T}$ has a one-dimensional f-non-degenerate Abelian ideal. Then we have a decomposition $\mathcal{T} = \mathcal{F}e_1 \oplus \cdots \oplus \mathcal{F}e_r \oplus \mathcal{I}$ of f-non-degenerate ideals, where every one-dimensional ideals of $\mathcal{I}$ is f-degenerate. Since $\mathcal{I} \cap \mathcal{I}^\perp = 0$, we have $\mathcal{I}^\perp = \mathcal{F}e_1 \oplus \cdots \oplus \mathcal{F}e_r$. Then $\dim \mathcal{F}(\mathcal{T}) = n(\mathcal{T}) = r + n(\mathcal{I})$ by hypothesis and the definition of $n(\mathcal{T})$. On the other hand, for given symmetric bilinear forms r_{ij} on $\mathcal{T}$ by $r_{ij}(e_i, e_j) = 1$ and zero otherwise, one can easily verify that r_{ij} are invariant for all $i, j \leq k$. In this way, we get $\frac{r^2+r}{2}$ symmetric invariant bilinear forms. Hence

$$\dim \mathcal{F}(\mathcal{T}) \geq \dim \mathcal{F}(\mathcal{I}) + \frac{r^2 + r}{2} \geq n(I) + \frac{r^2 + r}{2}.$$

We therefore get $n(T) = r + n(\mathcal{I}) \geq n(\mathcal{I}) + \frac{r^2+r}{2}$. Thus $r = 1$ for $r \geq \frac{r^2+r}{2}$. Moreover

$$\dim \mathcal{F}(\mathcal{I}) \leq n(\mathcal{T}) - 1 = n(\mathcal{I}) \leq \dim \mathcal{F}(\mathcal{I}).$$

It follows that $\mathcal{I}$ is semi-simple by Theorem 4.1(2) and the fact that every one-dimensional ideal of $\mathcal{I}$ is f-degenerate. Thus $\mathcal{T}$ is a direct sum of the semi-simple ideals $\mathcal{I}$ and $\mathcal{Z}(\mathcal{T})$ with $\dim \mathcal{Z}(\mathcal{T}) \leq 1$.

(2) Let $\mathcal{T}$ be a reductive Lie triple system over an algebraically closed field $\mathcal{F}$ with $\dim \mathcal{Z}(\mathcal{T}) \leq 1$. When $\dim \mathcal{Z}(\mathcal{T}) = 0$, the conclusion is immediate from Theorem 4.1(ii). We consider the case $\dim \mathcal{Z}(\mathcal{T}) = 1$ and $\mathcal{T} = \mathcal{F}e \oplus \mathcal{S}$, where $\mathcal{S}$ is semi-simple and $\mathcal{Z}(\mathcal{T}) = \mathcal{F}e$.

Since $[\mathcal{S},\mathcal{S}]=\mathcal{S}$, $r(\mathcal{F}e,\mathcal{S})=0$ for every $r\in\mathcal{F}(\mathcal{T})$. Hence, by the semi-simplicity of $\mathcal{S}$ and Theorem 4.1(ii),

$$\dim\mathcal{F}(\mathcal{T})=\dim\mathcal{F}(\mathcal{F}e)+\dim\mathcal{S}=1+n(\mathcal{S}).$$

Since $1+n(\mathcal{S})=n(\mathcal{T})$, we have $\dim\mathcal{F}(\mathcal{T})=n(\mathcal{T})$. With this we come to the end of the proof.

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