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HUA'S THEOREM WITH FIVE ALMOST EQUAL PRIME VARIABLES

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Abstract

It is proved that each sufficiently large integer $N \equiv 5$ (mod24) can be written as $N = p_1^2+p_2^2+p_3^2+p_4^2+p_5^2$ with $|p_j - \sqrt{N/5}| \le U = N^{\frac{1}{2}-\frac{1}{35}+\varepsilon}$, where p_j are primes. This result, which is obtained by an iterative method and a hybrid estimate for Dirichlet polynomial, improves the previous results in this direction.

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§ 1 . Introduction

In the additive theory of prime numbers, one studies the representation of positive integers by powers of primes. For the quadratic case, Hua [1] proved that each large integer congruent to 5 modulo 24 can be written as the sum of five squares of primes.

Under the General Riemann Hypothesis (GRH), Liu and Zhan [2] sharpened Hua's result by showing that each large integer N congruent to 5 modulo 24 can be written as sums of five almost equal prime squares. More precisely, they proved that under GRH,

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2, \tag{1.1}$$

where $|p_j - \sqrt{N/5}| \le U,\ j = 1, 2, \cdots, 5$ for $U = N^{\frac{1}{2}-\frac{1}{20}+\varepsilon}$. Bauer [3] showed that unconditionally the formula (1.1) holds true for $U = N^{\frac{1}{2}-\delta}$, where $\delta \ge 0$ and its exact value depends on the constants in the Deuring-Heilbronn phenomenon, and is not numerically determined. In 1998, Liu and Zhan [4] found the new approach to treat the enlarged major arcs, in which the possible existence of Siegel zero does not have special influence, and hence the Deuring-Heilbronn phenomenon can be avoided. This approach not only is technically simpler, but also gives, when applicable, substantially better results than Deuring-Heilbronn phenomenon. Due to this approach, they obtained that (1.1) is true for $U = N^{\frac{1}{2}-\frac{1}{50}+\varepsilon}$. This approach is subsequently improved and clarified in [5] and [6]. Recently Bauer [7] used the approach mentioned above to show that $U = N^{\frac{1}{2}-\frac{19}{850}+\varepsilon}$ is acceptable.

In this paper, we are able to modify the new ideas of Liu [8], so that they are capable of treating the present short interval case. This results in the following further improvement.

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Theorem 1.1. *Any sufficiently large integer N congruent to 5 modulo 24 can be written as* (1.1) *with* $|p_j - \sqrt{N/5}| \le U,\ j = 1, 2, \cdots, 5,$ *where*

$$U = N^{\frac{1}{2} - \frac{1}{35} + \varepsilon}. \tag{1.2}$$

This theorem is proved by the circle method. Here the main difficulty arises in treating the enlarged major arcs. The idea of the proof will be explained in Section 2.

Notation. As usual, $\varphi(n)$, $\mu(n)$ and $\Lambda(n)$ stand for the function of Euler, Möbius, and von Mangoldt respectively, and $\tau(n)$ is the divisor function. We use $\chi \bmod q$ and $\chi^0 \bmod q$ to denote a Dirichlet character and the principal character modulo q, and $L(s, \chi)$ is the Dirichlet L-function. For integers $a, b, \cdots$, we denote by $[a, b, \cdots]$ their least common multiple. N is a large integer, and $L = \log N$. And $r \sim R$ means $R < r \le 2R$. If there is no ambiguity, we express $\frac{a}{b} + \theta$ as $a/b + \theta$ or $\theta + a/b$. The same convention will be applied for quotients. The letter ε denotes a positive constant which is arbitrarily small, but not necessarily the same at different occurrences.

§ 2. Outline of the Method

Here we give an outline for the proof of Theorem 1.1. In order to apply the circle method, for U as in (1.2), we set

$$P = N^{2+8\varepsilon} U^{-4}, \qquad Q = U^7 N^{-5/2-6\varepsilon}. \tag{2.1}$$

By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q, 1 + 1/Q]$ may be written in the form

$$\alpha = a/q + \lambda, \qquad |\lambda| \le 1/(qQ) \tag{2.2}$$

for some integers a, q with $1 \le a \le q \le Q$ and $(a, q) = 1$. We denote by $\mathcal{M}(a, q)$ the set of α satisfying (2.2), and define the major arcs $\mathcal{M}$ and the minor arcs $C(\mathcal{M})$ as follows:

$$\mathcal{M} = \bigcup_{q \le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathcal{M}(a, q), \qquad C(\mathcal{M}) = \Big[\frac{1}{Q}, 1 + \frac{1}{Q}\Big] \Big\backslash \mathcal{M}. \tag{2.3}$$

Our Theorem 1.1 can be easily derived from the following theorem.

Theorem 2.1. *Let $\mathcal{M}$ be as above with P, Q determined by* (2.1)*. For*

$$N_1 = \sqrt{N/5} - U, \qquad N_2 = \sqrt{N/5} + U, \tag{2.4}$$

let

$$S(\alpha) = \sum_{N_1 < p \le N_2} (\log p) e(p^2 \alpha). \tag{2.5}$$

Then we have that for any $A > 0$,

$$\int_{\mathcal{M}} S^5(\alpha) e(-N\alpha) d\alpha = \frac{1}{32} P_0 \sum_{q \le P} A(N, q) + O(U^4 N^{-1/2} L^{-A}), \tag{2.6}$$

where

$$U^4N^{-\frac{1}{2}} \ll P_0 = \sum_{\substack{m_1+m_2+m_3+m_4+m_5=N \\ N_1^2<m_j\le N_2^2}} (m_1m_2m_3m_4m_5)^{-\frac{1}{2}} \ll U^4N^{-\frac{1}{2}} \tag{2.7}$$

and $A(n,q)$ *is defined by* (3.3).

The proof of Theorem 2.1 forms the bulk of the paper, which will be proved in Section 4.

Now we prove Theorem 1.1 from Theorem 2.1.

Proof of Theorem 1.1. Let N be a sufficiently large integer with $N\equiv 5 \pmod{24}$. Let

$$r(N) = \sum_{\substack{N=p_1^2+\cdots+p_5^2 \\ |p_j-\sqrt{N/5}|\le U}} (\log p_1)\cdots(\log p_5),$$

where $U = N^{\frac{1}{2}-\frac{1}{35}+\varepsilon}$. Then we have

$$r(N) = \int_0^1 S^5(\alpha)e(-N\alpha)d\alpha = \int_{\mathcal{M}} + \int_{C(\mathcal{M})} =: r_1(N) + r_2(N), \tag{2.8}$$

where $\mathcal{M}, C(\mathcal{M})$, and $S(\alpha)$ are as in (2.3) and (2.5).

To estimate the contribution from the minor arcs, we apply Liu and Zhan's result (see [2]), which states that

$$S(\alpha) \ll U^{1+\varepsilon}\Big(P^{-1/4} + \frac{N^{1/16}}{U^{1/4}} + \frac{N^{1/6}}{U^{1/2}} + \frac{Q^{1/4}N^{1/8}}{U^{3/4}}\Big) \ll U^2N^{-1/2-\varepsilon}. \tag{2.9}$$

Then we have that for any $A>0$,

$$r_2(N) \ll \Big\{\max_{\alpha\in C(\mathcal{M})}|S(\alpha)|\Big\}\int_0^1 |S(\alpha)|^4d\alpha \ll U^4N^{-1/2}L^{-A}. \tag{2.10}$$

From Theorem 2.1, we obtain that

$$r_1(N) = \frac{1}{32}P_0\sum_{q\le P}A(N,q) + O(U^4N^{-1/2}L^{-A}), \tag{2.11}$$

and

$$U^4N^{-\frac{1}{2}} \ll P_0 \ll U^4N^{-\frac{1}{2}}.$$

For the singular series $\sum_{q\le P} A(N,q)$ in (2.11), we quote Lemma 4.2 in [2], which states that

$$\sum_{q\le P} A(N,q) = (1+A(N,2)+A(N,4)+A(N,8))\prod_{3\le p}(1+A(N,p)) + O(P^{-1/2+\varepsilon}), \tag{2.12}$$

where the first term on the right hand side is convergent and satisfies $> c > 0$ for $N \equiv 5 \pmod{24}$.

From (2.8) and (2.10)–(2.12), Theorem 1.1 clearly follows.

§ 3 . Preliminaries

For χ mod q, define

$$C(\chi, a) = \sum_{h=1}^{q} \overline{\chi}(h) e\Big(\frac{ah^2}{q}\Big), \qquad C(q,a) = C(\chi^0, a). \tag{3.1}$$

If $\chi_1, \chi_2, \cdots, \chi_5$ are characters mod q, then we write

$$B(N, q, \chi_1, \cdots, \chi_5) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\Big(-\frac{aN}{q}\Big) C(\chi_1, a) C(\chi_2, a) C(\chi_3, a) C(\chi_4, a) C(\chi_5, a), \tag{3.2}$$

and

$$B(N, q) = B(N, q, \chi^0, \chi^0, \chi^0, \chi^0, \chi^0), \qquad A(N, q) = \frac{B(N, q)}{\varphi^5(q)}. \tag{3.3}$$

The following lemma is important when we prove Theorem 2.1.

Lemma 3.1. *Let* χ_j mod r_j *with* $j = 1, \cdots, 5$ *be primitive characters,* $r_0 = [r_1, \cdots, r_5]$, *and* χ^0 *the principal character* mod q. *Then*

$$\sum_{\substack{q \le x \\ r_0 | q}} \frac{1}{\varphi^5(q)} |B(n, q, \chi_1\chi^0, \chi_2\chi^0, \cdots, \chi_5\chi^0)| \ll r_0^{-3/2+\varepsilon} \log^c x.$$

Proof. The proof of this lemma is standard. See [6] for details.

The saving of $r_0^{-3/2+\varepsilon}$ on the right hand side will play a key role in our argument, and our result will depend on the magnitude of the exponent 3/2.

Recall N_1, N_2 as in (2.4), and define

$$\begin{aligned} V(\lambda) &= \sum_{N_1 < m \le N_2} e(m^2\lambda), \\ W(\chi, \lambda) &= \sum_{N_1 < p \le N_2} (\log p) \chi(p) e(p^2\lambda) - \delta_\chi \sum_{N_1 < m \le N_2} e(m^2\lambda), \end{aligned} \tag{3.4}$$

where $\delta_\chi = 1$ or 0 according as χ is principal or not. Define further

$$J(g) = \sum_{r \le P} [g, r]^{-3/2+\varepsilon} \sum_{\chi \bmod r}^{*} \max_{|\lambda| \le 1/(rQ)} |W(\chi, \lambda)|, \tag{3.5}$$

$$K(g) = \sum_{r \le P} [g, r]^{-3/2+\varepsilon} \sum_{\chi \bmod r}^{*} \Big(\int_{-1/(rQ)}^{1/(rQ)} |W(\chi, \lambda)|^2 d\lambda \Big)^{1/2}. \tag{3.6}$$

Our Theorem 2.1 depends on the following three lemmas, which will be proved in Sections 5 and 6.

Lemma 3.2. *For* P, Q *satisfying* (2.1), *we have*

$$J(g) \ll g^{-3/2+\varepsilon} U L^c. \tag{3.7}$$

Lemma 3.3. *Let P, Q be as in* (2.1). *For $g = 1$ Lemma* 3.2 *can be improved to*

$$J(1) \ll UL^{-A}, \tag{3.8}$$

where $A > 0$ is arbitrary.

Lemma 3.4. *For P, Q as in* (2.1), *we have*

$$K(g) \ll g^{-3/2+\varepsilon} U^{1/2} N^{-1/4} L^c. \tag{3.9}$$

§ 4. Proof of Theorem 2.1

With Lemmas 3.2–3.4 known, we can use the iterative idea to prove Theorem 2.1.

Proof of Theorem 2.1. For $q \le P$ and $N_1 < p \le N_2$, we have $(q,p) = 1$. Therefore we can rewrite the exponential sum $S(\alpha)$ as

$$S\Big(\frac{a}{q} + \lambda\Big) = \frac{C(q,a)}{\varphi(q)} V(\lambda) + \frac{1}{\varphi(q)} \sum_{\chi \bmod q} C(\chi, a) W(\chi, \lambda),$$

where $V(\lambda)$ and $W(\chi,\lambda)$ are as in (3.4). Thus

$$\int_{\mathcal{M}} S^5(\alpha) e(-N\alpha) d\alpha = I_0 + 5I_1 + 10I_2 + 10I_3 + 5I_4 + I_5, \tag{4.1}$$

where

$$I_j = \sum_{q \le P} \frac{1}{\varphi^5(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C^{5-j}(q,a) e\Big(-\frac{aN}{q}\Big)$$
$$\times \int_{-1/(qQ)}^{1/(qQ)} V^{5-j}(\lambda) \Big\{ \sum_{\chi \bmod q} C(\chi,a) W(\chi,\lambda) \Big\}^j e(-N\lambda) d\lambda.$$

We will prove that I_0 gives the main term, and $I_1, I_2, \cdots, I_5$ the error term.

The computation of I_0 is standard, and therefore we give the result directly

$$\begin{aligned} I_0 &= \frac{1}{32} P_0 \sum_{q \le P} \frac{B(N,q)}{\varphi^5(q)} + O((PQ)^4 N^{-5/2}) + O(U^4 N^{-1/2} L^{-A}) \\ &= \frac{1}{32} P_0 \sum_{q \le P} \frac{B(N,q)}{\varphi^5(q)} + O(U^4 N^{-1/2} L^{-A}). \end{aligned} \tag{4.2}$$

To bound the contributions of other terms, we begin with I_5, the most complicated one.

Reducing the characters in I_5 into primitive characters, we have

$$\begin{aligned}|I_5| &= \Big|\sum_{q\le P}\frac{1}{\varphi^5(q)}\sum_{\chi_1 \bmod q}\cdots\sum_{\chi_5 \bmod q}B(N,q,\chi_1,\cdots,\chi_5)\\ &\quad\times\int_{-1/(qQ)}^{1/(qQ)}W(\chi_1,\lambda)\cdots W(\chi_5,\lambda)e(-N\lambda)d\lambda\Big|\\ &\le\sum_{r_1\le P}\cdots\sum_{r_5\le P}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\cdots\sideset{}{^*}\sum_{\chi_5 \bmod r_5}\sum_{\substack{q\le P\\ r_0|q}}\frac{|B(N,q,\chi_1\chi^0,\cdots,\chi_5\chi^0)|}{\varphi^5(q)}\\ &\quad\times\int_{-1/(qQ)}^{1/(qQ)}|W(\chi_1\chi^0,\lambda)|\cdots|W(\chi_5\chi^0,\lambda)|d\lambda,\end{aligned}$$

where χ^0 is the principal character modulo q, $r_0=[r_1,r_2,\cdots,r_5]$ depending on $r_1,r_2,\cdots,r_5$, and the sum $\sum^*$ is over all primitive characters. For $q\le P$ and $N_1<p\le N_2$, we have $(q,p)=1$. Using this and (3.4), we have $W(\chi_j\chi^0,\lambda)=W(\chi_j,\lambda)$ for the primitive characters χ_j above. Thus by Lemma 3.1, we obtain

$$\begin{aligned}|I_5| &\le \sum_{r_1\le P}\cdots\sum_{r_5\le P}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\cdots\sideset{}{^*}\sum_{\chi_5 \bmod r_5}\int_{-1/(r_0Q)}^{1/(r_0Q)}|W(\chi_1,\lambda)|\cdots|W(\chi_5,\lambda)|d\lambda\\ &\quad\times\sum_{\substack{q\le P\\ r_0|q}}\frac{|B(N,q,\chi_1\chi^0,\cdots,\chi_5\chi^0)|}{\varphi^5(q)}\\ &\ll L^c\sum_{r_1\le P}\cdots\sum_{r_5\le P}r_0^{-3/2+\varepsilon}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\cdots\sideset{}{^*}\sum_{\chi_5 \bmod r_5}\int_{-1/(r_0Q)}^{1/(r_0Q)}|W(\chi_1,\lambda)|\cdots|W(\chi_5,\lambda)|d\lambda.\end{aligned}$$

In the last integral, we take out $|W(\chi_1,\lambda)|$, and then use Cauchy's inequality, to get

$$\begin{aligned}|I_5| &\ll L^c\sum_{r_1\le P}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\max_{|\lambda|\le 1/(r_1Q)}|W(\chi_1,\lambda)|\\ &\quad\times\sum_{r_2\le P}\sideset{}{^*}\sum_{\chi_1 \bmod r_2}\max_{|\lambda|\le 1/(r_2Q)}|W(\chi_2,\lambda)|\\ &\quad\times\sum_{r_3\le P}\sideset{}{^*}\sum_{\chi_3 \bmod r_3}\max_{|\lambda|\le 1/(r_3Q)}|W(\chi_3,\lambda)|\\ &\quad\times\sum_{r_4\le P}\sideset{}{^*}\sum_{\chi_4 \bmod r_4}\Big(\int_{-1/(r_4Q)}^{1/(r_4Q)}|W(\chi_4,\lambda)|^2d\lambda\Big)^{1/2}\\ &\quad\times\sum_{r_5\le P}r_0^{-3/2+\varepsilon}\sideset{}{^*}\sum_{\chi_5 \bmod r_5}\Big(\int_{-1/(r_5Q)}^{1/(r_5Q)}|W(\chi_5,\lambda)|^2d\lambda\Big)^{1/2}.\end{aligned}\tag{4.3}$$

Now we introduce an iterative procedure to bound the above sums over $r_5,\cdots,r_1$ consecutively.

We first estimate the above sum over r_5 in (4.3) via Lemma 3.4. Since

$$r_0=[r_1,\cdots,r_5]=[[r_1,\cdots,r_4],r_5],$$

the sum over r_5 in (4.3) is

$$\begin{aligned}&\sum_{r_5\le P}[[r_1,\cdots,r_4],r_5]^{-3/2+\varepsilon}\sideset{}{^*}\sum_{\chi_5 \bmod r_5}\Big(\int_{-1/(r_5Q)}^{1/(r_5Q)}|W(\chi_5,\lambda)|^2d\lambda\Big)^{1/2}\\&=K([r_1,\cdots,r_4])\ll[r_1,\cdots,r_4]^{-3/2+\varepsilon}U^{1/2}N^{-1/4}L^c.\end{aligned}$$

This contributes to the sum over r_4 of (4.3) in amount

$$\begin{aligned}&\ll U^{1/2}N^{-1/4}L^c\sum_{r_4\le P}[r_1,\cdots,r_4]^{-3/2+\varepsilon}\sideset{}{^*}\sum_{\chi_4 \bmod r_4}\Big(\int_{-1/(r_4Q)}^{1/(r_4Q)}|W(\chi_4,\lambda)|^2d\lambda\Big)^{1/2}\\&=U^{1/2}N^{-1/4}L^cK([r_1,r_2,r_3])\ll[r_1,r_2,r_3]^{-3/2+\varepsilon}UN^{-1/2}L^c,\end{aligned}$$

where we have used Lemma 3.4 again.

Inserting this last bound into (4.3), we can bound the sum over r_3 as

$$\begin{aligned}&\ll UN^{-1/2}L^c\sum_{r_3\le P}[r_1,r_2,r_3]^{-3/2+\varepsilon}\sideset{}{^*}\sum_{\chi_3 \bmod r_3}\max_{|\lambda|\le1/(r_3Q)}|W(\chi_3,\lambda)|\\&\ll UN^{-1/2}L^cJ([r_1,r_2])\ll U^2N^{-1/2}L^c[r_1,r_2]^{-3/2+\varepsilon}.\end{aligned}$$

Similarly we can bound the sums over r_2, r_1 and find that

$$\begin{aligned}|I_5|&\ll U^2N^{-1/2}L^c\sum_{r_1\le P}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\max_{|\lambda|\le1/(r_1Q)}|W(\chi_1,\lambda)|\\&\quad\times\sum_{r_2\le P}[r_1,r_2]^{-3/2+\varepsilon}\sideset{}{^*}\sum_{\chi_1 \bmod r_2}\max_{|\lambda|\le1/(r_2Q)}|W(\chi_2,\lambda)|\\&=U^2N^{-1/2}L^c\sum_{r_1\le P}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\max_{|\lambda|\le1/(r_1Q)}|W(\chi_1,\lambda)|J(r_1)\\&\ll U^3N^{-1/2}L^cJ(1)\ll U^4N^{-1/2}L^{-A},\end{aligned}\tag{4.4}$$

where we have used Lemma 3.3 and Lemma 3.2 consecutively.

Since the estimations of I_4, I_3, I_2, I_1 follow the similar procedure in terms of K and J in Lemmas 3.2–3.4, we shall omit the details. For example, we have

$$\begin{aligned}|I_4|&\ll L^c\max_{|\lambda|\le1/Q}|V(\lambda)|\sum_{r_1\le P}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\max_{|\lambda|\le1/(r_1Q)}|W(\chi_1,\lambda)|\\&\times\sum_{r_2\le P}\sideset{}{^*}\sum_{\chi_1 \bmod r_2}\max_{|\lambda|\le1/(r_2Q)}|W(\chi_2,\lambda)|\\&\times\sum_{r_3\le P}\sideset{}{^*}\sum_{\chi_3 \bmod r_3}\Big(\int_{-1/(r_3Q)}^{1/(r_3Q)}|W(\chi_3,\lambda)|^2d\lambda\Big)^{1/2}\\&\times\sum_{r_4\le P}[r_1,\cdots,r_4]^{-3/2+\varepsilon}\sideset{}{^*}\sum_{\chi_4 \bmod r_4}\Big(\int_{-1/(r_4Q)}^{1/(r_4Q)}|W(\chi_4,\lambda)|^2d\lambda\Big)^{1/2}.\end{aligned}$$

Obviously to get upper bounds for I_4, I_3, I_2, I_1, we need the estimates

$$\max_{|\lambda|\le1/Q}|V(\lambda)|\ll U$$

and

$$\Big(\int_{-1/Q}^{1/Q}|V(\lambda)|^2d\lambda\Big)^{1/2}\ll U^{1/2}N^{-1/4}.$$

The first estimate is trivial and the second estimate can be obtained by partial summation and the elementary estimate for exponential sums. Inserting these estimates into I_4, I_3, I_2, I_1, we get

$$I_1,\cdots,I_4\ll U^4N^{-1/2}L^{-A}. \tag{4.5}$$

From (4.1), (4.2), (4.4) and (4.5), we complete the proof of Theorem 2.1.

§ 5 . Estimation of $K(g)$

Let $Y\le X$ and $M_1,\cdots,M_{10}$ be positive integers such that

$$2^{-10}Y\le M_1\cdots M_{10}<X \qquad \text{and} \qquad 2M_6,\cdots,2M_{10}\le X^{1/5}. \tag{5.1}$$

For $j=1,\cdots,10$, define

$$a_j(m)=\begin{cases}\log m, & \text{if } j=1,\\ 1, & \text{if } j=2,\cdots,5,\\ \mu(m), & \text{if } j=6,\cdots,10,\end{cases} \tag{5.2}$$

where $\mu(n)$ is the Möbius function. Then we define the functions

$$\begin{aligned} f_j(s,\chi)&=\sum_{m\sim M_j}\frac{a_j(m)\chi(m)}{m^s},\\ F(s,\chi)&=f_1(s,\chi)\cdots f_{10}(s,\chi),\end{aligned} \tag{5.3}$$

where χ is a Dirichlet character, s a complex variable. The following hybrid estimate for $|F|$ is one of the key ingredients in carrying out the iterative procedure.

Lemma 5.1. *Let $F(s,\chi)$ be as in* (5.3). *Then for any $1\le R\le X^2$ and $T>0$,*

$$\sum_{\substack{r\sim R\\ d|r}}\sideset{}{^*}\sum_{\chi \bmod r}\int_T^{2T}\Big|F\Big(\frac12+it,\chi\Big)\Big|dt\ll\Big\{\frac{R^2}{d}T+\frac{R}{d^{1/2}}T^{1/2}X^{3/10}+X^{1/2}\Big\}\log^cX. \tag{5.4}$$

Proof. See [8] for details.

Now we prove Lemma 3.4, which determines our result in essence.

Proof of Lemma 3.4. Let

$$\widehat{W}(\chi,\lambda)=\sum_{N_1<m\le N_2}(\Lambda(m)\chi(m)-\delta_\chi)e(m^2\lambda).$$

Then

$$W(\chi,\lambda)-\widehat{W}(\chi,\lambda)\ll N^{1/4}. \tag{5.5}$$

This contributes to (3.6) in amount

$$\begin{aligned}
&\ll N^{1/4}\sum_{r\le P}[g,r]^{-3/2+\varepsilon}\frac{r^{1/2}}{Q^{1/2}}\\
&\ll g^{-3/2+\varepsilon}N^{1/4}Q^{-1/2}\sum_{r\le P}\Big(\frac{r}{(g,r)}\Big)^{-3/2+\varepsilon}r^{1/2}\\
&\ll g^{-3/2+\varepsilon}N^{1/4}Q^{-1/2}\sum_{r\le P}\Big(\frac{r}{(g,r)}\Big)^{-1+\varepsilon}r^{1/2}\\
&\ll g^{-3/2+\varepsilon}N^{1/4}Q^{-1/2}\sum_{\substack{d|g\\ d\le P}}d^{1-\varepsilon}\sum_{\substack{r\le P\\ d|r}}r^{-1/2+\varepsilon}\\
&\ll g^{-3/2+\varepsilon}N^{1/4}P^{1+\varepsilon}Q^{-1/2}\\
&\ll g^{-3/2+\varepsilon}U^{1/2}N^{-1/4}L^{c},
\end{aligned}$$

where we have used $[g,r](g,r)=gr$, (2.1) and (1.2).

Thus to establish Lemma 3.4, it suffices to show that

$$\sum_{r\sim R}[g,r]^{-3/2+\varepsilon}\sideset{}{^*}\sum_{\chi \bmod r}\Big(\int_{-1/(rQ)}^{1/(rQ)}|\widehat{W}(\chi,\lambda)|^2d\lambda\Big)^{1/2}\ll g^{-3/2+\varepsilon}U^{1/2}N^{-1/4}L^{c} \tag{5.6}$$

holds for $R\le P$.

By Gallagher's lemma (see [9, Lemma 1]), we have

$$\begin{aligned}
\int_{-1/(rQ)}^{1/(rQ)}|\widehat{W}(\chi,\lambda)|^2d\lambda &\ll \Big(\frac{1}{RQ}\Big)^2\int_{-\infty}^{\infty}\Big|\sum_{\substack{v<m^2\le v+rQ\\ N_1^2<m^2\le N_2^2}}(\Lambda(m)\chi(m)-\delta_\chi)\Big|^2dv\\
&\ll \Big(\frac{1}{RQ}\Big)^2\int_{N_1^2-rQ}^{N_2^2}\Big|\sum_{\substack{v<m^2\le v+rQ\\ N_1^2<m^2\le N_2^2}}(\Lambda(m)\chi(m)-\delta_\chi)\Big|^2dv\\
&\ll \Big(\frac{1}{RQ}\Big)^2\int_{N_1^2-rQ}^{N_2^2}\Big|\sum_{Y<m\le X}(\Lambda(m)\chi(m)-\delta_\chi)\Big|^2dv, \qquad (5.7)
\end{aligned}$$

where

$$Y=\max(v^{1/2},N_1),\qquad X=\min((v+rQ)^{1/2},N_2).$$

We argue exactly as Lemma 5.1 in [8] and see that the inner sum in (5.7) is a linear combination of $O(L^{10})$ terms, each of which has the form

$$\varSigma(u;\mathbf{M}):=\frac{1}{2\pi}\int_{-T}^{T}F\Big(\frac12+it,\chi\Big)\frac{X^{1/2+it}-Y^{1/2+it}}{1/2+it}dt+O\Big(\frac{N^{1/2}L^2}{T}\Big),$$

where T is a parameter satisfying $2\le T\le N^{1/2}$. One sees that

$$\frac{X^{1/2+it}-Y^{1/2+it}}{1/2+it}=\frac12\int_{Y^2}^{X^2}u^{-3/4+it/2}du=\frac12\int_{Y^2}^{X^2}u^{-3/4}e\Big(\frac{t}{4\pi}\log u\Big)du.$$

The integral can be estimated as

$$\ll X^{1/2} - Y^{1/2} \ll (v + rQ)^{1/4} - v^{1/4} \ll v^{1/4}\{(1 + rQ/v)^{1/4} - 1\} \ll N^{-3/4}RQ.$$

On the other hand, one has trivially

$$\frac{X^{1/2+it} - Y^{1/2+it}}{1/2 + it} \ll \frac{X^{1/2}}{|t|} \ll \frac{N_2^{1/2}}{|t|} \ll \frac{N^{1/4}}{|t|}.$$

Collecting the two upper bounds, we get

$$\frac{X^{1/2+it} - Y^{1/2+it}}{1/2 + it} \ll \min\Big(\frac{RQ}{N^{3/4}}, \frac{N^{1/4}}{|t|}\Big).$$

Taking

$$T = N^{1/2}, \qquad T_0 = 8\pi N/(QR),$$

we see that

$$\varSigma(u; \mathbf{M}) \ll \frac{RQ}{N^{3/4}} \int_{|t|\le T_0} \Big|F\Big(\frac{1}{2} + it, \chi\Big)\Big| dt + N^{1/4} \int_{T_0<|t|\le T} \Big|F\Big(\frac{1}{2} + it, \chi\Big)\Big| \frac{dt}{|t|} + O(L^2).$$

And consequently (5.7) becomes

$$\int_{-1/(rQ)}^{1/(rQ)} |\widehat{W}(\chi, \lambda)|^2 d\lambda \ll UN^{-1}L^{20} \max_{\mathbf{M}} \Big(\int_{|t|\le T_0} \Big|F\Big(\frac{1}{2} + it, \chi\Big)\Big| dt\Big)^2 + \frac{NUL^{20}}{(QR)^2} \max_{\mathbf{M}} \Big(\int_{T_0<|t|\le T} \Big|F\Big(\frac{1}{2} + it, \chi\Big)\Big| \frac{dt}{|t|}\Big)^2 + \frac{N^{1/2}UL^{24}}{(QR)^2},$$

where we have used $N_2^2 - N_1^2 \ll N^{1/2}U$ and $RQ \le PQ \ll N^{1/2}U$.

The last term above contributes to the left hand side of (5.6) in amount

$$\ll \sum_{r\sim R} r^{-3/2+\varepsilon} \sum_{\chi \bmod r} \frac{(N^{1/2}U)^{1/2}L^{12}}{RQ} \ll \frac{N^{1/4}U^{1/2}L^{12}}{Q} \ll P^{-3/2+\varepsilon}U^{1/2}N^{-1/4}L^c \ll g^{-3/2+\varepsilon}U^{1/2}N^{-1/4}L^c,$$

and therefore the left hand side of (5.6) is

$$\ll U^{1/2}N^{-1/2}L^{10} \max_{\mathbf{M}} \sum_{r\sim R} [g, r]^{-3/2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{|t|\le T_0} \Big|F\Big(\frac{1}{2} + it, \chi\Big)\Big| dt + \frac{N^{1/2}U^{1/2}L^{10}}{RQ} \max_{\mathbf{M}} \sum_{r\sim R} [g, r]^{-3/2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_0<|t|\le T} \Big|F\Big(\frac{1}{2} + it, \chi\Big)\Big| \frac{dt}{|t|} + g^{-3/2+\varepsilon}U^{1/2}N^{-1/4}L^c.$$

Thus, to prove (5.6) it suffices to show that the estimate

$$\sum_{r\sim R} [g, r]^{-3/2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_1}^{2T_1} \Big|F\Big(\frac{1}{2} + it, \chi\Big)\Big| dt \ll g^{-3/2+\varepsilon}N^{1/4}L^c \tag{5.8}$$

holds for $R \le P$ and $0 < T_1 \le T_0$, and

$$\sum_{r\sim R}[g,r]^{-3/2+\varepsilon}\sideset{}{^*}\sum_{\chi \bmod r}\int_{T_2}^{2T_2}\Big|F\Big(\frac{1}{2}+it,\chi\Big)\Big|dt \ll g^{-3/2+\varepsilon}(RQ)N^{-3/4}T_2L^c \tag{5.9}$$

holds for $R \le P$ and $T_0 < T_2 \le T$.

To get the estimate (5.8), we note that $[g,r](g,r) = gr$. Then the left hand side of (5.8) is

$$\begin{aligned}&\ll g^{-3/2+\varepsilon}\sum_{\substack{d|g\\ d\le R}}\Big(\frac{R}{d}\Big)^{-3/2+\varepsilon}\sum_{\substack{r\sim R\\ d|r}}\sideset{}{^*}\sum_{\chi \bmod r}\int_{T_1}^{2T_1}\Big|F\Big(\frac{1}{2}+it,\chi\Big)\Big|dt\\ &\ll g^{-3/2+\varepsilon}\sum_{\substack{d|g\\ d\le R}}\Big(\frac{R}{d}\Big)^{-1+\varepsilon}\sum_{\substack{r\sim R\\ d|r}}\sideset{}{^*}\sum_{\chi \bmod r}\int_{T_1}^{2T_1}\Big|F\Big(\frac{1}{2}+it,\chi\Big)\Big|dt.\end{aligned} \tag{5.10}$$

By Lemma 5.1, the above quantity can be estimated as

$$\begin{aligned}&\ll g^{-3/2+\varepsilon}\sum_{\substack{d|g\\ d\le R}}\Big(\frac{R}{d}\Big)^{-1+\varepsilon}\Big(\frac{R^2}{d}T_1+\frac{R}{d^{1/2}}T_1^{1/2}N^{3/20}+N^{1/4}\Big)L^c\\ &\ll g^{-3/2+\varepsilon}\tau(g)\{R^{1+\varepsilon}T_1+R^{1/2+\varepsilon}T_1^{1/2}N^{3/20}+N^{1/4}\}L^c\\ &\ll g^{-3/2+\varepsilon}N^{1/4}L^c,\end{aligned}$$

provided that $R \le P = U^{2+8\varepsilon}N^{-4}$. This establishes (5.8). Similarly we can prove (5.9) by taking $T = T_2$ in Lemma 5.1. Lemma 3.4 now follows.

§ 6. Estimation of $J(g)$ and $J(1)$

In this section, we prove Lemma 3.2 and Lemma 3.3.

Proof of Lemma 3.2. Recall that

$$W(\chi,\lambda) - \widehat{W}(\chi,\lambda) \ll N^{1/4}.$$

This contributes to (3.5) in amount

$$\begin{aligned}&\ll N^{1/4}\sum_{r\le P}[g,r]^{-3/2+\varepsilon}r \ll g^{-3/2+\varepsilon}N^{1/4}\sum_{r\le P}\Big(\frac{r}{(g,r)}\Big)^{-3/2+\varepsilon}r\\ &\ll g^{-3/2+\varepsilon}N^{1/4}\sum_{\substack{d|g\\ d\le P}}d^{1-\varepsilon}\sum_{\substack{r\le P\\ d|r}}r^{\varepsilon} \ll g^{-3/2+\varepsilon}N^{1/4}P^{1+\varepsilon} \ll g^{-3/2+\varepsilon}UL^c,\end{aligned}$$

where we have used $[g,r](g,r) = gr$ and (2.1). Thus Lemma 3.2 is a consequence of the estimate

$$\sum_{r\sim R}[g,r]^{-3/2+\varepsilon}\sideset{}{^*}\sum_{\chi \bmod r}\max_{|\lambda|\le 1/(rQ)}|\widehat{W}(\chi,\lambda)| \ll g^{-3/2+\varepsilon}UL^c, \tag{6.1}$$

where $R \le P$ and $c > 0$ is some constant.

It is easy to establish (6.1) for $r = 1$. In fact, for $r = 1$ the left hand side of (6.1) is

$$\ll g^{-3/2+\varepsilon} \sum_{N_1 < m \le N_2} \log m \ll g^{-3/2+\varepsilon} UL,$$

which is obviously acceptable. It therefore remains to show (6.1) in the case $r > 1$.

In this case we have $\delta_\chi = 0$ for all $\chi \bmod r$. Thus arguing similarly as in the section before, we find that

$$|\widehat{W}(\chi,\lambda)| \ll L^{10} \max_{\mathbf{M}} \Big| \int_{-T}^{T} F\Big(\frac{1}{2}+it,\chi\Big) \int_{N_1^2}^{N_2^2} v^{-3/4} e\Big(\frac{t}{4\pi}\log v + \lambda v\Big) dv dt \Big| + UN^{-\varepsilon} P^{-2},$$

where the maximum is taken over all $\mathbf{M} = (M_1, M_2, \cdots, M_{10})$ and we have taken

$$T = N^{1/2+2\varepsilon} U^{-1} P^2 (1 + |\lambda| N). \tag{6.2}$$

Since

$$\frac{d}{dv}\Big(\frac{t}{4\pi}\log v + \lambda v\Big) = \frac{t}{4\pi v} + \lambda, \qquad \frac{d^2}{dv^2}\Big(\frac{t}{4\pi}\log v + \lambda v\Big) = -\frac{t}{4\pi v^2},$$

by Lemmas 4.4 and 4.3 in [10], the inner integral above can be estimated as

$$\ll N^{-3/4} \min\Bigg\{ UN^{1/2}, \frac{N}{(|t|+1)^{1/2}}, \frac{N}{\min\limits_{N_1^2 < v \le N_2^2} |t + 4\pi\lambda v|} \Bigg\}. \tag{6.3}$$

Take

$$T_0 = NU^{-2}, \qquad \widehat{T_0} = 8\pi N/(RQ). \tag{6.4}$$

Here the choice of $\widehat{T_0}$ is to ensure that $|t + 4\pi\lambda v| > |t|/2$ whenever $|t| > \widehat{T_0}$. Thus, in order to prove Lemma 3.2, it is enough to show that for $R \le P$ and $0 < T_1 \le T_0$,

$$\sum_{r \sim R} [g,r]^{-3/2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_1}^{2T_1} \Big| F\Big(\frac{1}{2}+it,\chi\Big) \Big| dt \ll g^{-3/2+\varepsilon} N^{1/4} L^c, \tag{6.5}$$

for $R \le P$ and $T_0 < T_2 \le \widehat{T_0}$,

$$\sum_{r \sim R} [g,r]^{-3/2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_2}^{2T_2} \Big| F\Big(\frac{1}{2}+it,\chi\Big) \Big| dt \ll g^{-3/2+\varepsilon} UN^{-1/4} T_2^{1/2} L^c, \tag{6.6}$$

while for $R \le P$ and $\widehat{T_0} < T_3 \le T$,

$$\sum_{r \sim R} [g,r]^{-3/2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_3}^{2T_3} \Big| F\Big(\frac{1}{2}+it,\chi\Big) \Big| dt \ll g^{-3/2+\varepsilon} UN^{-1/4} T_3 L^c. \tag{6.7}$$

Following the same procedure that used to prove (5.8) and (5.9), we can establish these estimates by taking $T = T_1$, T_2, T_3 in Lemma 5.1 respectively. Thus Lemma 3.2 follows.

Now we complete the proof of Lemma 3.3.

Proof of Lemma 3.3. The proof of Lemma 3.3 is the same as that of Lemma 3.2 except for the saving L^{-A} on the right hand side. In order to save this factor, we have to distinguish two cases $L^B < R \le P$ and $R \le L^B$ where B is a constant depending on A. The proof of the first case is the same as that of Lemma 3.2. Here for a certain sufficiently large B, $L^B < R \le P$ guarantees that the term $g^{-3/2+\varepsilon}UL^c$ can be replaced with $g^{-3/2+\varepsilon}UL^{-A}$. So we omit the details.

Now we prove the second case $R \le L^B$. We use the explicit formula (see [11, p.313])

$$\sum_{m\le u}\Lambda(m)\chi(m) = \delta_\chi u - \sum_{|\gamma|\le T}\frac{u^\rho}{\rho} + O\Big\{\Big(\frac{u}{T}+1\Big)\log^2(ruT)\Big\}, \tag{6.8}$$

where $\rho = \beta + i\gamma$ is a non-trivial zero of the function $L(s,\chi)$, and $2 \le T \le u$ is a parameter. Taking $T = N^{26/125}$ in (6.8), and then inserting it into $\widehat{W}(\chi,\lambda)$, we get

$$\begin{aligned}\widehat{W}(\chi,\lambda) &= \int_{N_1}^{N_2} e(u^2\lambda)d\Big\{\sum_{n\le u}(\Lambda(m)\chi(m)-\delta_\chi)\Big\}\\ &= \int_{N_1}^{N_2} e(u^2\lambda)\sum_{|\gamma|\le N^{26/125}} u^{\rho-1}du + O\{N^{73/250}(1+|\lambda|N^{1/2}U)L^2\}\\ &\ll U\sum_{|\gamma|\le N^{26/125}} N^{(\beta-1)/2} + O(N^{99/125}UQ^{-1}L^2)\\ &\ll U\sum_{|\gamma|\le N^{26/125}} N^{(\beta-1)/2} + O(UN^{-\varepsilon}),\end{aligned}$$

where we have used (2.1).

Now let $\eta(T) = c_2\log^{-4/5}T$. By Staz VII.6.2 in [12], $\prod_{\chi \bmod r} L(s,\chi)$ is zero-free in the region $\sigma \ge 1-\eta(T), |t| \le T$ except for the possible Siegel zero. But by Siegel's theorem (see for example [13, §21]) the Siegel zero does not exist in the present situation, since $r \sim R \le L^B$. Thus by the large-sieve type zero-density estimates for Dirichlet L-functions (see for example [14]),

$$\begin{aligned}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\sum_{|\gamma|\le N^{26/125}} N^{(\beta-1)/2} &\ll L^c\int_0^{1-\eta(N^{26/125})}(N^{26/125})^{12(1-\alpha)/5}N^{(\alpha-1)/2}d\alpha\\ &\ll L^cN^{-0.0008\eta(N^{26/125})} \ll \exp(-c_3L^{1/5}).\end{aligned}$$

Consequently

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\max_{|\lambda|\le 1/(rQ)}|\widehat{W}(\chi,\lambda)| \ll UL^{-A},$$

where $A > 0$ is arbitrary. This proves Lemma 3.3 in the second case.

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References

[1] Hua, L. K., Some results in the additive prime number theory, *Quart. J. Math.,* **9**(1938), 68–80.

[2] Liu, J. Y. & Zhan, T., On sums of five almost equal prime squares, *Acta Arith.,* **77**(1996), 369–383.

[3] Bauer, C., A note on sums of five almost equal prime squares, *Arch. Math.,* **69**(1997), 20–30.

[4] Liu, J. Y. & Zhan, T., Sums of five almost equal prime squares, *Sci. in China, Series A,* **41**(1998), 710–722.

[5] Bauer, C., Liu, M. C. & Zhan, T., On sums of three prime squares, *J. Number Theory,* **85**(2000), 336–359.

[6] Liu, J. Y. & Liu, M. C., The exceptional set in four prime squares problem, *Illinois J. Math.,* **44**(2000), 272–293.

[7] Bauer, C., Sums of five almost equal prime squares, *Acta Math. Sin.,* to appear.

[8] Liu, J. Y., On lagrange's theorem with prime variables, *Quart. J. Math.,* **54**(2003), 453–462.

[9] Gallagher, P. X., A large sieve density estimate near $\sigma = 1$, *Invent. Math.,* **11**(1970), 329–339.

[10] Titchmarsh, E. C., The Theory of the Riemann Zeta-function, 2nd edition, University Press, Oxford, 1986.

[11] Pan, C. D. & Pan, C. B., Fundamentals of Analytic Number Theory (in Chinese), Science Press, Beijing, 1991.

[12] Prachar, K., Primzahlverteilung, Springer, Berlin, 1957.

[13] Davenport, H., Multiplicative Number Theory, 2nd edition, Springer, Berlin, 1980.

[14] Huxley, M. N., Large values of Dirichlet polynomials (III), *Acta Arith.,* **26**(1974/75), 435–444.