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ADJACENCY PRESERVING MAPS ON THE SPACE OF SELF-ADJOINT OPERATORS***

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Abstract

The authors extend Hua's fundamental theorem of the geometry of Hermitian matrices to the infinite-dimensional case. An application to characterizing the corresponding Jordan ring automorphism is also presented.

Keywords Self-adjoint operators, Adjacency, Non-linear maps
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§ 1 . Introduction and Statement of Results

The study of the geometry of matrices was initiated by Hua [11–18] in the middle of twenty century. In this geometry, the points of the space are a certain kind of matrices of a given size (rectangular matrices, symmetric matrices, Hermitian matrices, etc.). With each such space of matrices, we associate a group of motions. The main problem is to characterize this group by a geometric invariant called adjacency (see [23]). Recently, some authors have generalized a part of Hua's results (see, for example, [19–22]) and given other proofs of the Hua's fundamental theorems of the geometry of rectangular matrices (see [19]) and Hermitian matrices (see [21]). Motivated by [19], where the fundamental theorem of the geometry of rectangular matrices is extended to the infinite-dimensional case, we consider the question of extending the fundamental theorem of the geometry of Hermitian matrices over the complex fields to the infinite dimensional case in this paper.

The idea of the geometry of matrices has been recently generalized to the study of general preserver problems, that is, the problems of characterizing the maps between operator algebras which preserve certain properties (see, for example, [1–3]). If the maps are linear, the general preserver problems become the linear preserver problems which have attracted much attention in last decades (for some papers on this topic, see [4–9]). We remark here that the results in this paper were used in [10] to get the classification of the zero-product preserving additive maps on self-adjoint operator spaces.

Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{R}$ be the field of real numbers. Denote $\mathcal{M}_n(\mathbb{C})$ and $\mathcal{H}_n(\mathbb{C})$ the algebra of all $n \times n$ matrices over $\mathbb{C}$ and the real linear space of all

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Hermitian matrices in $\mathcal{M}_n(\mathbb{C})$, respectively. As usual, $\mathrm{GL}_n(\mathbb{C})$ denotes the general linear group of degree n over $\mathbb{C}$, that is, the group of all nonsingular matrices in $\mathcal{M}_n(\mathbb{C})$.

Let $X_1, X_2 \in \mathcal{H}_n(\mathbb{C})$. X_1 and X_2 are said to be of arithmetic distance r, denoted by $\rho(X_1, X_2) = r$, if $\mathrm{rank}(X_1 - X_2) = r$. In the case that $r = 1$, we say that X_1 and X_2 are adjacent. It is easy to verify that ρ satisfies the requirements for a distance function in a metric space. With the space $\mathcal{H}_n(\mathbb{C})$ we associate naturally a group of motions which consists of transformations of the form

$$X \mapsto PXP^* + H_0$$

for all $X \in \mathcal{H}_n(\mathbb{C})$, where $P \in \mathrm{GL}_n(\mathbb{C})$, $H_0 \in \mathcal{H}_n(\mathbb{C})$ and P^* denotes the conjugate transpose of P, i.e., $P^* = \overline{P}^t$, where $\overline{X} = [\bar{x}_{ij}]$ if $X = [x_{ij}] \in \mathcal{M}_n(\mathbb{C})$. Obviously the elements of the group of motions leave the arithmetic distance between any pair of points of $\mathcal{H}_n(\mathbb{C})$ invariant. Hua's remarkable fundamental theorem states that the adjacency alone is sufficient to characterize the motions to automorphisms of the underlying field. More precisely, we have

Theorem 1.1. *Let n be an integer ≥ 2, and Φ be a bijective map from $\mathcal{H}_n(\mathbb{C})$ onto itself. Assume that Φ preserves adjacency in both directions, that is, for every pair $X_1, X_2 \in \mathcal{H}_n(\mathbb{C})$, X_1 and X_2 are adjacent if and only if $\Phi(X_1)$ and $\Phi(X_2)$ are adjacent. Then there exist a non-zero real number c, a nonsingular matrix $P \in \mathrm{GL}_n(\mathbb{C})$ and a matrix $X_0 \in \mathcal{H}_n(\mathbb{C})$ such that either Φ has the form $\Phi(X) = cPXP^* + X_0$, for all $X \in \mathcal{H}_n(\mathbb{C})$, or has the the form $\Phi(X) = cP\overline{X}P^* + X_0$, for all $X \in \mathcal{H}_n(\mathbb{C})$.*

For the general result that $\mathbb{C}$ is replaced by a division ring with involution, we refer to Wan's book [23]. We also point out that the bijectivity assumption of the fundamental theorem of the geometry of Hermitian matrices can be relaxed to the surjectivity assumption. The proof is similar to that in the proof of Theorem 1.2 below.

The purpose of this paper is to extend the above fundamental theorem of the geometry of Hermitian matrices to the infinite dimensional case.

Let H be a Hilbert space over $\mathbb{C}$ and $\mathcal{B}(H)$ the algebra of all bounded linear operators acting on H. We denote by $\mathcal{S}_F^a(H) \subseteq \mathcal{B}(H)$ the real linear space of all finite rank self-adjoint operators and $\mathcal{S}^a(H) \subseteq \mathcal{B}(H)$ the real linear space of all self-adjoint operators. As in the finite dimension case, we say that two operators $X_1, X_2 \in \mathcal{S}^a(H)$ are adjacent if $X_1 - X_2$ is a rank-one operator.

Note that a rank-1 operator X is self-adjoint if and only if there exists a vector $x \in H$ and a nonzero real number c such that $X = cx \otimes x$. Moreover, every element in $\mathcal{S}_F^a(H)$ can be expressed as a sum of finite many rank-one elements in $\mathcal{S}_F^a(H)$. A map $A : H \to H$ is said to be conjugate-linear if A is additive and $A(tx) = \bar{t}Ax$ for all $x \in H$ and $t \in \mathbb{C}$. Note that, the dual operator A^* of a conjugate-linear operator A is defined by $\langle A^*x, y\rangle = \langle Ay, x\rangle$ for every $x, y \in H$.

The following are our main results.

Theorem 1.2. *Let $\Phi : \mathcal{S}_F^a(H) \to \mathcal{S}_F^a(H)$ be a surjective map. Then Φ preserves the adjacency in both directions if and only if there exist an $X_0 \in \mathcal{S}_F^a(H)$, a bijective linear or* *conjugate-linear operator A on H and a scalar $c \in \mathbb{R} \setminus \{0\}$ such that $X \mapsto \Phi(X) - X_0$ is a linear or conjugate-linear bijective map and*

$$\Phi(x \otimes x) = cAx \otimes Ax + X_0$$

for all $x \in H$.

Theorem 1.3. *Let $\Phi : \mathcal{S}_F^a(H) \to \mathcal{S}_F^a(H)$ be a surjective continuous map. Then Φ preserves adjacency in both directions if and only if there exists an operator $X_0 \in \mathcal{S}_F^a(H)$, a bounded bijective linear or conjugate linear operator A on H and a scalar $c \in \mathbb{R} \setminus \{0\}$ such that*

$$\Phi(X) = cAXA^* + X_0$$

for all $X \in \mathcal{S}_F^a(H)$.

Theorem 1.4. *Let $\Phi : \mathcal{S}^a(H) \to \mathcal{S}^a(H)$ be a surjective strongly continuous map. Then Φ preserves adjacency in both directions if and only if there exist a bounded bijective linear or conjugate-linear operator A on H, a scalar $c \in \mathbb{C} \setminus \{0\}$ and an operator $X_0 \in \mathcal{S}^a(H)$ such that*

$$\Phi(X) = cAXA^* + X_0$$

for all $X \in \mathcal{S}^a(H)$.

It is clear that $\mathcal{S}^a(H)$ is a Jordan ring with respect to the addition $(X, Y) \mapsto X + Y$, and the Jordan multiplication $(X, Y) \mapsto XY + YX$. Recall that a bijective map $\Phi : \mathcal{S}^a(H) \to \mathcal{S}^a(H)$ is called a Jordan ring automorphism if

$$\Phi(X + Y) = \Phi(X) + \Phi(Y)$$

and

$$\Phi(XY + YX) = \Phi(X)\Phi(Y) + \Phi(Y)\Phi(X)$$

for every $X, Y \in \mathcal{S}^a(H)$. As an application of Theorem 1.2, we get easily a maybe known characterization of Jordan ring automorphism of $\mathcal{S}^a(H)$.

Corollary 1.1. *A map $\Phi : \mathcal{S}^a(H) \to \mathcal{S}^a(H)$ is a Jordan ring automorphism if and only if there exists a unitary or conjugate unitary operator U on H such that*

$$\Phi(X) = UXU^*$$

for every $X \in \mathcal{S}^a(H)$.

§ 2. Proofs of the Results

In this section we give the proofs of Theorems 1.2–1.4 and Corollary 1.1.

Proof of Theorem 1.2. The "if" part is obvious. To check the "only if" part we assume that Φ is surjective and preserves adjacency in both directions. There is no loss of generality in assuming that $\Phi(0) = 0$. Otherwise we let $\Psi(X) = \Phi(X) - \Phi(0)$ and then consider Ψ. We proceed in steps.

Step 1. Φ is injective.

Assume that $\Phi(X_1) = \Phi(X_2)$, $X_1, X_2 \in \mathcal{S}_F^a(H)$ and denote $Y = X_2 - X_1$. Define a new map $\Psi : \mathcal{S}_F^a(H) \to \mathcal{S}_F^a(H)$ by $\Psi(X) = \Phi(X + X_1) - \Phi(X_1)$. Then Ψ maps both 0 and Y into 0 and preserves the adjacency in both directions. In particular, Φ maps rank-1 self-adjoint operators into rank-1 self-adjoint operators. If $Y \neq 0$, then Y can be expressed as $Y = \sum\limits_{i=1}^{n} \epsilon_i y_i \otimes y_i$ for some $y_i \in H$ and $\epsilon_i \in \{-1, 1\}$, $i = 1, 2, \cdots, n$. Choose $y \in H$ such that y is linearly independent of $y_1, \cdots, y_n$. Obviously, $\mathrm{rank}(y \otimes y - Y) \neq 1$. But $1 = \mathrm{rank}(\Psi(y \otimes y) - \Psi(Y)) = \mathrm{rank}(\Psi(y \otimes y))$, contradicting the fact that Ψ maps rank-1 self-adjoint operators into rank-1 self-adjoint operators.

Step 2. We assert that both Φ and Φ^{-1} preserve the arithmetic distance, that is,

$$\mathrm{rank}(X - Y) = \mathrm{rank}(\Phi(X) - \Phi(Y))$$

for any $X, Y \in \mathcal{S}_F^a(H)$.

If $\mathrm{rank}(X - Y) = r$, then there exist $t_i \in \mathbb{R} \setminus \{0\}$ and unit vectors $x_i \in H$ such that $X - Y = \sum\limits_{i=1}^{r} t_i x_i \otimes x_i$. Let $Y_0 = Y$, $Y_1 = Y + t_1 x_1 \otimes x_1$, $Y_2 = Y + t_1 x_1 \otimes x_1 + t_2 x_2 \otimes x_2, \cdots, Y_r = Y + \sum\limits_{i=1}^{r} t_i x_i \otimes x_i = X$. Since Φ preserves adjacency, we have $\mathrm{rank}(\Phi(X) - \Phi(Y)) \leq \sum\limits_{i=1}^{r} \mathrm{rank}(\Phi(Y_i) - \Phi(Y_{i-1})) = r$. The same argument applied to Φ^{-1} ensures that $\mathrm{rank}(\Phi(X) - \Phi(Y)) = \mathrm{rank}(X - Y)$.

By Step 2, for any non-zero vector $x \in H$, there exist nonzero $y \in H$ and $a \in \mathbb{R}\setminus\{0\}$, such that $\Phi(x \otimes x) = a y \otimes y$. Indeed, we have $\Phi(\mathbb{R} x \otimes x) = \mathbb{R} y \otimes y$, where $\mathbb{R} x \otimes x = \{\alpha x \otimes x \mid \alpha \in \mathbb{R}\}$.

Denote by $\mathrm{ran}\, T$ the range of a map T and $\mathrm{span}\, \mathcal{S}$ the linear space spanned by $\mathcal{S}$. Some times, we also denote the linear span of vectors $x_1, x_2, \cdots, x_n$ by $[x_1, x_2, \cdots, x_n]$.

Step 3. Suppose that $x_1, x_2, \cdots, x_n$ are linearly independent vectors in H and write $\Phi(x_i \otimes x_i) = a_i y_i \otimes y_i$, $y_i \in H$, $i = 1, 2, \cdots, n$. Then for any $X \in S_F^a(H)$ with $\mathrm{ran} X \subseteq [x_1, x_2, \cdots, x_n]$, we have $\mathrm{ran}(\Phi(X)) \subseteq [y_1, y_2, \cdots y_n]$.

When $n = 1$, the assertion is obvious. In the following we assume $n \geq 2$.

Claim 1. $y_1, y_2, \cdots, y_n$ are linearly independent.

Assume, on the contrary, that $y_1, y_2, \cdots, y_n$ are linearly dependent. Without loss of generality, we may assume that $y_1, y_2, \cdots, y_{n-1}$ are linearly independent and $y_n = y_1 + y_2 + \cdots + y_{n-1}$. Then, for any $j \in \{1, 2, \cdots, n-1\}$, $\mathrm{rank}\Big(\sum\limits_{i=1}^{n-1} y_i \otimes y_i - y_j \otimes y_j\Big) = n - 2$. So $\Phi^{-1}\Big(\sum\limits_{i=1}^{n-1} y_i \otimes y_i\Big) = \Phi^{-1}(y_j \otimes y_j) + R_j$, where $R_j \in \mathcal{S}_F^a(H)$ and $\mathrm{rank}(R_j) = n - 2$. Hence $\mathrm{ran}(\Phi^{-1}(y_j \otimes y_j)) \subseteq \mathrm{ran}\Big(\Phi^{-1}\Big(\sum\limits_{i=1}^{n-1} y_i \otimes y_i\Big)\Big)$, i.e.

$$\{x_1, x_2, \cdots, x_{n-1}\} \subseteq \mathrm{ran}\Big(\Phi^{-1}\Big(\sum_{i=1}^{n-1} y_i \otimes y_i\Big)\Big).$$

Since $\operatorname{rank}\Big(\Phi^{-1}\Big(\sum_{i=1}^{n-1} y_i \otimes y_i\Big)\Big) = n-1$, we have

$$\operatorname{ran}\Big(\Phi^{-1}\Big(\sum_{i=1}^{n-1} y_i \otimes y_i\Big)\Big) = [x_1, x_2, \cdots, x_{n-1}].$$

On the other hand, $\operatorname{rank}(\Phi^{-1}(y_n \otimes y_n)) = 1$, $\operatorname{rank}\Big(y_n \otimes y_n - \sum_{i=1}^{n-1} y_i \otimes y_i\Big) = n-1$, rank $\Big(\Phi^{-1}(y_n \otimes y_n) - \Phi^{-1}\Big(\sum_{i=1}^{n-1} y_i \otimes y_i\Big)\Big) = n-1$. So

$$\operatorname{ran}(\Phi^{-1}(y_n \otimes y_n)) \subseteq \operatorname{ran}\Big(\Phi^{-1}\Big(\sum_{i=1}^{n-1} y_i \otimes y_i\Big)\Big) \subseteq [x_1, x_2, \cdots, x_{n-1}],$$

arriving at a contradiction.

Claim 2. If $x' \in [x_1, x_2, \cdots x_n]$ and if $\Phi(x' \otimes x') = a'y' \otimes y'$, then $y' \in [y_1, y_2, \cdots, y_n]$.

Let $x' = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$, where $\alpha_i \in \mathbb{C}$. Without loss of generality, we assume that $\alpha_i = 1$, $i = 1, 2, \cdots, n$. For any $j \in \{1, 2, \cdots, n\}$, $\operatorname{rank}\Big(\sum_{i=1}^{n} x_i \otimes x_i - x_j \otimes x_j\Big) = n-1$. So $\Phi\Big(\sum_{i=1}^{n} x_i \otimes x_i\Big) = \Phi(x_j \otimes x_j) + S_j$, where $S_j \in \mathcal{S}_F^a(H)$ and $\operatorname{rank}(S_j) = n-1$. It follows that $\operatorname{ran}\Big(\Phi\Big(\sum_{i=1}^{n} x_i \otimes x_i\Big)\Big) = \operatorname{span}\{\operatorname{ran}(\Phi(x_j \otimes x_j)), j = 1, 2, \cdots, n\} = [y_1, y_2, \cdots, y_n]$. Since $\operatorname{rank}(\Phi(x' \otimes x')) = 1$ and $\operatorname{rank}\Big(x' \otimes x' - \sum_{i=1}^{n} x_i \otimes x_i\Big) = n$, we have

$$\operatorname{rank}\Big(\Phi(x' \otimes x') - \Phi\Big(\sum_{i=1}^{n} x_i \otimes x_i\Big)\Big) = n.$$

It is easily seen now that $\operatorname{ran}(\Phi(x' \otimes x')) \subseteq [y_1, y_2, \cdots y_n]$.

Claim 3. If $X \in \mathcal{S}_F^a(H)$ and $\operatorname{ran}(X) \subseteq [x_1, x_2, \cdots, x_n]$, then

$$\operatorname{ran}(\Phi(X)) \subseteq [y_1, y_2, \cdots, y_n].$$

Assume $\operatorname{rank}(X) = s$, $1 \le s \le n$; then there exist linearly independent vectors $x'_1, x'_2, \cdots, x'_s \in [x_1, x_2, \cdots, x_n]$ such that $X = \sum_{i=1}^{s} a'_i x'_i \otimes x'_i$, $0 \ne a_i \in \mathbb{R}$ $(i = 1, 2, \cdots, n)$. Let $\Phi(x'_i \otimes x'_i) = b'_i y'_i \otimes y'_i$, $i = 1, 2, \cdots, s$. By Claim 2, we see that $y'_i \in [y_1, y_2, \cdots, y_n]$. Then, by the proof of Claim 1, we have $\operatorname{ran}(\Phi(X)) \subseteq [y'_1, y'_2, \cdots, y'_s] \subseteq [y_1, y_2, \cdots, y_n]$, as desired.

Let Ω be the direct set consisting of all finite dimensional linear subspaces of H, ordered by inclusion.

Step 4. If $\lambda = [x_1, x_2, \cdots, x_n] \in \Omega$ and $\Phi(x_i \otimes x_i) = a_i y_i \otimes y_i$, $i = 1, 2, \cdots, n$, then there exist a nonzero $c_\lambda \in \mathbb{R}$ and a linear or conjugate-linear operator $A_\lambda : H \to H$ with $\operatorname{ran}(A_\lambda) = \mu = [y_1, y_2, \cdots, y_n]$ and $\ker(A_\lambda) = \lambda^\perp$ such that, for every $X \in S_F^a(H)$ with $\operatorname{ran}(X) \subseteq \lambda$, we have $\Phi(X) = c_\lambda A_\lambda X A_\lambda^*$.

We may assume that $\dim \lambda = n$. By Claim 1 in Step 3, $\mu = [y_1, y_2, \cdots, y_n]$ has dimension n, too. Let P_λ and Q_λ be the projections with ranges λ and μ, respectively. Let ϕ_1 :

$P_\lambda \mathcal{S}_F^a(H)P_\lambda \to H_n(\mathbb{C})$ be the real linear isomorphism determined by $\phi_1(x_j \otimes x_j) = e_j \otimes e_j$, $\phi_1(x_k \otimes x_j + x_j \otimes x_k) = e_k \otimes e_j + e_j \otimes e_k$ and $\phi_1(ix_k \otimes x_j - ix_j \otimes x_k) = ie_k \otimes e_j - ie_j \otimes e_k$, where i is the imaginary unit, e_j denotes the jth standard unit row vector in $\mathbb{C}^n$ and $e_k \otimes e_j$ denotes the $n \times n$ matrix with 1 in position (k, j) and 0 elsewhere. We define similarly $\phi_2 : Q_\lambda \mathcal{S}_F^a(H)Q_\lambda \to H_n(\mathbb{C})$. Let φ be the map from $H_n(\mathbb{C})$ onto $H_n(\mathbb{C})$ defined by $\varphi = \phi_2\phi\phi_1^{-1}$, which is a surjective map preserving adjacency in both directions, where $\phi = \Phi|_{P_\lambda \mathcal{S}_F^a(H)P_\lambda}$. Applying the fundamental theorem of the geometry of Hermitian matrices due to Hua (i.e., Theorem 1.1 in Section 1), it is easily seen that there exist some $c \in \mathbb{R}\setminus\{0\}$ and nonsingular $n \times n$ matrix A such that $\varphi(T) = c_\lambda AT^f A^\star$ for every $T \in H_n^1(\mathbb{C})$, where f is the identity or the conjugation of $\mathbb{C}$. Thus $\phi(X) = \phi_2^{-1}\varphi\phi_1(X) = \phi_2^{-1}(c_\lambda A\phi_1(X)^f A^\star)$ for every $X \in P_\lambda \mathcal{S}_F^a(H)P_\lambda$. We first consider the case that f is the identity of $\mathbb{C}$, that is,

$$\phi(X) = \phi_2^{-1}(c_\lambda A\phi_1(X)A^\star).$$

Write A in

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Note that by Step 3 we have $\phi(x_j \otimes x_j) = \alpha_j y_j \otimes y_j$ $(j = 1, \cdots, n)$. A computation of $\phi(x_j \otimes x_j) = \phi_2^{-1}(c_\lambda A\phi_1(x_j \otimes x_j)A^\star)$ shows that $A = \mathrm{diag}\{a_{11}, a_{22}, \cdots, a_{nn}\}$ and $\phi(x_j \otimes x_j) = c_\lambda |a_{jj}|^2 y_j \otimes y_j = \alpha_j y_j \otimes y_j$. Define a linear operator A_λ on H by $A_\lambda x_j = \beta_j y_j$ $(j = 1, 2, \cdots, n)$, $A_\lambda x = 0$ if $x \in \lambda^\perp$, where $\beta_j \in \mathbb{C}$ satisfies $|\beta_i|^2 = |a_{ii}|^2$. Then A_λ satisfies that $\Phi(X) = c_\lambda A_\lambda X A_\lambda^\star$ for every $X \in \mathcal{S}_F^a(H)$ with $\mathrm{ran}(X) \subseteq \lambda$. For the case that $\phi(X) = \phi_2^{-1}(c_\lambda A\overline{\phi_1(X)}A^\star)$ for every $X \in P_\lambda \mathcal{S}_F^a(H)P_\lambda$, ϕ is conjugate linear. We define a map $J_n : \mathbb{C}^n \to \mathbb{C}^n$ by $J_n(\begin{matrix} \xi_1 & \xi_2 & \cdots & \xi_n \end{matrix})^t = (\begin{matrix} \bar{\xi}_1 & \bar{\xi}_2 & \cdots & \bar{\xi}_n \end{matrix})^t$. Then $\overline{T} = JTJ$ and hence $\phi(X) = \phi_2^{-1}(c_\lambda B\phi_1(X)B^\star)$ with $B = AJ$ a conjugate linear map from $\mathbb{C}^n$ onto itself. Similarly, $\phi(x_j \otimes x_j) = \alpha_j y_j \otimes y_j$ $(j = 1, \cdots, n)$ implies that $Be_j = a_{jj}e_j$ and $\phi(x_j \otimes x_j) = c_\lambda |a_{jj}|^2 y_j \otimes y_j = \alpha_j y_j \otimes y_j$. In the same way as linear case, we find a conjugate linear operator $A_\lambda : H \to H$ with $\mathrm{ran}(A_\lambda) = \mu$ and $(\ker(A_\lambda))^\perp = \lambda$ such that $\Phi(X) = c_\lambda A_\lambda X A_\lambda^\star$ for every $X \in \mathcal{S}_F^a(H)$ with $\mathrm{ran}(X) \subseteq \lambda$.

As a consequence of Step 4, Φ is additive.

Step 5. If there exists $\lambda_0 \in \Omega$ with $\dim \lambda_0 \geq 2$ such that A_{λ_0} is linear (resp., conjugate-linear), then for every $\lambda \in \Omega$, A_λ is linear (resp., conjugate-linear).

Otherwise, assume that A_{λ_0} is linear but there exists $\lambda_1 \in \Omega$ so that A_{λ_1} is conjugate-linear. Take $\lambda' \in \Omega$ such that $\lambda_1 \cup \lambda_0 \subset \lambda'$. For any $x \in \lambda_0 \subset \lambda'$, $\Phi(x \otimes x) = c_{\lambda_0} A_{\lambda_0} x \otimes A_{\lambda_0} x = c_{\lambda'} A_{\lambda'} x \otimes A_{\lambda'} x$. Hence $A_{\lambda_0} x, A_{\lambda'} x$ are linear dependent. It follows that $A_{\lambda'}|_{\lambda_0} = \alpha_0 A_{\lambda_0}$ for some $\alpha_0 \in \mathbb{C} \setminus \{0\}$. With the same reason, we have $A_{\lambda'}|_{\lambda_1} = \alpha_1 A_{\lambda_1}$ for some $\alpha_1 \in \mathbb{C} \setminus \{0\}$. Thus we get a contradiction since, by Step 4, $A_{\lambda'}$ is linear or conjugate-linear.

Step 6. There exist a nonzero real number c and a bijective linear or conjugate-linear operator $A : H \to H$ such that

$$\Phi(x \otimes x) = cAx \otimes Ax.$$

Without loss of generality, we assume that, for any $\lambda \in \Omega$, A_λ is linear. Fix $\lambda_0 \in \Omega$ with $\dim \lambda_0 \geq 2$. For any $\lambda \in \Omega$, if $\lambda \subseteq \lambda_0$, as in Step 4, there exists a nonzero $a \in \mathbb{C}$ such that $A_\lambda = aA_{\lambda_0}|_\lambda$. Absorbing a scalar vector, we can assume that $A_\lambda = A_{\lambda_0}|_\lambda$ and consequently, $c_\lambda = c_{\lambda_0}$

With the same reason, if $\lambda_0 \subset \lambda$ or if $\lambda \cap \lambda_0 \neq \{0\}$, we can assume $A_\lambda|_{\lambda_0} = A_{\lambda_0}$ or $A_\lambda|_{\lambda\cap\lambda_0} = A_{\lambda_0}|_{\lambda\cap\lambda_0}$ and $c_\lambda = c_{\lambda_0}$.

If $\lambda \cap \lambda_0 = \{0\}$, let $\lambda + \lambda_0 = \mu \in \Lambda$. Then we have $A_\mu|_{\lambda_0} = \alpha_0 A_{\lambda_0}$ for some $\alpha_0 \in \mathbb{C} \setminus \{0\}$. Let $A'_\lambda = \alpha_0^{-1} A_\mu|_\lambda$. For every $x \in \lambda_0 \subseteq \mu$, we have

$$\begin{aligned}\Phi(x \otimes x) &= c_{\lambda_0} A_{\lambda_0} x \otimes A_{\lambda_0} x = c_\mu A_\mu x \otimes A_\mu x \\ &= c_\mu \alpha_0 A_{\lambda_0} x \otimes \alpha_0 A_{\lambda_0} x = c_\mu |\alpha_0|^2 A_{\lambda_0} x \otimes A_{\lambda_0} x.\end{aligned}$$

And for every $x \in \lambda \subseteq \mu$ we have

$$\begin{aligned}\Phi(x \otimes x) &= c_\lambda A_\lambda x \otimes A_\lambda x = c_\mu A_\mu x \otimes A_\mu x = c_\mu \alpha_0 A'_\lambda x \otimes \alpha_0 A'_\lambda x \\ &= c_\mu |\alpha_0|^2 A'_\lambda x \otimes A'_\lambda x = c_0 A'_\lambda x \otimes A'_\lambda x.\end{aligned}$$

So, we can assume $c_\lambda = c_0$ and $A_\lambda = A'_\lambda$.

In conclusion, for any $\lambda_1, \lambda_2 \in \Omega$, we have $A_{\lambda_1}|_{\lambda_1\cap\lambda_2} = A_{\lambda_2}|_{\lambda_1\cap\lambda_2}$. So we can find a linear operator $A : H \to H$ such that $Ax = A_\lambda x$, if $x \in \lambda$ for some $\lambda \in \Omega$, and there is a $c \in \mathbb{R} \setminus \{0\}$ such that $\Phi(x \otimes x) = cAx \otimes Ax$, for all $x \in H$.

Proof of Theorem 1.3. The "if" part is obvious. For the "only if" part, we use Theorem 1.2 to see that Φ has the form stated in Theorem 1.2. Since Φ is continuous, the linear or conjugate linear operator A must be bounded and A^* exists. Now the desired conclusion follows.

Proof of Theorem 1.4. We only need to check the "only if" part. Assume that Φ is surjective and preserves adjacency in both directions. Without loss of generality we may require that $\Phi(0) = 0$.

For any $X, Y \in \mathcal{S}^a(H)$, it is easy to see that $\operatorname{rank}(X - Y) = r$ implies the existence of operators $X = X_0, X_1, \cdots, X_r = Y$ in $\mathcal{S}^a(H)$ such that X_{i-1} and X_i are adjacent, $i = 1, 2, \cdots, r$. By the triangle inequality, we get $\operatorname{rank}(\Phi(X) - \Phi(Y)) \leq r = \operatorname{rank}(X - Y)$. Considering Φ^{-1} instead of Φ, we arrive at $\operatorname{rank}(\Phi(X) - \Phi(Y)) = \operatorname{rank}(X - Y)$. In particular, we have $\Phi(\mathcal{S}^a_F(H)) = \mathcal{S}^a_F(H)$. Since, by the closed graph theorem, the strong continuity of Φ implies the (norm) continuity, it follows from Theorem 1.3 that there exist a bounded invertible linear or conjugate-linear operator A and a scalar $c \in \mathbb{R} \setminus 0$ such that $\Phi(X) = cAXA^*$ for all $X \in \mathcal{S}^a_F(H)$. For any $X \in \mathcal{S}^a(H)$, there exist a net $\{\alpha_\gamma \mid \gamma \in \Gamma\}$ of complex numbers and a net $\{P_\gamma \mid \gamma \in \Gamma\}$ of projections such that $\left\{ \sum_{\gamma\in\Gamma'} \alpha_\gamma P_\gamma \;\middle|\; \Gamma' \text{ is finite subset in } \Gamma \right\}$ converges to X uniformly. And for every finite subset Γ' in Γ, $\sum_{\lambda\in\Gamma'} \alpha_\lambda P_\lambda$ is a strong limit of some monotone increasing net of operators in $\mathcal{S}^a_F(H)$. Now the strong continuity of Φ is used to complete the proof.

Now let us give a proof of Corollary 1.1 by use of Theorems 1.2–1.4.

Proof of Corollary 1.1. Assume that Φ is a Jordan ring isomorphism of $\mathcal{S}^a(H)$. We have to show that there exists a unitary or conjugate unitary operator U on H such that $\Phi(X) = UXU^*$, for every $X \in \mathcal{S}^a(H)$. It is trivial to check that

$$\Phi(0) = \Phi(0), \tag{2.1}$$

$$\Phi(X^2) = \Phi(X)^2, \tag{2.2}$$

$$\Phi(I) = I, \tag{2.3}$$

$$\Phi(XYX) = \Phi(X)\Phi(Y)\Phi(X) \tag{2.4}$$

hold for every $X, Y \in \mathcal{S}^a(H)$. We divide the rest of proof into several steps.

Step 1. If $X, Y \in \mathcal{S}^a(H)$ satisfy $X^2 = X \neq 0$, $Y^2 = Y \neq 0$ and $XY = YX = 0$, then

$$\Phi(X)\Phi(Y) = \Phi(Y)\Phi(X) = 0.$$

By (2.1), $0 = \Phi(0) = \Phi(XY + YX) = \Phi(X)\Phi(Y) + \Phi(Y)\Phi(X)$. Thus $\Phi(X)\Phi(Y) = -\Phi(Y)\Phi(X)$. It follows from (2.2) and (2.4) that

$$0 = \Phi(XYX) = -\Phi(X)^2\Phi(Y) = -\Phi(X)\Phi(Y).$$

Therefore

$$\Phi(X)\Phi(Y) = \Phi(Y)\Phi(X) = 0.$$

Step 2. Φ maps rank-one projections into rank-one projections.

By (2.2), Φ maps projections into projections.

Assume that there exists a rank-one projection $P \in \mathcal{S}^a(H)$ such that $\mathrm{rank}(\Phi(P)) > 1$. Write $Q = \Phi(P)$. Then there exist projections $Q_1, Q_2 \in \mathcal{S}^a(H)$ with $Q_1Q_2 = Q_2Q_1 = 0$ such that $Q = Q_1 + Q_2$. Thus we have $P = \Phi^{-1}(Q) = \Phi^{-1}(Q_1) + \Phi^{-1}(Q_2)$. By Step 1, $\Phi^{-1}(Q_1)\Phi^{-1}(Q_2) = \Phi^{-1}(Q_2)\Phi^{-1}(Q_1) = 0$. This implies that $\mathrm{rank}(P) = \mathrm{rank}(\Phi^{-1}(Q_1) + \Phi^{-1}(Q_2)) > 1$, a contradiction.

Step 3. Φ maps rank-1 self-adjoint operators into rank-1 self-adjoint operators.

For any rank-1 self-adjoint operator $X = tx \otimes x \in \mathcal{S}^a(H)$ ($t \in \mathbb{R} \setminus \{0\}$ and $x \in H \setminus \{0\}$), $b^{-1}X$ is a rank-one projection, where $b = t\langle x, x\rangle \neq 0$. By Step 2, $\mathrm{rank}(\Phi(b^{-1}X)) = 1$. As $X = (b^{-1}X)X(b^{-1}X)$, by (2.4), we get $\Phi(X) = \Phi(b^{-1}X)\Phi(X)\Phi(b^{-1}X)$. Therefore $\Phi(X)$ is of rank-one.

Step 4. There exists a real number $c \neq 0$ and a bijective linear or conjugate-linear operator $B : H \to H$ such that $\Phi(x \otimes x) = cBx \otimes Bx$ holds for all $x \in H$.

As in the proof of Theorem 1.4, it is easily checked that $\Phi(\mathcal{S}^a_F(H)) = \mathcal{S}^a_F(H)$. So the assertion is obtained by Step 3 and Theorem 1.2. Particularly, Φ is linear or conjugate-linear when restricted to $\mathcal{S}^a_F(H)$.

Let us first consider the case that Φ is linear, i.e., B is linear.

Step 5. $\langle Bx, By\rangle = c^{-1}\langle x, y\rangle$ for all $x, y \in H$.

Firstly, we show that $\langle Bx, Bx\rangle = c^{-1}\langle x, x\rangle$ for all $x \in H$. This is obviously true if $x = 0$. Assume that $x \neq 0$, then $\Phi(\|x\|^{-2}x \otimes x) = \|x\|^{-2}\Phi(x \otimes x) = \|x\|^{-2}cBx \otimes Bx$.

Since $\|x\|^{-2}x\otimes x$ is a rank-one projection, $(\|x\|^2)^{-1}cBx\otimes Bx$ is a rank-one projection, too. Therefore $\langle x,x\rangle^{-1}\langle cBx,Bx\rangle=1$, i.e., $\langle Bx,Bx\rangle=c^{-1}\langle x,x\rangle$. As a result, we have $c>0$.

Now the general equation $\langle Bx,By\rangle=c^{-1}\langle x,y\rangle$ follows from the polarization identity.

Let $U=\sqrt{c^{-1}}B$. Then U is a unitary operator.

Step 6. $\Phi(X)=UXU^*$ holds for all $X\in\mathcal{S}_F^a(H)$.

By the additivity of Φ, we only need to check the case that X is of rank-one . Let $X=x\otimes x$. For any $y\in H$,

$$\begin{aligned}\Phi(x\otimes x)y&=(Ux\otimes Ux)y=\langle y,Ux\rangle Ux=\langle U^*y,x\rangle Ux\\&=U(\langle U^*y,x\rangle)x=U(x\otimes x)U^*y.\end{aligned}$$

So $\Phi(x\otimes x)=U(x\otimes x)U^*$.

Step 7. $\Phi(X)=AXA^*$ holds for all $X\in\mathcal{S}^a(H)$.

For any $X\in\mathcal{S}^a(H)$ and $y\in H$, we have

$$\begin{aligned}\Phi(X(y\otimes y)X)&=\Phi(X)\Phi(y\otimes y)\Phi(X)\\&=\Phi(X)U(y\otimes y)U^*\Phi(X)\\&=\Phi(X)Uy\otimes\Phi(X)Uy.\end{aligned}$$

On the other hand

$$\Phi(X(y\otimes y)X)=\Phi(Xy\otimes Xy)=UXy\otimes UXy.$$

So there exists a number λ_X with $|\lambda_X|=1$ such that

$$\Phi(X)U=\lambda_X UX.$$

Thus

$$\Phi(X)=\lambda_X UXU^*.$$

By Step 6 and (2.3), we see that $\lambda_X=1$ and hence

$$\Phi(X)=UXU^*$$

for every $X\in\mathcal{S}^a(H)$.

The proof is similar when Φ is conjugate-linear.

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