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BOUNDS ON COINCIDENCE INDICES ON NON-ORIENTABLE SURFACES**

D. VENDRÚSCOLO*

Abstract

This paper presents some results about bounds for coincidence indices of Nielsen coincidence classes for maps between nonorientable surfaces. Denoting by K_n the nonorientable surface constructed by a connected sum of n torus with a Klein bottle, the author proves: (i) for pairs of maps between two Klein bottles or for pairs of maps from a Klein bottle to a surface K_n the coincidence class index is bounded. (ii) for pairs of maps from K_n to the Klein bottle the coincidence class index is unbounded. Other boundedness results are given for more technical conditions, including one for self maps.

Keywords Nielsen theory, Coincidence theory, Coincidence index, Surfaces
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§ 1. Introduction

Questions about bounds for indices first appeared in the fixed point context. The first results appeared in studies of surface homeomorphisms (see [13, 18, 19]). In [12, 14] and [15] some results about bounds for Nielsen fixed point class indices for self-maps of surfaces are given.

In the coincidence context, bounds for coincidence class indices for pairs of maps in the torus may be found in [1] and [2] and, more recently, in [8] we find many results about bounds for coincidence class indices for orientable surfaces. In particular the authors studied maps $f_1, f_2 : S_h \to S_g$ with $h \geq g$ and where S_i is the compact orientable surface of genus i.

Using the notation of [8], we define

$$B(g, h, d_1, d_2) = \sup\{|\mathrm{ind}(C)| \mid |\deg(f_1)| = d_1,\ |\deg(f_2)| = d_2\},$$

where the supremum is taken over all coincidence classes C of all pairs of maps $(f_1, f_2) : S_h \to S_g$ with the given degrees.

We recall the following results

Theorem 1.1. (cf. [8, Proposition 9]) $B(1, h, d_1, d_2) = \infty$ *for* $h > 1$.

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*Departamento de Matemática, Universidade Federal de São Carlos. Rod. Washington Luiz, Km 235 - C.P. 676 CEP:13565-905 São Carlos, SP, Brazil. **E-mail:** daniel@dm.ufscar.br
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Theorem 1.2. (cf. [8, Theorem 10]) *When* $2 \leq g \leq h$ *and* $0 \leq d_1, d_2 < \dfrac{h-1}{g-1}$, *hence neither of* f_1, f_2 *is homotopic to a covering map, then*

$$B(g, h, d_1, d_2) = \infty.$$

In general, the case

$$d_2 = \frac{h-1}{g-1}$$

remains open, despite the fact that under certain conditions on the map f_2 there is a bound for these indices (see [8, p.75 and Proposition 11]).

To obtain similar results for maps between nonorientable surfaces we need a definition for the index of a Nielsen coincidence classes in this context.

In [4] we find a definition of a semi-index (a non-negative integer) of a coincidence classe for maps between smooth manifolds, and in [11] a similar construction is described for topological manifolds. These definitions are very "geometric".

In [6] a coincidence index for maps from a complex into a manifold was obtained (without hypotheses on orientation) in an algebraic way, using obstruction theory.

The equivalence between the two definitions is shown in [7] under the condition that one of the maps is orientation true. Then a Lefschetz number is defined with the usual properties.

Definition 1.1. *Let* M, N *be two manifolds. A map* $f : M \to N$ *is orientation true when each* $\alpha \in \pi_1(M)$ *preserves a local orientation of* M *if and only if*

$$f(\alpha) \in \pi_1(N)$$

preserves a local orientation of N.

For a map between orientable manifolds of the same dimension, the notion of degree is well known, it is an integer, a homological invariant denoted by $\deg(f)$. In the nonorientable case we have only a homological definition of degree with coefficients in $\mathbb{Z}_2$. Here, for maps between closed (compact and without boundary) surfaces, we use the definition of the absolute degree of a map, as presented in [20, p.416] and, by abuse of notation, we call this the degree of f, and we also denote this by $\deg(f)$. This definition is equivalent to the one given in [5] (see also [3, 9]).

In this paper we prove some results about the bounds for the coincidence class index on some nonorientable surfaces, under the condition that one of the maps is orientation true.

We denote the Klein bottle by K and the nonorientable surface constructed by a connected sum of n torus with a Klein bottle by K_n ($K_0 = K$). In the same way we denote the projective plane by P and the nonorientable surface constructed by connected sum of n tori with a projective plane by P_n.

Some results are stated only for the surfaces K_n. We fix a presentation of $\pi_1(K_n)$ (see Fig. 1):

$$\pi_1(K_n) = \left\langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, \alpha, \beta \;\middle|\; \prod_{i=1}^{n} [a_i, b_i] \cdot \alpha \cdot \beta \cdot \alpha \cdot \beta^{-1} \right\rangle,$$

where $[a_i, b_i] = a_i \cdot b_i \cdot a_i^{-1} \cdot b_i^{-1}$.

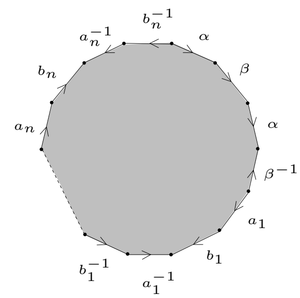

Fig. 1. K_n described as a polygon with identified edges

In this presentation, β is a loop that reverses a local orientation of K_n and $\pi_1(K) = \langle \alpha, \beta \mid \alpha \cdot \beta \cdot \alpha \cdot \beta^{-1} \rangle$.

In the same way as in the orientable case (see [8]), we define

$$B_K(g, h, d_1, d_2) = \sup\{|\mathrm{ind}(f_1, f_2, C)| \mid \deg(f_1) = d_1,\ \deg(f_2) = d_2\},$$

where the supremum is taken over all coincidences classes C of all pairs of maps $(f_1, f_2) : K_h \to K_g$ with given degrees and with f_2 orientation true.

In the next section we prove that for pairs of maps between two Klein bottles or for pairs of maps from the Klein bottle to a surface K_g, the coincidence index is bounded. In Section 3 we present examples of pairs of maps from K_h to K showing that coincidence index is not bounded in this case. In Section 4 we prove some boundedness results including one for self maps.

In [22] we can find examples showing that coincidence index is not bounded for pairs of maps from K_h to K_g such that either both maps are degree zero or both maps have nonzero degree and both maps are not homotopic to a covering map. We can also find similar results for other nonorientable surfaces. In particular there exist examples of pairs of maps with zero or nonzero degree that have coincidence classes with nonbounded index for maps from a surface of type K_n to surfaces of type P_n. We believe that similar results can be proved in other similar situations (for example for maps between two surfaces of type P_n or maps from an orientable to a nonorientable surface).

We note that for maps from a nonorientable to an orientable surface we do not have orientation true maps, and so it is not possible to apply the techniques used here. The same occurs for maps of nonzero degree from a surface of type P_h to one of type K_g (see [9, Theorem 2.5(c)]).

§ 2. Maps from the Klein Bottle

2.1. Self maps of the Klein bottle

This situation was carefully studied in [4] where we may find the calculation of the Nielsen number for such maps. That paper uses the notion of semi-index mentioned above. For the purposes of this paper we state

Theorem 2.1. $B_K(0,0,d_1,d_2) \leq 1, \forall\, d_1, d_2 \geq 0$.

Proof. In the proof of [4, Lemma 3.7], it was shown that for a pair $f_1, f_2 : K \to K$ either $N(f_1, f_2) = 0$ (and all coincidence classes have semi-index zero) or the pair (f_1, f_2) satisfies the hypotheses of [4, Theorem 2.5].

In the proof of [4, Theorem 2.5] it was shown that the semi-index of a coincidence class C of the pair (f_1, f_2) is the same as the semi-index of a coincidence class of some lift pair $(\widetilde{f}_1, \widetilde{f}_2)$ of self maps of torus.

Using the facts that the semi-index is equal to the index used here (see [7]) and that the index of coincidence classes of self maps of torus is bounded (see [1, p.125]), we have $|\mathrm{ind}(f_1, f_2, C)| \leq 1$.

2.2. Maps from Klein bottle to K_g

We take the Klein bottle as in Fig. 2 and the circle S^1 as the interval $[0,1]$ with the relation $0 \sim 1$.

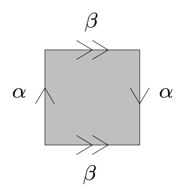

Fig. 2. The Klein bottle as the square $[0,1] \times [0,1]$ with identified edges

Let $p : K \to S^1$ be the map defined by $p(x,y) = x$.

Lemma 2.1. *Let $f : K \to K_g$ be a map. If $g \geq 1$ then there exists $\bar{f} : S^1 \to K_g$ such that f is homotopic to a map $\bar{f} \circ p$.*

Proof. We note that $f_\#(\pi_1(K)) = \langle f_\#(\alpha), f_\#(\beta) \rangle \subset \pi_1(K_g)$. Moreover,

$$f_\#(\alpha) \cdot f_\#(\beta) \cdot f_\#(\alpha) \cdot f_\#(\beta)^{-1} = 1.$$

Now using Freiheitssatz (cf. [17, Theorem 4.10]), we have that $f_\#(\pi_1(K))$ is free with only one generator, so there exists $w \in \pi_1(K_g)$ such that $f_\#(\alpha) = w^r$ and $f_\#(\beta) = w^s$, $r, s \in \mathbb{Z}$ and we have, using the above relation,

$$w^r \cdot w^s \cdot w^r \cdot w^{-s} = 1,$$

so $r = 0$ and $f_\#(\alpha) = 1$, the trivial element.

Let $\overline{w}$ be a curve on K_g such that $\langle \overline{w} \rangle = w \in \pi_1(K_g)$ and $\bar{f} : S^1 \to K_g$ be a map such that if e is the generator of $\pi_1(S^1)$ then $\bar{f}_\#(e) = w^s$. We have $f_\# = (\bar{f} \circ p)_\#$. Since K is an Eilenberg-MacLane space of type $K(\pi, 1)$, f is homotopic to $\bar{f} \circ p$.

We note that we can take $\bar{f}$ as a local homeomorphism over its image, and in this case the inverse image of any point of K_g by $\bar{f} \circ p$ is the union of a finite number of circles or it is empty.

Theorem 2.2. $B_K(g,0,0,0) = 0$ *for all* $g \geq 1$.

Proof. Let $f_1, f_2 : K \to K_g$ be a pair of maps. Then by Lemma 2.1, $f_1 \sim \bar{f}_1 \circ p$ and $f_2 \sim \bar{f}_2 \circ p$ so that if we choose curves $\overline{w}_1, \overline{w}_2$ on K_g such that $\bar{f}_1 \circ p(K) \subset \overline{w}_1$, $\bar{f}_2 \circ p(K) \subset \overline{w}_2$ and $\overline{w}_1 \cap \overline{w}_2 = \{p_1, p_2, \cdots, p_n\}$ is finite, then under these conditions,

$$\mathrm{Coin}(\bar{f}_1 \circ p, \bar{f}_2 \circ p) = \bigcup_{i=1}^{n} ((\bar{f}_1 \circ p)^{-1}(p_i) \cap (\bar{f}_2 \circ p)^{-1}(p_i)).$$

In fact, if $\mathrm{Coin}(\bar{f}_1 \circ p, \bar{f}_2 \circ p) \neq \emptyset$, then using a small translation of $\bar{f}_1 \circ p$ in the direction of β, we obtain $\mathrm{Coin}(\bar{f}_1 \circ p, \bar{f}_2 \circ p) = \emptyset$. This shows that if C is a coincidence class of the pair $(\bar{f}_1 \circ p, \bar{f}_2 \circ p)$ then $\mathrm{ind}(\bar{f}_1 \circ p, \bar{f}_2 \circ p, C) = 0$; we know that $f_1 \sim \bar{f}_1 \circ p$ and $f_2 \sim \bar{f}_2 \circ p$, so the same occurs for the pair (f_1, f_2).

Remark 2.1. In fact this proof shows that all classes are "geometrically" inessential, in the sense that they can disappear under homotopy. Note that we do not use the fact that f_2 is orientation true, i.e., this result is still true for any pair of maps.

§ 3. Maps from K_h to the Klein Bottle

In this section we use some formulae presented in [10] to calculate the Lefschetz number of pairs of maps. We give some examples to show that certain indices are not bounded.

Theorem 3.1. *$B_K(0, h, d_1, d_2) = \infty$ with $h > 1$, $\forall\, d_1, d_2 \geq 0$.*

Proof. We define homomorphisms $\bar{f}_1, \bar{f}_2 : \pi_1(K_h) \to \pi_1(K)$ such that

$$\begin{array}{ccc}
 & \bar{f}_1 & \\
a_1 & \to & 1 \\
b_1 & \to & \alpha \\
a_2 & \to & 1 \\
b_2 & \to & \beta^2 \\
a_3 & \to & 1 \\
b_3 & \to & 1 \\
\vdots & \vdots & \vdots \\
a_h & \to & 1 \\
b_h & \to & 1 \\
\alpha & \to & \alpha^{d_1} \\
\beta & \to & \beta^{-1}
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & \bar{f}_2 & \\
a_1 & \to & 1 \\
b_1 & \to & 1 \\
a_2 & \to & \alpha^n \\
b_2 & \to & 1 \\
a_3 & \to & 1 \\
b_3 & \to & 1 \\
\vdots & \vdots & \vdots \\
a_h & \to & 1 \\
b_h & \to & 1 \\
\alpha & \to & \alpha^{d_2} \\
\beta & \to & \beta
\end{array}$$

Since K_h is an Eilenberg-MacLane space of type $K(\pi, 1)$, there exists $f_i : K_h \to K$ such that $f_{i\#} = \bar{f}_i$.

Both f_1 and f_2 are orientation true and they lift to $\tilde{f}_i : S_{2h+1} \to S_1$ between the double orientable coverings.

We may check that $\tilde{f}_i$ has degree d_i and then, using [9, Definition 1.3], we have $\deg(f_i) = d_i$.

Using the notation of [10], we have $f_2^*(\tilde{\alpha}) = n\tilde{a}_2 + d_2\tilde{\alpha}$, $f_2^*(\beta) = \beta$ and $f_1^*(\beta) = 2b_2 - \beta$. By [10, Theorem 2.5], we have

$$L(f_1, f_2) = 2 \cdot (d_1 - n).$$

We know that if $w \in \pi_1(K)$ then $w = \alpha^p \cdot \beta^q$, and we have

$$w = \alpha^p \cdot \beta^q \sim f_1(a_2^{-p}) \cdot \alpha^p \cdot \beta^q \cdot (f_2(a_2^{-p}))^{-1} = \alpha^{-p} \cdot \alpha^p \cdot \beta^q = \beta^q.$$

Since $q = 2q'$ or $q = 2q' + 1$ and

$$\beta^q \sim f_1(\beta^{-q'}) \cdot \beta^q \cdot (f_2(\beta^{-q'}))^{-1} = \beta^{-q'} \cdot \beta^q \cdot \beta^{-q'},$$

it follows that if $q = 2q'$ then $\beta^q \sim 1$ and if $q = 2q' + 1$ then we have $\beta^q \sim \beta$.

There exist only two Reidemeister classes and the sum of their indices is $|2 \cdot (d_1 - n)|$ (see [7, Theorem 4.5, Definition 5.1 and Theorem 5.5]). Thus $B_K(0, h, d_1, d_2) = \infty$.

§ 4. Some Boundedness Results

4.1. Covering maps

The result that we will show in this subsection is an adaptation, including the non-orientable case, of [8, Proposition 11] where one of the maps is a covering map. Under the same assumptions of [8] we will obtain a boundary for the index of a coincidence class of surfaces, without the orientable condition. We recall that a covering map is orientation true.

Theorem 4.1. *Let S_A and S_B be two compact surfaces, $f : S_A \to S_B$ a map and $p : S_A \to S_B$ a covering map. If there exists a subgroup $K \subset \pi_1(S_A)$ such that*

- $[\pi_1(S_A) : K] = n < \infty$,
- $f_\pi(K) \subset p_\pi(K)$,

then for all coincidence Nielsen classes C of the pair (f, p), $|\mathrm{ind}(f, p, C)| \leq |2n\chi(S_A) - 1|$.

Proof. If $\mathrm{Coin}(f, p) = \emptyset$, then we have nothing to do. Suppose then that $\mathrm{Coin}(f, p) \neq \emptyset$. Let $x_0 \in \mathrm{Coin}(f, p) \subset S_A$, and let C be the coincidence class such that $x_0 \in C$.

Since $[\pi_1(S_A) : K] = n < \infty$, there exist a compact surface S_K and a covering map $q : (S_K, x_0') \to (S_A, x_0)$ such that $\chi(S_K) = n\chi(S_A)$ and $q_\#(\pi_1(S_K, x_0')) = K$.

If $\bar{x}_0 = f(x_0) = p(x_0)$, then there exists a map $f' : S_K \to S_K$ which makes the following diagram commute:

$$\begin{tikzcd}
(S_K, x_0') \arrow[r, "f'"] \arrow[dd, "q"'] & (S_K, x_0') \arrow[d, "q"] \\
 & (S_A, x_0) \arrow[d, "p"] \\
(S_A, x_0) \arrow[r, "f"] & (S_B, \bar{x}_0)
\end{tikzcd}$$

We will denote by C' the coincidence class of the pair (f', id) such that $x_0' \in C'$.

Observe that $\mathrm{Coin}(f', \mathrm{id})$ is included naturally in $\mathrm{Coin}(p \circ q \circ f', p \circ q)$ and since $p \circ q$ is a covering map, by [8, Lemma 1(iv)], C' is a coincidence class of pair $(p \circ q \circ f', p \circ q)$. Using the commutativity of the above diagram, we have that C' is a coincidence class of pair $(f \circ q, p \circ q)$ and $|\mathrm{ind}(f', \mathrm{id}, C)| = |\mathrm{ind}(f \circ q, p \circ q, C')|$.

Since q is a covering map by [8, Lemma 2(iii)], C' covers C, n' times with $1 \leq n' \leq n$. Using [4, Lemma 2.1, Corollary 2.2 and Lemma 2.3] and [7, Lemma 5.3] we obtain $|\mathrm{ind}(f \circ q, p \circ q, C')| = n'|\mathrm{ind}(f, p, C)|$, in the non-orientable case also. So we have $|\mathrm{ind}(f, p, C)| \leq$

$|\mathrm{ind}(f \circ q, p \circ q, C')| = |\mathrm{ind}(f', \mathrm{id}, C')|$. It is useful to remember the equivalence between the semi-index defined in [4] and the absolute value of the index used here (see [7]).

Since C' is a coincidence class of the pair (f', id), C' is a fixed point class of $f' : S_K \to S_K$, and using [12, Theorem 2] we obtain $|\mathrm{ind}(f', \mathrm{id}, C')| \leq |2\chi(S_K) - 1|$ and then $|\mathrm{ind}(f, p, C)| \leq |2n\chi(S_A) - 1|$.

4.2. Self maps

By analogy with $B_K(g, h, d_1, d_2)$ we define

$$B_P(g, h, d_1, d_2) = \sup\{|\mathrm{ind}(f_1, f_2, C)| \mid \deg(f_1) = d_1,\ \deg(f_2) = d_2\},$$

where the supremum is taken over all coincidence classes C of all pairs of maps $(f_1, f_2) : P_h \to P_g$ with given degrees and with f_2 orientation true.

Theorem 4.2. *$B_K(g, g, 0, 1) = 2g - 1$ and $B_P(g, g, 0, 1) = 2g$ for $g \geq 1$.*

Proof. Let M_n be the non-orientable surface that admits the surface S_n (of genus n) as a double orientable covering. Given a pair $f_1, f_2 : M_n \to M_n$, such that f_2 has degree 1, we know that f_2 is homotopic to a homeomorphism f_2' and the inclusion $\mathrm{Coin}(f_1, f_2) \to \mathrm{Fix}((f_2')^{-1} \circ f_1)$ preserves the Nielsen classes indices. Further $|\deg(f_1)| = |\deg((f_2')^{-1} \circ f_1)|$. We can take the pair (f_1, f_2) to be $((f_2')^{-1} \circ f_1, \mathrm{id})$. Now we can use the definition and the properties of the Nielsen fixed point class index. For simplicity, we will denote $(f_2')^{-1} \circ f_1$ by f.

First we will show that given $C \subset \mathrm{Fix}(f)$, a Nielsen class, then $|\mathrm{ind}(C)| \leq n$. For this we observe that if $\deg(f) = 0$ then f can be factored through the 1-skeleton of M_n. By [16, Proposition 6.6] we have that $f_\#(\pi_1(M_n))$ is a free subgroup with rank at most $\frac{n+1}{2}$, so we can describe f as a map between two wedges of circles, $\bigvee S^1$, of at most $\frac{n+1}{2}$ copies of S^1. Using the results of [12] and the commutativity property of the fixed point class we see that $|\mathrm{ind}(C)| \leq n$.

Now we will construct an example satisfying the conditions of the theorem with $|\mathrm{ind}(C)| = n$. Given n odd, we consider

$$f : \bigvee_i^{\frac{n+1}{2}} S^1 \to \bigvee_i^{\frac{n+1}{2}} S^1,$$

the map that sends each loop of S^1 over itself of degree 2, and has only one fixed point in the intersection of all the copies of S^1. We can take

$$\bigvee_i^{\frac{n+1}{2}} S^1 \subset M_n,$$

and we have that f can be extended to a map $\tilde{f} : M_n \to M_n$ of degree zero which has only one fixed point class C. By the Lefschetz theorem, we obtain $|\mathrm{ind}(C)| = n$.

Now we observe

$$M_n = \begin{cases} K_{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \\ P_{\frac{n}{2}}, & \text{if } n \text{ is even,} \end{cases}$$

which completes the proof.

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