Chin. Ann. Math.
26B:3(2005),323–334.

THE EXPONENTIAL STABILIZATION FOR A SEMILINEAR WAVE EQUATION WITH LOCALLY DISTRIBUTED FEEDBACK***

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Abstract

This paper considers the exponential decay of the solution to a damped semilinear wave equation with variable coefficients in the principal part by Riemannian multiplier method. A differential geometric condition that ensures the exponential decay is obtained.

Keywords Wave equations, Exponential decay, Distributed damping, Variable coefficients, Riemannian manifold
2000 MR Subject Classification 35A, 35L05, 35Q, 58J45, 93C20

§ 1 . Introduction

In this paper, we will consider the following semilinear damped wave equation with variable coefficients in the principal part:

$$\begin{cases} u_{tt} - \sum\limits_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\Big(a_{ij}(x)\dfrac{\partial u}{\partial x_j}\Big) + f(u) + a(x)u_t = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(0) = u^0, \quad u_t(0) = u^1 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $a_{ij}(x) = a_{ji}(x)$ are C^∞ functions in $\mathbb{R}^n$ satisfying

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2, \qquad \forall\, x \in \Omega,\ \forall\, \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n \tag{1.2}$$

for some positive constant α. Ω is assumed to be a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega = \Gamma$. $f \in C^1(\mathbb{R})$ is such a function that

$$f(s)s \geq 0, \qquad \forall\, s \in \mathbb{R}, \tag{1.3}$$

Manuscript received May 8, 2004. Revised November 23, 2004.
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***Project supported by the National Natural Science Foundation of China (No.60334040, No.60225003).

and satisfies the following growth condition

$$|f(s_1) - f(s_2)| \le C^*(1 + |s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2|, \qquad \forall\, s_1, s_2 \in \mathbb{R} \tag{1.4}$$

for some constant $C^* > 0$ and $p > 1$ with $(n-2)p \le n$. In addition, f is assumed to be superlinear in this paper, i.e.,

$$\exists \delta > 0: \quad f(s)s \ge (2+\delta)\int_0^s f(z)dz, \qquad \forall\, s \in \mathbb{R}. \tag{1.5}$$

Let ω be a neighbourhood of the whole boundary Γ (here and in what follows by a neighbourhood of the boundary or of a portion $\Gamma_0 \subset \Gamma$ we mean the intersection of Ω with some neighbourhood of those sets in $\mathbb{R}^n$). $a(x) \in L^\infty(\Omega)$ is a nonnegative bounded function such that

$$a(x) \ge a_0 > 0, \qquad \text{a.e. in } \ \omega \tag{1.6}$$

for some constant $a_0 > 0$. The condition (1.6) implies that the damping is not effective in the whole Ω, but only in the subset $\omega \subset \Omega$.

Under the above conditions, the system (1.1) is well posed in the space $H_0^1(\Omega) \times L^2(\Omega)$, i.e., for any initial data $\{u^0, u^1\} \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a unique weak solution u of (1.1) such that

$$u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$$

(see [1]). Then we define the energy of u at instant t by

$$E(t) = \frac{1}{2}\int_\Omega \Big(|u_t|^2 + \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)dx + \int_\Omega \Phi(u(x,t))dx, \tag{1.7}$$

where

$$\Phi(s) = \int_0^s f(z)dz, \qquad \forall\, s \in \mathbb{R}, \tag{1.8}$$

and it is easy to check that $\Phi(s)$ is non-negative for any $s \in \mathbb{R}$. Multiplying $(1.1)_1$ (the first equation of the system (1.1)) by u_t and integrating over $\Omega \times (s, s+T)$ with $s \ge 0, T > 0$, we have

$$E(s+T) - E(s) = -\int_s^{s+T}\!\!\int_\Omega a(x)|u_t|^2 dxdt \le 0, \tag{1.9}$$

which signifies that the energy $E(t)$ is a non-increasing function of the time variant t. In fact, it is just our purpose to show that $E(t)$ decays exponentially to 0 as $t \to \infty$ under certain differential geometric conditions which will be specified later on, i.e., to prove for every energy finite solution u of (1.1),

$$E(t) \le Ce^{-\lambda t}E(0), \qquad \forall\, t \ge 0 \tag{1.10}$$

with some positive constants C and λ under certain conditions.

On the exponential decay of the energy for semilinear damped wave equations, there have been a plenty of literatures. Here we cite [1] among others. In [1], Zuazua considered the following semilinear damped wave equation with constant coefficients in the principal part:

$$\begin{cases} u_{tt} - \Delta u + f(u) + a(x)u_t = 0 & \text{in } \ \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(0) = u^0, \quad u_t(0) = u^1 & \text{in } \ \Omega. \end{cases} \tag{1.11}$$

The author in [1] investigated:

(1) $f \in C^1(\mathbb{R})$ satisfies (1.3) and (1.4), and is globally Lipschitz, i.e., $f' \in L^\infty(\mathbb{R})$.

(2) $f \in C^1(\mathbb{R})$ satisfies the conditions (1.3)–(1.5).

For the second case, by multiplier methods, the author proved that the energy of (1.11) decays exponentially, i.e.,

$$E_1(t) \triangleq \frac{1}{2} \int_\Omega (|u_t|^2 + |\nabla u|^2) dx + \int_\Omega \Phi(u(x,t)) dx \le Ce^{-\lambda t} E_1(0)$$

for some positive constants C and λ. And in §3.2 of [1], Zuazua thought that for the system (1.1) with variable coefficients in the principal part, the exponential decay would hold under a restrictive condition:

$$\sum_{i,j=1}^n (2a_{ij} - (x - x_0) \cdot \nabla a_{ij}) \, \xi_i \xi_j \ge 2\beta |\xi|^2, \qquad \forall \, x \in \Omega, \ \forall \, \xi \in \mathbb{R}^n \tag{1.12}$$

for some $x_0 \in \mathbb{R}^n$ and $\beta > 0$, but the general case remained open.

In this paper, motivated by the Riemann multiplier method developed by Yao [2], we discuss the exponential decay for the system (1.1) by Riemannian geometry method. A general differential geometric condition which is sufficient for the exponential decay for the system (1.1) is obtained.

The paper is organized as follows: In Section 2, we first introduce some notations and geometry identities we work on, and then state the main result. Section 3 is devoted to the proof of the main result.

§ 2. Preliminaries and Main Result

We first introduce some notations in Riemann geometry and multiplier identities developed in [2] (see also [5, 6]) which are needed in the proof of our main result.

Let $\mathbb{R}^n$ have the usual topology and $x = (x_1, x_2, \cdots, x_n)$ be the natural coordinate system. For each $x \in \mathbb{R}^n$, we denote by $A(x)$ an $n \times n$ matrix and $G(x)$ its inverse, i.e.,

$$A(x) = (a_{ij}(x))_{n \times n} \ \text{ and } \ G(x) = (g_{ij}(x))_{n \times n} \triangleq A(x)^{-1}.$$

We define the inner product $\langle \cdot, \cdot \rangle_g$ and norm $| \cdot |_g$ over the tangent space $\mathbb{R}^n_x$ by

$$\langle X, Y \rangle_g = g(X, Y) \triangleq \sum_{i,j=1}^n g_{ij}(x) \alpha_i \beta_j, \tag{2.1}$$

$$|X|_g = \langle X, X \rangle_g^{\frac{1}{2}}, \qquad \forall \, X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \ Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x. \tag{2.2}$$

It is easy to check that $(\mathbb{R}^n, g)$ is a Riemann manifold with Riemann metric g defined by (2.1). Let D be the Levi-Civita connection in metric g. For a vector field H on $(\mathbb{R}^n, g)$, the covariant differential DH of H determines a bilinear form on $\mathbb{R}^n_x \times \mathbb{R}^n_x$ for each $x \in \mathbb{R}^n$ by

$$DH(X, Y) = \langle D_X H, Y\rangle_g, \qquad \forall\, X, Y \in \mathbb{R}^n_x, \tag{2.3}$$

where $D_X H$ is the covariant derivative of vector field H with respect to X.

At the same time, we denote the usual Euclidean inner product and norm by $\langle\cdot, \cdot\rangle_0$ and $|\cdot|_0$ respectively, i.e.,

$$\langle X, Y\rangle_0 = X \cdot Y = \sum_{i=1}^n \alpha_i\beta_i, \tag{2.4}$$

$$|X|_0 = \Big(\sum_{i=1}^n \alpha_i^2\Big)^{\frac{1}{2}}, \qquad \forall\, X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i},\ Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x. \tag{2.5}$$

For a vector field $X = \sum\limits_{i=1}^n \alpha_i(x)\frac{\partial}{\partial x_i}$ on $\mathbb{R}^n$, we denote the divergence of X in the Euclidean metric by $\mathrm{div}_0 X$,

$$\mathrm{div}_0 X = \sum_{i=1}^n \frac{\partial \alpha_i(x)}{\partial x_i}.$$

It is clear that

$$\mathrm{div}_0(\varphi X) = \varphi \mathrm{div}_0 X + X(\varphi) \qquad \text{for any} \ \ \varphi \in C^1(\mathbb{R}^n), \tag{2.6}$$

$$\int_\Omega \mathrm{div}_0 X dx = \int_{\partial\Omega} X \cdot \nu d\sigma, \tag{2.7}$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ is the unit normal vector of $\partial\Omega$ pointing towards the exterior of Ω, and $d\sigma$ is the Euclidean surface element on $\partial\Omega$.

In the following Lemma 2.1, we introduce some elementary identities. For the proof, we refer the readers to [2] and omit it here.

Lemma 2.1. *Let $x = (x_1, x_2, \cdots, x_n)$ be the natural coordinate system in $\mathbb{R}^n$, $\varphi, \psi \in C^1(\overline{\Omega})$ and H, X vector fields. We denote by ∇_g and ∇_0 the gradient operators in the Riemannian metric and in the Euclidean metric respectively. Then for all $x \in \overline{\Omega}$,*

$$\langle H(x), A(x)X(x)\rangle_g = H(x) \cdot X(x), \tag{2.8}$$

$$\nabla_g \varphi(x) = \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij}(x)\frac{\partial \varphi}{\partial x_j}\Big)\frac{\partial}{\partial x_i}, \tag{2.9}$$

$$\langle \nabla_g \varphi, \nabla_g \psi\rangle_g = \nabla_g \varphi(\psi) = \nabla_0 \varphi \cdot A(x)\nabla_0 \psi, \tag{2.10}$$

$$|\nabla_g \varphi|_g^2 = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial \varphi}{\partial x_i}\frac{\partial \varphi}{\partial x_j}, \tag{2.11}$$

$$\begin{aligned}\langle \nabla_g \varphi, \nabla_g (H(\varphi))\rangle_g(x) = {}& DH(\nabla_g \varphi, \nabla_g \varphi)(x) + \frac{1}{2}\mathrm{div}_0(|\nabla_g \varphi|_g^2 H)(x) \\ & - \frac{1}{2}|\nabla_g \varphi|_g^2(x)\mathrm{div}_0 H(x).\end{aligned} \tag{2.12}$$

For the sake of convenience, we set the differential operator

$$\mathcal{A}u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big).$$

Now we list some multiplier identities needed later.

Lemma 2.2. *Let u solve the equation*

$$u_{tt} + \mathcal{A}u + f(u) + a(x)u_t = 0 \tag{2.13}$$

on $\Omega \times (s, s+T)$ with $s \geq 0, T > 0$, and let H be a vector field on $\overline{\Omega}$. Then we have

$$\int_s^{s+T}\!\!\int_\Omega u_{tt}H(u)dxdt = \int_\Omega u_t H(u)dx\Big|_s^{s+T} + \frac{1}{2}\int_s^{s+T}\!\!\int_\Omega u_t^2 \mathrm{div}_0 H dxdt - \frac{1}{2}\int_s^{s+T}\!\!\int_{\partial\Omega} u_t^2 H\cdot\nu d\sigma dt \tag{2.14}$$

and

$$\int_s^{s+T}\!\!\int_\Omega \mathcal{A}(u)H(u)dxdt = \int_s^{s+T}\!\!\int_\Omega DH(\nabla_g u, \nabla_g u)dxdt + \frac{1}{2}\int_s^{s+T}\!\!\int_{\partial\Omega} |\nabla_g u|_g^2 H\cdot\nu d\sigma dt - \frac{1}{2}\int_s^{s+T}\!\!\int_\Omega |\nabla_g u|_g^2 \mathrm{div}_0 H dxdt - \int_s^{s+T}\!\!\int_{\partial\Omega} \frac{\partial u}{\partial\nu_{\mathcal{A}}}H(u)d\sigma dt, \tag{2.15}$$

where

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\nu_i$$

is the conormal derivative. Moreover, if $\psi \in C^2(\overline{\Omega})$ then

$$\int_s^{s+T}\!\!\int_\Omega \psi\big(u_t^2 - \big|\nabla_g u\big|_g^2\big)dxdt = \int_\Omega \psi u u_t dx\Big|_s^{s+T} + \int_s^{s+T}\!\!\int_\Omega \psi u(f(u) + a(x)u_t)dxdt + \frac{1}{2}\int_s^{s+T}\!\!\int_\Omega u^2\mathcal{A}\psi dxdt - \int_s^{s+T}\!\!\int_{\partial\Omega}\frac{\partial u}{\partial\nu_{\mathcal{A}}}\psi u d\sigma dt + \frac{1}{2}\int_s^{s+T}\!\!\int_{\partial\Omega} u^2\nabla_g\psi\cdot\nu d\sigma dt. \tag{2.16}$$

Proof. (2.14) and (2.15) can be obtained directly from Lemma 2.1 by integrating by parts. We now prove (2.16). It is obvious that

$$\mathcal{A}\psi = -\mathrm{div}_0(\nabla_g\psi)$$

by (2.9) of Lemma 2.1, and we have

$$\langle\nabla_g u, \nabla_g(\psi u)\rangle_g = \psi|\nabla_g u|_g^2 + u\langle\nabla_g u, \nabla_g\psi\rangle_g = \psi|\nabla_g u|_g^2 + \frac{1}{2}\nabla_g\psi(u^2) = \psi|\nabla_g u|_g^2 + \frac{1}{2}\mathrm{div}_0(u^2\nabla_g\psi) + \frac{1}{2}u^2\mathcal{A}\psi.$$

Then by (2.7) and Green's formula we deduce that

$$
\begin{aligned}
-\int_s^{s+T}\!\!\int_\Omega \psi|\nabla_g u|_g^2 dxdt &= \frac{1}{2}\int_s^{s+T}\!\!\int_\Omega \operatorname{div}_0(u^2\nabla_g\psi)dxdt + \frac{1}{2}\int_s^{s+T}\!\!\int_\Omega u^2\mathcal{A}\psi dxdt \\
&\quad - \int_s^{s+T}\!\!\int_\Omega \langle\nabla_g u, \nabla_g(\psi u)\rangle_g dxdt \\
&= \frac{1}{2}\int_s^{s+T}\!\!\int_{\partial\Omega} u^2\nabla_g\psi\cdot\nu dxdt + \frac{1}{2}\int_s^{s+T}\!\!\int_\Omega u^2\mathcal{A}\psi dxdt \\
&\quad - \int_s^{s+T}\!\!\int_\Omega \psi u\mathcal{A}u dxdt - \int_s^{s+T}\!\!\int_{\partial\Omega} \psi u\frac{\partial u}{\partial\nu_\mathcal{A}} d\sigma dt.
\end{aligned}
$$

Moreover, integrating by parts yields

$$
\int_s^{s+T}\!\!\int_\Omega \psi u_t^2 dxdt = \int_\Omega \psi u u_t dx\Big|_s^{s+T} - \int_s^{s+T}\!\!\int_\Omega \psi u u_{tt} dxdt.
$$

Combining the above two identities, and noting (2.13) we then complete the proof.

Based on the above preparations, we now state our main result as follows.

Theorem 2.1. *Let ω be a neighbourhood of the whole boundary Γ and let f satisfy (1.3)–(1.5). Let $H = (H^1, H^2, \cdots, H^n)$ be a vector field on the Riemannian manifold $(\mathbb{R}^n, g)$ such that*

$$
\inf\{\ \operatorname{div}_0 H\ ;\ x\in\overline{\Omega}\ \} > 0, \tag{2.17}
$$

$$
DH(X,X) \ge \gamma|X|_g^2, \qquad \forall\, X\in\mathbb{R}_x^n,\ \forall\, x\in\Omega \tag{2.18}
$$

for some constant $\gamma > 0$, and set

$$
\Gamma_0 = \{x\in\partial\Omega \mid H\cdot\nu > 0\}, \qquad \Gamma_1 = \{x\in\partial\Omega \mid H\cdot\nu \le 0\}.
$$

Then for any given initial data $\{u^0, u^1\} \in H_0^1(\Omega)\times L^2(\Omega)$, the energy of the system (1.1) with variable coefficients decays exponentially to 0 as $t\to\infty$, i.e., there exist two constants $C > 0$ and $\lambda > 0$, such that

$$
E(t) = \frac{1}{2}\int_\Omega (u_t^2 + |\nabla_g u|_g^2)dx + \int_\Omega \Phi(u)dx \le Ce^{-\lambda t}E(0), \qquad \forall\, t\ge 0. \tag{2.19}
$$

Remark 2.1. The existence of the vector field H on the Riemannian manifold $(\mathbb{R}^n, g)$ satisfying the geometric condition (2.18) can be guaranteed in some cases by curvature conditions on Riemannian manifold (see [2] where some examples are given).

Remark 2.2. The differential geometric condition (2.18) is more general than the condition (1.12) given by Zuazua [1]. In [5], the authors have shown that (2.18) is equivalent to the following condition

$$
\exists\eta > 0, \quad \sum_{i,j=1}^n \Big(\sum_{k=1}^n a_{ik}\frac{\partial H^j}{\partial x_k} + \sum_{k=1}^n a_{jk}\frac{\partial H^i}{\partial x_k} - \nabla_0 a_{ij}\cdot H\Big)\xi_i\xi_j \ge \eta|\xi|^2,
$$

$$
\forall\, x\in\Omega,\ \forall\,\xi = (\xi_1, \xi_2, \cdots, \xi_n)\in\mathbb{R}^n.
$$

Thus if we take $H = (x_1 - x_1^0, x_2 - x_2^0, \cdots, x_n - x_n^0)$ for some $x^0 \in \mathbb{R}^n$, then (1.12) can be deduced from (2.18).

§ 3 . Proof of Theorem 2.1

In the present section, we adopt Riemannian multiplier method introduced by Yao in [2] to prove Theorem 2.1. Inspired by the work by J. Rauch and M. Taylor [4] we aim at to establish the energy estimate of type

$$E(s+T) \leq C \int_s^{s+T}\int_\Omega a(x)|u_t(x,t)|^2 dxdt.$$

We first introduce the well-known Sobolev-Poincaré inequality which will be needed later (see for example [9]).

Lemma 3.1. (Sobolev-Poincaré Inequality) *Let q be a number with $2 \leq q < +\infty$ ($n = 1, 2$) or $2 \leq q \leq \dfrac{2n}{n-2}$ ($n \geq 3$). Then there is a constant $c_* = c(\Omega, q)$ such that*

$$\|u\|_q \leq c_* \|\nabla u\|_2 \qquad \textit{for} \ \ u \in H_0^1(\Omega).$$

In addition, the following inequality is obvious,

$$\exists C > 0, \quad \int_\Omega u^2 dx \leq C \int_\Omega |\nabla_g u|_g^2 dx, \qquad \forall\, u \in H_0^1(\Omega).$$

In fact, it is just the consequence of Poincaré inequality in $H_0^1(\Omega)$ in the case of (1.2).

In the following, we will denote by C constant independent of s and T which may be varying at different places. Otherwise we will employ subscripts.

Proof of Theorem 2.1. Multiplying $(1.1)_1$ by $H(u)$ and integrating over $\Omega \times (s, s+T)$ with arbitrary $s \geq 0$ and $T > 0$, by (2.14) and (2.15) of Lemma 2.2, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_s^{s+T}\int_\Omega \operatorname{div}_0 H(u_t^2 - |\nabla_g u|_g^2)dxdt + \int_s^{s+T}\int_\Omega DH(\nabla_g u, \nabla_g u)dxdt \\
&+ \int_s^{s+T}\int_\Omega a(x)u_t H(u)dxdt + \int_s^{s+T}\int_\Omega f(u)H(u)dxdt + \int_\Omega u_t H(u)dx)\Big|_s^{s+T} \\
=& \int_s^{s+T}\int_{\partial\Omega} \frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u)d\sigma dt - \frac{1}{2}\int_s^{s+T}\int_{\partial\Omega} |\nabla_g u|^2 H\cdot\nu d\sigma dt.
\end{aligned} \tag{3.1}$$

Referring to the proof of Lemma 2.3 in [2], we know that on the boundary $\partial\Omega$,

$$H(u) = \frac{1}{|\nu_{\mathcal{A}}|_g^2}\frac{\partial u}{\partial \nu_{\mathcal{A}}} H\cdot\nu, \quad |\nabla_g u|_g^2 = \frac{1}{|\nu_{\mathcal{A}}|_g^2}\Big|\frac{\partial u}{\partial \nu_{\mathcal{A}}}\Big|^2, \qquad \forall\, x \in \partial\Omega.$$

Then (3.1) gives

$$\begin{aligned}
&\frac{1}{2}\int_s^{s+T}\int_\Omega \operatorname{div}_0 H(u_t^2 - |\nabla_g u|_g^2)dxdt + \int_s^{s+T}\int_\Omega DH(\nabla_g u, \nabla_g u)dxdt \\
&+ \int_s^{s+T}\int_\Omega a(x)u_t H(u)dxdt - \int_s^{s+T}\int_\Omega \Phi(u)\operatorname{div}_0 H dxdt + \int_\Omega u_t H(u)dx\Big|_s^{s+T}
\end{aligned}$$

$$= \frac{1}{2}\int_s^{s+T}\int_{\partial\Omega}\frac{1}{|\nu_{\mathcal{A}}|_g^2}\Big|\frac{\partial u}{\partial\nu_{\mathcal{A}}}\Big|^2 H\cdot\nu d\sigma dt. \tag{3.2}$$

Now we multiply $(1.1)_1$ by ζu with some $\zeta\in W^{1,\infty}(\Omega)$ and integrate by parts to achieve

$$\int_\Omega \zeta u\Big(u_t+\frac{a}{2}u\Big)dx\Big|_s^{s+T} - \int_s^{s+T}\int_\Omega \zeta(u_t^2-|\nabla_g u|_g^2)dxdt$$
$$+\int_s^{s+T}\int_\Omega u\langle\nabla_g u,\nabla_g\zeta\rangle_g dxdt + \int_s^{s+T}\int_\Omega \zeta u f(u)dxdt = 0. \tag{3.3}$$

When we take $\psi=\mathrm{div}_0 H$ in (2.16) of Lemma 2.2 and take into account that $u=0$ on $\partial\Omega\times(0,\infty)$, the following identity holds:

$$\int_s^{s+T}\int_\Omega \mathrm{div}_0 H(u_t^2-|\nabla_g u|_g^2)dxdt$$
$$=\int_\Omega u_t u\,\mathrm{div}_0 H dx\Big|_s^{s+T}+\frac{1}{2}\int_s^{s+T}\int_\Omega u^2\mathcal{A}(\mathrm{div}_0 H)dxdt$$
$$+\int_s^{s+T}\int_\Omega (f(u)u+a(x)u_t u)\mathrm{div}_0 H dxdt. \tag{3.4}$$

It follows from (3.2)–(3.4) with $\zeta\equiv 1$ that

$$\tau\int_s^{s+T}\int_\Omega (u_t^2-|\nabla_g u|_g^2)dxdt+\int_s^{s+T}\int_\Omega DH(\nabla_g u,\nabla_g u)dxdt$$
$$=\frac{1}{2}\int_s^{s+T}\int_{\partial\Omega}\frac{1}{|\nu_{\mathcal{A}}|_g^2}\Big|\frac{\partial u}{\partial\nu_{\mathcal{A}}}\Big|^2 H\cdot\nu d\sigma dt-\int_s^{s+T}\int_\Omega a(x)u_t H(u)dxdt$$
$$-\frac{1}{2}\int_s^{s+T}\int_\Omega a(x)u_t u\,\mathrm{div}_0 H dxdt-\frac{1}{4}\int_s^{s+T}\int_\Omega u^2\mathcal{A}(\mathrm{div}_0 H)dxdt$$
$$+\int_s^{s+T}\int_\Omega\Big(\Big(\tau-\frac{1}{2}\mathrm{div}_0 H\Big)uf(u)+\Phi(u)\mathrm{div}_0 H\Big)dxdt$$
$$-\int_\Omega\Big(u_t H(u)+\frac{1}{2}u_t u\,\mathrm{div}_0 H-\tau\Big(u_t u+\frac{a}{2}u^2\Big)\Big)dx\Big|_s^{s+T} \tag{3.5}$$

for any $\tau\in\mathbb{R}$. Now we take $0<\tau<\min\big\{\gamma,\frac{1}{2}\inf\{\mathrm{div}_0 H; x\in\overline{\Omega}\}\big\}$. On one hand, we have

$$C\int_s^{s+T}\int_\Omega (u_t^2+|\nabla_g u|_g^2)dxdt\le\tau\int_s^{s+T}\int_\Omega (u_t^2-|\nabla_g u|_g^2)dxdt$$
$$+\int_s^{s+T}\int_\Omega DH(\nabla_g u,\nabla_g u)dxdt. \tag{3.6}$$

On the other hand, there exists a $\theta>0$ such that

$$2<\frac{\mathrm{div}_0 H}{\frac{1}{2}\mathrm{div}_0 H-\tau}+\theta\le 2+\delta$$

for τ small enough. Then by this choice of τ and (2.17), we have

$$\int_s^{s+T}\int_\Omega\Big(\Big(\tau-\frac{1}{2}\mathrm{div}_0 H\Big)uf(u)+\Phi(u)\mathrm{div}_0 H\Big)dxdt\le 0. \tag{3.7}$$

From (3.5)–(3.7) we deduce that

$$\begin{aligned} & C\int_s^{s+T}\!\!\int_\Omega (u_t^2+|\nabla_g u|_g^2)dxdt \\ \le\; & \frac{1}{2}\int_s^{s+T}\!\!\int_{\Gamma_0}\frac{1}{|\nu_{\mathcal{A}}|_g^2}\Big|\frac{\partial u}{\partial \nu_{\mathcal{A}}}\Big|^2 H\cdot\nu d\sigma dt+\Big|\int_s^{s+T}\!\!\int_\Omega a(x)u_tH(u)dxdt\Big| \\ & +\frac{1}{2}\Big|\int_s^{s+T}\!\!\int_\Omega a(x)u_tu\mathrm{div}_0Hdxdt\Big|+\frac{1}{4}\int_s^{s+T}\!\!\int_\Omega u^2|\mathcal{A}(\mathrm{div}_0H)|dxdt+\mathcal{N}, \end{aligned} \tag{3.8}$$

where

$$\mathcal{N}=\Big|\int_\Omega\Big(u_tH(u)+\frac{1}{2}u_tu\mathrm{div}_0H-\tau u\Big(u_t+\frac{a}{2}u\Big)\Big)dx\Big|_s^{s+T}\Big|.$$

Noting that

$$0\le f(s)s\le C^*(|s|^2+|s|^{p+1}),\qquad \forall\, s\in\mathbb{R}$$

by (1.3) and (1.4), and using Poincaré inequality, we have the following estimate

$$\int_s^{s+T}\!\!\int_\Omega \Phi(u)dxdt\le C\int_s^{s+T}\!\!\int_\Omega|\nabla_g u|_g^2dxdt \tag{3.9}$$

for some positive constant C dependent on u^0 and u^1 but independent of s and T. Thus it is easy to see from (3.8) and (3.9) that

$$\begin{aligned} C\int_s^{s+T}E(t)dt\le\; & \frac{1}{2}\int_s^{s+T}\!\!\int_{\Gamma_0}\frac{1}{|\nu_{\mathcal{A}}|_g^2}\Big|\frac{\partial u}{\partial \nu_{\mathcal{A}}}\Big|^2 H\cdot\nu d\sigma dt+\Big|\int_s^{s+T}\!\!\int_\Omega a(x)u_tH(u)dxdt\Big| \\ & +\frac{1}{2}\Big|\int_s^{s+T}\!\!\int_\Omega a(x)u_tu\mathrm{div}_0Hdxdt\Big|+\mathcal{N}+\frac{1}{4}\int_s^{s+T}\!\!\int_\Omega u^2|\mathcal{A}(\mathrm{div}_0H)|dxdt. \end{aligned} \tag{3.10}$$

Now we estimate the right hand of (3.10) term by term. By Hölder inequality, we have

$$\Big|\int_s^{s+T}\!\!\int_\Omega a(x)u_tH(u)dxdt\Big|\le\varepsilon\int_s^{s+T}\!\!\int_\Omega a(x)|H(u)|^2dxdt+\frac{1}{2\varepsilon}\int_s^{s+T}\!\!\int_\Omega a(x)|u_t|^2dxdt \tag{3.11}$$

for any $\varepsilon>0$, and

$$\int_s^{s+T}\!\!\int_\Omega|a(x)u_tu\mathrm{div}_0H|dxdt\le\frac{K}{2}\int_s^{s+T}\!\!\int_\Omega a(x)u_t^2dxdt+\frac{K}{2}\|a\|_\infty\int_s^{s+T}\!\!\int_\Omega u^2dxdt, \tag{3.12}$$

where

$$K=\sup\{\ |\mathrm{div}_0H|\ ;\ x\in\overline{\Omega}\ \}.$$

Combining (3.10)–(3.12) for $\varepsilon>0$ small enough yields

$$\begin{aligned} C\int_s^{s+T}E(t)dt\le\; & \frac{1}{2}\int_s^{s+T}\!\!\int_{\Gamma_0}\frac{1}{|\nu_{\mathcal{A}}|_g^2}\Big|\frac{\partial u}{\partial \nu_{\mathcal{A}}}\Big|^2 H\cdot\nu d\sigma dt+\int_s^{s+T}\!\!\int_\Omega a(x)u_t^2dxdt \\ & +\int_s^{s+T}\!\!\int_\Omega u^2dxdt+\mathcal{N}. \end{aligned} \tag{3.13}$$

Following the methods of Lions [3] and Zuazua [1] we now estimate the quantity

$$\int_s^{s+T}\int_{\Gamma_0}\frac{1}{|\nu_{\mathcal{A}}|_g^2}\Big|\frac{\partial u}{\partial\nu_{\mathcal{A}}}\Big|^2 H\cdot\nu d\sigma dt$$

in terms of

$$\int_s^{s+T}\int_\Omega a(x)|u_t|^2.$$

First we can construct a neighborhood $\widehat{\omega}$ of $\overline{\Gamma}_0$ such that

$$\overline{\widehat{\omega}}\cap\Omega\subset\omega,$$

and $Z=(z_1,z_2,\cdots,z_n)\in(W^{1,\infty}(\Omega))^n$ such that

$$\begin{cases}Z=\nu & \text{on } \Gamma_0,\\ Z\cdot\nu>0 & \text{a.e. in } \Gamma,\\ Z=0 & \text{on } \Omega\setminus\widehat{\omega}.\end{cases}$$

Now we take vector field $Z=\sum\limits_{i=1}^n z_i\frac{\partial}{\partial x_i}\in\mathbb{R}^n$, and replace H with Z in (3.2) to obtain

$$\begin{aligned}\int_s^{s+T}\int_{\Gamma_0}\frac{1}{|\nu_{\mathcal{A}}|_g^2}\Big|\frac{\partial u}{\partial\nu_{\mathcal{A}}}\Big|^2 d\sigma dt&\le\int_s^{s+T}\int_{\partial\Omega}\frac{1}{|\nu_{\mathcal{A}}|_g^2}\Big|\frac{\partial u}{\partial\nu_{\mathcal{A}}}\Big|^2 Z\cdot\nu d\sigma dt\\ \le C\int_s^{s+T}\int_{\widehat{\omega}}(|u_t|^2+|\nabla_g u|_g^2+\Phi(u))dxdt&+2\int_\Omega u_t Z(u)dx\Big|_s^{s+T}.\end{aligned}\tag{3.14}$$

We then construct a function $\eta\in W^{1,\infty}(\Omega)$ satisfying

$$\begin{cases}0\le\eta\le1 & \text{a.e. in } \Omega,\\ \eta=1 & \text{a.e. in } \widehat{\omega},\\ \eta=0 & \text{a.e. in } \Omega\setminus\omega.\end{cases}$$

Applying (3.3) with $\zeta=\eta$, we have

$$\begin{aligned}&\int_s^{s+T}\int_\Omega\eta(|\nabla_g u|_g^2+uf(u))dxdt\\ =&\int_s^{s+T}\int_\Omega\eta u_t^2dxdt-\int_s^{s+T}\int_\Omega u\langle\nabla_g u,\nabla_g\eta\rangle_g dxdt-\int_s^{s+T}\eta u\Big(u_t+\frac{a}{2}u\Big)dx\Big|_s^{s+T}\\ \le&\int_s^{s+T}\int_\omega u_t^2dxdt+\Big|\int_s^{s+T}\int_\Omega u\langle\nabla_g u,\nabla_g\eta\rangle_g dxdt\Big|+\mathcal{P}\end{aligned}\tag{3.15}$$

with

$$\mathcal{P}=\Big|\int_\Omega\eta u\Big(u_t+\frac{a}{2}u\Big)dx\Big|_s^{s+T}\Big|.$$

On the other hand, by Hölder inequality

$$\begin{aligned}&\Big|\int_s^{s+T}\int_\Omega u\langle\nabla_g u,\nabla_g\eta\rangle_g dxdt\Big|\\ \le&\frac{1}{2\varepsilon}\int_s^{s+T}\int_\Omega u^2dxdt+\varepsilon\int_s^{s+T}\int_\Omega|\langle\nabla_g u,\nabla_g\eta\rangle_g|^2dxdt.\end{aligned}\tag{3.16}$$

Then by (1.5), (1.8), (3.15) and (3.16) with $\varepsilon > 0$ small enough we obtain

$$\begin{aligned}\int_s^{s+T}\int_{\widehat{\omega}}(|\nabla_g u|_g^2 + \Phi(u))dxdt &\le \int_s^{s+T}\int_\Omega \eta(|\nabla_g u|_g^2 + \Phi(u))dxdt \\ &\le C\int_s^{s+T}\int_\Omega \eta(|\nabla_g u|_g^2 + uf(u))dxdt \\ &\le C\Big(\int_s^{s+T}\int_\omega u_t^2 dxdt + \int_s^{s+T}\int_\Omega u^2 dxdt + \mathcal{P}\Big). \end{aligned} \tag{3.17}$$

From (3.14) and (3.17) we conclude that

$$\begin{aligned}&\int_s^{s+T}\int_{\Gamma_0}\frac{1}{|\nu_{\mathcal{A}}|_g^2}\Big|\frac{\partial u}{\partial \nu_{\mathcal{A}}}\Big|^2 H\cdot\nu d\sigma dt \\ &\le C\Big(\int_s^{s+T}\int_\Omega a(x)u_t^2 dxdt + \int_s^{s+T}\int_\Omega u^2 dxdt\Big) + C\Big(\Big|\int_\Omega u_t Z(u)dx\Big|_s^{s+T}\Big| + \mathcal{P}\Big). \end{aligned} \tag{3.18}$$

By (3.13) and (3.18) we then get

$$\begin{aligned}&TE(s+T) \le \int_s^{s+T} E(t)dt \\ &\le C\Big(\int_s^{s+T}\int_\Omega a(x)u_t^2 dxdt + \int_s^{s+T}\int_\Omega u^2 dxdt\Big) + C\Big(\Big|\int_\Omega u_t Z(u)dx\Big|_s^{s+T}\Big| + \mathcal{N} + \mathcal{P}\Big). \end{aligned} \tag{3.19}$$

It is not difficult to obtain the estimation

$$\begin{aligned}\Big|\int_\Omega u_t Z(u)dx\Big|_s^{s+T}\Big| + \mathcal{N} + \mathcal{P} &\le C(E(s) + E(s+T)) \\ &= C\Big(2E(s+T) + \int_s^{s+T}\int_\Omega a(x)u_t^2 dxdt\Big). \end{aligned} \tag{3.20}$$

(3.19) together with (3.20) yields

$$TE(s+T) \le C\Big(\int_s^{s+T}\int_\Omega a(x)u_t^2 dxdt + \int_s^{s+T}\int_\Omega u^2 dxdt\Big) + CE(s+T). \tag{3.21}$$

Then if we select some T large enough such that $T > C$, the following estimate

$$E(s+T) \le C_*\Big(\int_s^{s+T}\int_\Omega a(x)u_t^2 dxdt + \int_s^{s+T}\int_\Omega u^2 dxdt\Big) \tag{3.22}$$

holds for some positive constant C_* dependent on T but independent of s. By the standard compactness-uniqueness argument we can absorb the lower term, i.e.,

$$\exists C > 0, \quad \int_s^{s+T}\int_\Omega u^2 dxdt \le C\int_s^{s+T}\int_\Omega a(x)u_t^2 dxdxt, \tag{3.23}$$

and we refer to [1] for the details. (3.22) combined with (3.23), (1.9) implies

$$E(s+T) \le \frac{C_T}{1+C_T}E(s), \qquad \forall\, s \ge 0, \tag{3.24}$$

where C_T is a positive constant dependent on T but independent of s. Therefore we deduce by iteration from (3.24) that

$$E(kT) \leq \Big(\frac{C_T}{1+C_T}\Big)^k E(0), \qquad \forall\, k \in \mathbb{N}, \tag{3.25}$$

where $\mathbb{N}$ denotes the set of natural numbers. Thus we achieve (2.19) with $C = 1 + \frac{1}{C_T}$ and $\lambda = \frac{1}{T}\ln(1 + \frac{1}{C_T})$. This completes the proof.

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