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LOOP GROUP ACTIONS AND THE RIBAUCOUR TRANSFORMATIONS FOR FLAT LAGRANGIAN SUBMANIFOLDS***

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Abstract

The Ribaucour transformations for flat Lagrangian submanifolds in C^n and CP^n via loop group actions are given. As a consequence, the authors obtain a family of new flat Lagrangian submanifolds from a given one via a purely algebraic algorithm. At the same time, it is shown that such Ribaucour transformation always comes with a permutability formula.

Keywords Lagrangian submanifold, Loop group action, Darboux transformation, Ribaucour transformation, Permutability formula
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§ 1. Introduction

The theory of integrable system has been widely used to study some differential geometric objects such as surfaces with some geometric property in R^3, harmonic maps, and the isometric immersions in space forms, etc. (see [1, 5, 10, 13]). In particular, the Lagrangian surfaces in C^2 and flat Lagrangian submanifolds in complex space forms have been studied in various papers (see [3, 7]). These geometric objects are often in correspondence with the solutions of nonlinear partial differential equations, which admit Lax pairs, so one can give the construction methods of solutions of these PDEs and these geometric objects by using the integrable system theory and soliton theory. In particular, Bäcklund, Darboux and Ribaucour transformations for these geometric objects can be constructed (see [1, 3, 6, 8, 12, 14]). The importance of these geometric transformations is twofold: geometrically, as a tool for obtaining a family of new examples from a given one with the same geometric property; analytically, as a method for generating new solutions of the associated PDEs from a given one.

The main purpose of this paper is to give the explicit construction for flat Lagrangian submanifolds in C^n and CP^n by means of generalized Ribaucour transformations (resp. Ribaucour transformations) via loop group actions. The contents of the paper are arranged

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as follows. We review the concrete constructions of the solutions of the $\frac{U(n)}{O(n)}$-system via loop group action in §2. In §3, we explain the connection between Darboux transformation and the loop group action for the $\frac{U(n)}{O(n)}$-system. In fact, they are consistent. In §4, we give generalized Ribaucour transformations (resp. Ribaucour transformations) for flat Lagrangian submanifolds of C^n and CP^n by means of the results in §2. In §5, we show the permutability formulas of these transformations for flat Lagrangian submanifolds in C^n (resp. CP^n). Finally, the examples to construct flat Lagrangian submanifolds in C^n (resp. CP^n) via a purely algebraic algorithm are given in §6.

§ 2. Loop Group Dressing Action for $\frac{U(n)}{O(n)}$-System

Let $G = U(n)$ be a real form with regard to a conjugate linear involution τ of $\mathrm{GL}(n, C)$, σ a complex linear involution of $\mathrm{GL}(n, C)$, where $\tau(\xi) = (\xi^*)^{-1}$, $\sigma(\xi) = (\xi^t)^{-1}$ for $\xi \in \mathrm{GL}(n, C)$. σ and τ induce involutions on Lie algebra $\mathrm{gl}(n, C)$ respectively, also denoted as σ and τ. Namely, $\tau(\xi) = -\xi^*$, $\sigma(\xi) = -\xi^t$ for $\xi \in \mathrm{gl}(n, C)$. Let $\mathcal{G} = u(n) = \mathcal{K} + \mathcal{P}$ be the Cartan decomposition of the symmetric space $U(n)/O(n)$, where $\mathcal{K} = o(n)$ and

$$\mathcal{P} = \{iF \mid F = (f_{jk}) \in \mathrm{gl}(n, R),\ f_{jk} = f_{kj} \text{ for } 1 \le j,\, k \le n\} \qquad (i = \sqrt{-1})$$

are the $+1$ and -1 eigenspaces of σ, respectively. Moreover, the linear subspace

$$\mathcal{A} = \mathrm{span}\{a_j = ic_j \mid 1 \le j \le n\}$$

is a maximal abelian linear subspace of $\mathcal{P}$, and

$$\mathcal{A}^\perp \cap \mathcal{P} = \{iF \mid F = (f_{jk}) \in \mathrm{gl}(n, R),\ f_{jk} = f_{kj},\ f_{jj} = 0 \text{ for } 1 \le j, k \le n\},$$

where c_j is the diagonal matrix such that all entries are zero except the j^{th} entry, which is equal to 1, $\mathcal{A}^\perp = \{y \in \mathcal{G} \mid \mathrm{tr}(xy) = 0,\ x \in \mathcal{A}\}$ is the orthogonal complement of $\mathcal{A}$ with respect to the killing form on $\mathcal{G}$. We say that $g(\lambda)$ satisfies $\frac{U(n)}{O(n)}$-reality condition, if $\tau(g(\bar\lambda)) = g(\lambda), \sigma(g(-\lambda)) = g(\lambda)$, i.e.,

$$g(\bar\lambda)^* g(\lambda) = I, \qquad \overline{g(\bar\lambda)} = g(-\lambda). \tag{2.1}$$

It is evident that

Lemma 2.1. *If $g(\lambda)$ satisfies the $\frac{U(n)}{O(n)}$-reality condition* (2.1)*, then*

(1) $g(-\lambda)^t g(\lambda) = I$ and $\overline{g(\lambda)} = g(-\bar\lambda)$;

(2) If λ is a pure imaginary, then $\overline{g(\lambda)} = g(\lambda)$.

Let O_∞ be an open neighborhood of ∞ in $S^2 = C \cup \{\infty\}$. Define loop groups

$$G_+ = \left\{ g : C \to \mathrm{GL}(n, C) \;\middle|\; \begin{array}{l} g(\lambda) \text{ is holomorphic, satisfies} \\ \text{the reality condition (2.1)} \end{array} \right\},$$

$$G_- = \left\{ g : O_\infty \to \mathrm{GL}(n, C) \;\middle|\; \begin{array}{l} g(\lambda) \text{ is holomorphic, } g(\infty) = 0, \\ \text{and satisfies the reality condition (2.1)} \end{array} \right\}.$$

We also define the corresponding Lie algebras in an analogous way

$$\mathcal{G}_+ = \left\{ \xi : C \to \mathrm{gl}(n,C) \,\middle|\, \begin{array}{l} \xi(\lambda) \text{ is holomorphic,} \\ \xi(\bar\lambda)^* + \xi(\lambda) = 0,\ \overline{\xi(\bar\lambda)} = \xi(-\lambda) \end{array} \right\},$$

$$\mathcal{G}_- = \left\{ \xi : O_\infty \to \mathrm{gl}(n,C) \,\middle|\, \begin{array}{l} \xi(\lambda) \text{ is holomorphic, } \xi(\infty) = I, \\ \xi(\bar\lambda)^* + \xi(\lambda) = 0,\ \overline{\xi(\bar\lambda)} = \xi(-\lambda) \end{array} \right\}.$$

It is well known (cf. [13]) that the group of rational maps $g : S^2 \to \mathrm{GL}(n, C)$ satisfying $g(\bar\lambda)^* g(\lambda) = I$ is generated by the simple element

$$h_{z,\pi}(\lambda) = \pi + \frac{\lambda - z}{\lambda - \bar z}\pi', \tag{2.2}$$

where $z \in C$, π is a Hermitian projection of C^n and $\pi' = I - \pi$. Obviously, $h_{z,\pi}$ satisfies the reality condition (2.1) if and only if $\bar z = -z, \bar\pi = \pi$.

Let $z = is$ $(s \in R \setminus \{0\})$ and L a $k \times n$ constant real matrix such that the columns of L form a basis of a k-plane (also denoted by L) in C^n. Then we have a real symmetric projection of C^n,

$$\pi = L^t(LL^t)^{-1}L, \tag{2.3}$$

i.e., $\bar\pi = \pi^t = \pi = \pi^2$. Obviously, $h_{is,\pi} \in G_-$, $h^{-1}_{is,\pi} = h_{-is,\pi}$ and $h_{-is,\pi}(\lambda) = h_{is,\pi}(-\lambda)$.

For any $\lambda \in C^* = C \setminus \{0\}$, we consider the linear system

$$d\Psi_\lambda = \Psi_\lambda \sum_{j=1}^n (\lambda a_j + [a_j, v])dx_j, \tag{2.4}$$

where $\{a_j\}$ is a basis of $\mathcal{A}$, $v : R^n \to \mathcal{A}^\perp \cap \mathcal{P}$ is a smooth function. Thus, $\theta_\lambda = \sum_j (\lambda a_j + [a_j, v])dx_j$ is a $\mathcal{G}_+$-valued one form and $\Psi_\lambda(x) = \Psi(x, \lambda) \in G_+$.

A direct computation shows that

Lemma 2.2. *If* Ψ_λ *is a solution of* (2.4)*, then*

(1) *For any* $n \times n$ *complex matrix* $g(\lambda)$, $g(\lambda)\Psi_\lambda$ *is a solution of* (2.4) *if and only if* $g(\lambda)$ *satisfies the reality condition* (2.1)*. In particular,*

(2) $g_0\Psi_\lambda$ *is a solution of* (2.4) *if and only if the constant matrix* $g_0 \in K$*, where* K *is the Lie subgroup of* G *corresponding to the Lie algebra* $\mathcal{K}$*.*

(3) *For any* $\lambda \in C^*$ *and a complex function* $f(\lambda)$, $f(\lambda)\Psi_\lambda$ *is a solution of* (2.4) *if and only if* $\overline{f(\bar\lambda)}f(\lambda) = 1$ *and* $\overline{f(\bar\lambda)} = f(-\lambda)$.

Without loss of generality, we consider the normalized linear system

$$\begin{cases} d\Psi_\lambda = \Psi_\lambda\theta_\lambda, \\ \Psi_\lambda(0) = I. \end{cases} \tag{2.5}$$

We take $a_j = ic_j \in \mathcal{A}$ $(1 \le j \le n)$ and $v = -iF : R^n \to \mathcal{A}^\perp \cap \mathcal{P}$ in (2.5). The integrability condition of the system (2.4) or (2.5) is

$$\begin{cases} (f_{jk})_{x_j} + (f_{jk})_{x_k} + \sum_l f_{jl}f_{kl} = 0 & \text{for } j \neq k, \\ (f_{jk})_{x_l} = f_{jl}f_{lk} & \text{for } j,\ k,\ l \text{ which are distinct.} \end{cases} \tag{2.6}$$

System (2.6) is called $\frac{U(n)}{O(n)}$-system (cf. [11]), which is also called the symmetric generalized wave equation in [3].

From Lemma 2.1 and Theorem 4.3 in [12], we get the following

Theorem 2.1. *Let* $\Psi_\lambda : R^n \to G_+$ *be a solution of* (2.5) *and* L *a* $k \times n$ *constant real matrix such that* $\det(LL^t) \neq 0$. *Denote* $\pi = L^t(LL^t)^{-1}L$. *For any* $s \in R \setminus \{0\}$, *set* $V = L\Psi_{is}$. *Then*

(1) $\det(VV^t) \neq 0$ *and* $\widetilde{\pi} = V^t(VV^t)^{-1}V$ *is a real symmetric projection onto the* k*-plane* V *of* C^n.

(2) *Let* $\widetilde{\Psi}_\lambda = h_{is,\pi}\Psi_\lambda h_{-is,\widetilde{\pi}}$. *Then there is a neighborhood* $\mathcal{O}$ *of the origin in* R^n *such that* $\widetilde{\Psi}_\lambda : U \to G_+$ *is a new solution of* (2.5), *and* $\widetilde{F} = F + 2s\widetilde{\pi}_{\mathcal{A}^\perp}$ *is a new solution of* (2.6), *where* $\widetilde{\pi}_{\mathcal{A}^\perp}$ *is a projection to* $\mathcal{A}^\perp$.

Remark 2.1. If we consider the system (2.4) without the initial condition $\Psi_\lambda(0) = I$, set $\widetilde{\widetilde{\Psi}}_\lambda = h_{-is,\pi}\widetilde{\Psi} = \Psi_\lambda h_{-is,\widetilde{\pi}}$ and let L, V be as in Theorem 2.1, then $\widetilde{\widetilde{\Psi}}_\lambda$ is a new solution of the system (2.4). But the solutions obtained in this way may have polar points.

§ 3. Darboux Transformation for the $\frac{U(n)}{O(n)}$-System

In this section, we give Darboux transformation for the $\frac{U(n)}{O(n)}$-system with the method in [6]. In fact, this transformation is consistent with the loop group action for the $\frac{U(n)}{O(n)}$-system in §2.

Let $\Psi_\lambda(x)$ be a solution of the linear system (2.4), that is,

$$(\Psi_\lambda)_{x_j} = \Psi_\lambda(\lambda a_j + [a_j, v]). \tag{3.1}$$

We will construct an $n \times n$ complex matrix S and a smooth function $\tilde{v} : R^n \to \mathcal{A}^\perp \cap \mathcal{P}$ such that $\widetilde{\Psi}_\lambda := \Psi_\lambda(\lambda I - S)$ is also a solution of (3.1). From (3.1), we know that $\widetilde{\Psi}_\lambda$ is a solution of (3.1) if and only if S and $\tilde{v}$ satisfy

$$S_{x_j} = [S, [a_j, v]] - S[a_j, S], \tag{3.2}$$

$$[a_j, \tilde{v}] = [a_j, v] + [S, a_j]. \tag{3.3}$$

Choose nonzero complex numbers $\lambda_1, \cdots, \lambda_n$, such that $\lambda_j \neq \lambda_k$ for some $1 \leq j \neq k \leq n$ and constant row vectors $\ell_1, \cdots, \ell_n$ such that the $n \times n$ matrix $H = [(\ell_1\Psi_{\lambda_1})^t, \cdots, (\ell_n\Psi_{\lambda_n})^t]^t$ is non-degenerate. Then the matrix $H = [(\ell_1\Psi_{\lambda_1})^t, \cdots, (\ell_n\Psi_{\lambda_n})^t]^t$ satisfies

$$H_{x_j} = \Lambda H a_j + H[a_j, v], \tag{3.4}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$.

Let $S = H^{-1}\Lambda H$. From (3.4), we get that S is a solution of (3.2). This proves the following lemma.

Lemma 3.1. *Let* $h_j = \ell_j\Psi(\lambda_j)$ *be row vector solutions of* (3.1) *corresponding to* λ_j *for* $1 \leq j \leq n$ *such that the* $n \times n$ *matrix* $H = (h_1^t, \cdots, h_n^t)^t$ *is non-degenerate. Then* $S = H^{-1}\Lambda H$ *is a solution of* (3.2).

Since $\tilde{v}$ obtained from (3.3) is generally not in $\mathcal{A}^\perp \cap \mathcal{P}$, we choose λ_j and h_j again as follows:

(1) for any $s \in R \setminus \{0\}$, $\lambda_1 = \cdots = \lambda_k = is$, $\lambda_{k+1} = \cdots = \lambda_n = -is$;

(2) choose constant row vectors ℓ_j $(1 \le j \le n)$ such that $\det L_1 L_1^t \neq 0$, $\det L_2 L_2^t \neq 0$ and $L_1 L_2^t = 0$, where $L_1 = (\ell_1^t, \cdots, \ell_k^t)^t$, $L_2 = (\ell_{k+1}^t, \cdots, \ell_n^t)^t$ are $k \times n$, $(n-k) \times n$ matrices, respectively.

Let Ψ_λ be a solution of (2.4). Then $H_1 = L_1 \Psi_{is}$, $H_2 = L_2 \Psi_{-is}$ are matrices with rank k, $n-k$, respectively. Moreover, by Lemma 2.1(2) and $\Psi(is)\Psi(-is)^t = I$, we get that $H = [H_1^t, H_2^t]^t$ is a real invertible matrix with $\det H_1 H_1^t \neq 0$, $\det H_2 H_2^t \neq 0$ and $H_1 H_2^t = 0$. From the definition of S, similarly to the proof of the theorem 1 in [6], we can prove $S^t = S$. Hence, iS is a real symmetric matrix. This implies that $\tilde{v} = v - (S)_{\mathcal{A}^\perp}$ obtained from (3.3) lies in $\mathcal{A}^\perp \cap \mathcal{P}$. Especially, if we take $a_j = ic_j$, $v = -iF$, $\tilde{v} = -i\widetilde{F}$, then $\widetilde{F} = F - (iS)_{\mathcal{A}^\perp}$ is a new solution of (2.6).

Furthermore, if we write

$$S = H^{-1}\Lambda H = isH^{-1}\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} H - isH^{-1}\begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \end{bmatrix} H = is\widetilde{\pi} - is\widetilde{\pi}', \tag{3.5}$$

where $\widetilde{\pi} = H^{-1}\mathrm{diag}(I_k, 0)H$, then from (3.5) we have $D_\lambda = \lambda I - S = (\lambda - is)h_{-is,\widetilde{\pi}}$.

If we consider the solution of (2.5), then $\widetilde{\Psi}_\lambda = D_\lambda(0)^{-1}\Psi_\lambda D_\lambda = h_{is,\pi}\Psi_\lambda h_{-is,\widetilde{\pi}}$ is a new solution of (2.5), where $\pi = L^{-1}\mathrm{diag}(I_k, 0)L$. As §5 in [6], we can get

$$\widetilde{\pi} = \Psi_{is}^t L_1^t (L_1 \Psi_{is} \Psi_{is}^t L_1^t)^{-1} L_1 \Psi_{is}.$$

Let $V = H_1 = L_1 \Psi_{is}$. Then $\widetilde{\pi} = V^t (VV^t)^{-1} V$ is a real symmetric projection of C^n.

Summing up, we have the following

Theorem 3.1. *Let L_1, L_2 be $k \times n$, $(n-k) \times n$ matrices such that* $\det(L_1 L_1^t) \neq 0$, $\det(L_2 L_2^t) \neq 0$, $L_1 L_2^t = 0$ *and Ψ_λ a solution of* (2.5). *For $s \in R \setminus \{0\}$, set $H_1 = L_1 \Psi_{is}$, $H_2 = L_2 \Psi_{-is}$ and $S = H^{-1}\Lambda H$, where $H = [H_1^t, H_2^t]$, $\Lambda = \mathrm{diag}(isI_k, -isI_{n-k})$. Then*

(1) $\det(H_1 H_1^t) \neq 0$, $\det(H_2 H_2^t) \neq 0$, $H_1 H_2^t = 0$.

(2) *The Darboux matrix $D_\lambda = \lambda I - S = (\lambda - is)h_{-is,\widetilde{\pi}}$.*

(3) *$\widetilde{\Psi}_\lambda = D_\lambda(0)^{-1}\Psi_\lambda D_\lambda$ is a solution of* (2.5), *and $\widetilde{F} = F - (iS)_{\mathcal{A}^\perp} = F + 2s(i\widetilde{\pi})_{\mathcal{A}^\perp}$ is a solution of* (2.6).

§ 4. Ribaucour Transformations for Flat Lagrangian Submanifolds

Let $X : M^n \to R^{2n}$ be an n-dimensional flat submanifold of R^{2n} with the first fundamental form $I = \sum_j b_j^2 dx_j^2$. Denote $f_{jk} = \frac{(b_j)_{x_k}}{b_k}$ for $1 \le j \neq k \le n$, $F = (f_{jk})$. In [11], Terng C. L. proved that X is a flat Lagrangian submanifold if and only if $F = F^t$ if and only if F is a solution of (2.6).

Let X be a flat Lagrangian submanifold of R^{2n}. Then there is a unitary frame field $\Phi = (e_1, \cdots, e_n, Je_1, \cdots, Je_n) : M \to U(n)$ in $R^{2n} \cong C^n$ satisfying

$$d\Phi = \Phi \begin{bmatrix} \omega & \delta \\ -\delta & \omega \end{bmatrix}, \tag{4.1}$$

where $\omega = (\omega_{jk}) = [\delta, F]$, $\delta = \mathrm{diag}(dx_1, \cdots, dx_n)$ (cf. [11]). Let $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ be the standard complex structure on C^n. We regard $u(n)$ as the subalgebra of $o(2n)$, i.e.,

$$A = A_1 + iA_2 \in u(n) \to \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix} \in o(2n),$$

where $A_1 \in o(n)$, $A_2 \in \mathrm{gl}(n, R)$ is symmetric. Thus, we identify the position vector $X \in R^{2n}$ with $X + iJX \in C^n$ and $e_j \in R^{2n}$ with $e_j + iJe_j \in C^n$ $(1 \le j \le n)$. Moreover, $u(n)$-valued real frame $\Phi = (e_1, \cdots, e_n, Je_1, \cdots, Je_n)$ is identified with $u(n)$-valued complex frame $\Phi + iJ\Phi := (e_1 + iJe_1, e_2 + iJe_2, \cdots, e_n + iJe_n)$, $\theta = \Phi^{-1}d\Phi = \begin{bmatrix} \omega & \delta \\ -\delta & \omega \end{bmatrix}$ with $\theta =: i\delta + \omega = i\delta + [\delta, F]$. Then $dX = \sum\limits_{j=1}^{n} \omega_j e_j \in R^{2n}$ is identified with

$$d(X + iJX) = \sum_{j=1}^{n} \omega_j (e_j + iJe_j) \in C^n, \tag{4.2}$$

and (4.1) with

$$d(\Phi + iJ\Phi) = (\Phi + iJ\Phi)(i\delta + [\delta, F]). \tag{4.3}$$

Introduce a spectral parameter λ in (4.1), that is,

$$d\Phi_\lambda = \Phi_\lambda \begin{bmatrix} \omega & \lambda\delta \\ -\lambda\delta & \omega \end{bmatrix}. \tag{4.4}$$

This is equivalent to

$$d(\Phi_\lambda + iJ\Phi_\lambda) = (\Phi_\lambda + iJ\Phi_\lambda)(i\lambda\delta + [\delta, F]). \tag{4.5}$$

Evidently, $\theta_\lambda = i\lambda\delta + [\delta, F] = \sum\limits_j (i\lambda c_j + [c_j, F]) dx_j$ is $\mathcal{G}_+$-valued one form and (4.5) is solvable if and only if $d\theta_\lambda + \theta_\lambda \wedge \theta_\lambda = 0$ if and only if F satisfies (2.6).

Denote $\mathrm{sym}_* = \{F = (f_{jk}) \in \mathrm{gl}(n, R) \mid f_{jk} = f_{kj},\ f_{jj} = 0, 1 \le j,\ k \le n\}$. Let $F \in \mathrm{sym}_*$ be a solution of (2.6). Then there is a solution $\Phi_\lambda + iJ\Phi_\lambda = (e_1 + iJe_1, \cdots, e_n + iJe_n)$ of (4.5) corresponding to F. Suppose that the nonzero smooth functions b_j satisfy $(b_j)_{x_k} = f_{jk} b_k$ for $1 \le j \ne k \le n$. Then X identified with $X + iJX$, which is defined by

$$d(X + iJX) = \sum_{j=1}^{n} b_j (e_j + iJe_j) dx_j,$$

is a local flat Lagrangian submanifold of C^n with the first fundamental form $I = \sum\limits_j b_j^2 dx_j^2$.

Combining the above argument with Theorem 2.1 and noting Remark 2.1, Lemma 2.2, we have

Theorem 4.1. *Let $X : M^n \to R^{2n} \cong C^n$ be a flat Lagrangian submanifold with an unitary frame $\Phi = (e_1, \cdots, e_n, Je_1, \cdots, Je_n)$ and the first fundamental form $I = \sum\limits_j b_j^2 dx_j^2$, $\Phi_\lambda + iJ\Phi_\lambda$ a solution of (4.5) with $\Phi_1 + iJ\Phi_1 = \Phi + iJ\Phi$. For any $s \in R \setminus \{0\}$, let*

L be a $k\times n$ constant real matrix such that $\det(LL^t)\neq 0$, *$V=L(\Phi_{is}+iJ\Phi_{is})$, and $\widetilde{\Phi}_\lambda+iJ\widetilde{\Phi}_\lambda=(\Phi_\lambda+iJ\Phi_\lambda)h_{-is,\widetilde{\pi}}$, where $\widetilde{\pi}=V^t(VV^t)^{-1}V$. Then*

(1) *There is a neighborhood $\mathcal{O}$ of the origin such that $(\widetilde{\Phi}_\lambda+iJ\widetilde{\Phi}_\lambda)(x)$ is a solution of* (4.5) *for $x\in\mathcal{O}$ and $\widetilde{F}=(\tilde{f}_{jk}):\mathcal{O}\to\mathrm{sym}_*$ is a solution of* (2.6), *where $\widetilde{\Phi}_\lambda+iJ\widetilde{\Phi}_\lambda=(\Phi_\lambda+iJ\Phi_\lambda)(I-\frac{2is}{\lambda+is}\widetilde{\pi})$, $\widetilde{F}=(\tilde{f}_{jk})=F+2s(\widetilde{\pi})_{off}$, $(\cdot)_{off}$ is a matrix all of whose diagonal elements are zero.*

(2) *If $\tilde{b}_1,\cdots,\tilde{b}_n$ are nonzero smooth functions satisfying $(\tilde{b}_k)_{x_j}=\tilde{f}_{kj}\tilde{b}_j$ for $1\leq j\neq k\leq n$, then there exists a local flat Lagrangian submanifold $\widetilde{X}:\mathcal{O}\to R^{2n}\cong C^n$ with a unitary frame $\widetilde{\Phi}_1=(\tilde{e}_1,\cdots,\tilde{e}_n,J\tilde{e}_1,\cdots,J\tilde{e}_n)$ and the first fundamental form $\widetilde{I}=\sum_j\tilde{b}_j^2dx_j^2$, where $\widetilde{\Phi}_1$ is identified with $\widetilde{\Phi}_1+iJ\widetilde{\Phi}_1=(\Phi+iJ\Phi)(I-\frac{2s^2+2is}{1+s^2}\widetilde{\pi})$.*

Furthermore, we can give the explicit expression of $\widetilde{X}$ and give the geometric interpretation of the action of $h_{is,\pi}$ on the solution space of (4.5). This needs the following definition.

Definition 4.1. (cf. [3]) *An isometric immersion $\tilde{f}:\widetilde{M}^n\to R^{n+p}$ is called a Ribaucour transformation of an isometric immersion $f:M^n\to R^{n+p}$ if there exists a vector bundle isometry $P:f^*TR^{n+p}\to\tilde{f}^*TR^{n+p}$ covering a diffeomorphism $p:M^n\to\widetilde{M}^n$, a smooth section $\omega\in\Gamma((f^*TR^{n+p})^*)$ and a symmetric tensor D on M^n such that $\|f-\tilde{f}\circ p\|\neq 0$ everywhere,*

(a) *$P(Z)-Z=\omega(Z)(f-\tilde{f}\circ p)$, $Z\in\Gamma(f^*TR^{n+p})$, and*

(b) *$P\circ f_*\circ D=\tilde{f}_*\circ p_*$.*

Below we restrict the above definition of the Ribaucour transformation to flat Lagrangian submanifolds of C^n, and show that the action of $h_{is,\pi}$ on solutions of (4.5) corresponds to a Ribaucour transformation for flat Lagrangian submanifolds of C^n.

Theorem 4.2. *Let $X:M\to R^{2n}\cong C^n$ be a flat Lagrangian submanifold with first fundamental form $I=\sum_j b_j^2dx_j^2$, where $\{e_1,\cdots,e_n\}$ is a tangent frame on M. Φ, Φ_λ, L, V, s as in Theorem* 4.1. *Then*

(1) *There exists a vector valued function $\gamma:M\to R^k$ such that $d\gamma=V\delta b$, where $b=(b_1,\cdots,b_n)^t$.*

(2) *Let $\widetilde{X}+iJ\widetilde{X}=(X+iJX)-\frac{2s(s+i)}{1+s^2}(\Phi+iJ\Phi)V^tW\gamma$, where $W=(VV^t)^{-1}$. Then $\widetilde{X}$ identified with $\widetilde{X}+iJ\widetilde{X}$ is a local flat Lagrangian submanifold in $R^{2n}\cong C^n$ with the first fundamental form $\widetilde{I}=\sum_j\tilde{b}_j^2dx_j^2$, where $(\tilde{e}_1,\cdots,\tilde{e}_n,J\tilde{e}_1,\cdots,J\tilde{e}_n)$ is identified with $\widetilde{\Phi}_1+iJ\widetilde{\Phi}_1=(\Phi_1+iJ\Phi_1)(I-\frac{2s^2+2is}{1+s^2}V^tWV)$, and $\tilde{b}=b+2sV^tW\gamma$.*

(3) *The solution $\widetilde{F}$ of* (2.6) *corresponding to $\widetilde{X}$ is $\widetilde{F}=F+2s(V^tWV)_{off}$.*

(4) *$(\widetilde{X}+iJ\widetilde{X})-(\widetilde{\Phi}+iJ\widetilde{\Phi})V^tW\gamma=(X+iJX)-(\Phi+iJ\Phi)V^tW\gamma$.*

Proof. (1) Denote $V=L\Phi_{is}=(v_1,\cdots,v_n)$. It follows that $dV=V\theta_{is}$, i.e.,

$$\begin{cases}(v_k)_j=f_{jk}v_j & \text{for } j\neq k,\\ (v_k)_k=-sv_k-\sum_{j\neq k}f_{jk}v_j\end{cases}\tag{4.6}$$

from (4.4) and $\omega = [\delta, F]$. Noting $(b_j)_{x_k} = f_{jk} b_k$ $(j \neq k)$, we have $(b_j v_j)_{x_k} = (b_k v_k)_{x_j}$. So, there is a function γ, such that $d\gamma = \sum\limits_j b_j v_j dx_j = V\delta b$.

(2) Since $V = L\Phi_{is}$, $dV = V\theta_{is}$ and F is symmetric. From (4.5), we get

$$d((\Phi_1 + iJ\Phi_1)V^t) = (i - s)(\Phi_1 + iJ\Phi_1)\delta V^t, \tag{4.7}$$

$$dW = d(VV^t)^{-1} = -Wd(VV^t)W = 2sWV\delta V^t W. \tag{4.8}$$

From (4.2), we get

$$d(X + iJX) = (\Phi_1 + iJ\Phi_1)\delta b. \tag{4.9}$$

It follows from (4.7)–(4.9) and (1) that

$$\begin{aligned} d(\widetilde{X} + iJ\widetilde{X}) &= d(X + iJx) - \frac{2s(i+s)}{1+s^2} d[(\Phi + iJ\Phi)V^t W\gamma] \\ &= (\Phi_1 + iJ\Phi_1)\Big(I - \frac{2s^2 + 2si}{1+s^2} V^t W V\Big)\delta(b + 2sV^t W\gamma) \\ &= (\widetilde{\Phi}_1 + iJ\widetilde{\Phi}_1)\delta\tilde{b}, \end{aligned} \tag{4.10}$$

where $\tilde{b} = b + 2sV^t W\gamma$. So, (2) follows.

(3) It follows from Theorem 4.1 directly.

(4) From (2), we get

$$\begin{aligned} (\widetilde{X} + iJ\widetilde{X}) - (X + iJX) &= -\frac{2s^2 + 2si}{1+s^2}(\Phi + iJ\Phi)V^t W(VV^t)(VV^t)^{-1}\gamma \\ &= [(\widetilde{\Phi} + iJ\widetilde{\Phi}) - (\Phi + iJ\Phi)]V^t W\gamma, \end{aligned} \tag{4.11}$$

that is,

$$(\widetilde{X} + iJ\widetilde{X}) - (\widetilde{\Phi} + iJ\widetilde{\Phi})V^t W\gamma = (X + iJX) - (\Phi + iJ\Phi)V^t W\gamma. \tag{4.12}$$

Corollary 4.1. *In Theorem 4.2, let L be a nonzero constant vector in R^n and $V = L\Phi_{is} = (v_1, \cdots, v_n)$. Then*

(1) There exists a function $\phi : M \to R$ such that $d\phi = \sum\limits_j b_j v_j dx_j$.

(2) Let $\widetilde{X} + iJ\widetilde{X} = (X + iJX) - \frac{2s(s+i)\phi}{(1+s^2)\sum v_j^2} \sum\limits_j v_j(e_j + iJe_j)$. Then $\widetilde{X}$ identified with $\widetilde{X} + iJ\widetilde{X}$ is a local flat Lagrangian submanifold in $R^{2n} \cong C^n$ with the first fundamental form $\widetilde{I} = \sum\limits_j \tilde{b}_j^2 dx_j^2$, where $\tilde{e}_j$ is identified with $\tilde{e}_j + iJ\tilde{e}_j = e_j + iJe_j - \frac{(2s^2+2is)v_j}{(1+s^2)\sum v_j^2} \sum\limits_k (e_k + iJe_k)v_k$ and $\tilde{b}_j = b_j + \frac{2s\phi v_j}{\sum v_j^2}$ for $1 \le j \le n$.

(3) The solution $\widetilde{F} = (\tilde{f}_{jk})$ of (2.6) corresponding to $\widetilde{X}$ is $(\tilde{f}_{jk}) = (f_{jk}) + \big(\frac{2s}{\sum v_j^2} v_j v_k\big)_{off}$.

(4) $\widetilde{X} + iJ\widetilde{X} - \dfrac{\phi}{v_j}(\tilde{e}_j + iJ\tilde{e}_j) = X + iJX - \dfrac{\phi}{v_j}(e_j + iJe_j)$, and $\widetilde{X}$ is a Ribaucour transformation of X.

Proof. (1), (2), (3) are obvious.

(4) By (2) and Theorem 4.1, we define a vector bundle isometry $P\colon X^*TR^{2n}\to\widetilde{X}^*TR^{2n}$, $P(e_1,\cdots,e_n,Je_1,\cdots,Je_n)=(\tilde e_1,\cdots,\tilde e_n,J\tilde e_1,\cdots,J\tilde e_n)$ covering the map $p: X(x)\to\widetilde{X}(x)$. Then P satisfies the condition (b) in Definition 4.1 with $p=\text{id}$. On the other hand,

$$(\tilde e_j+iJ\tilde e_j)-(e_j+iJe_j)=-\frac{2s^2+2si}{(1+s^2)\sum v_j^2}\sum_k(e_k+iJe_k)v_kv_j. \tag{4.13}$$

Hence

$$\begin{aligned}(\widetilde{X}_j+iJ\widetilde{X}_j)-(X_j+iJX_j)&=-\frac{(2s^2+2si)\phi}{(1+s^2)\sum v_j^2}\sum_k(e_k+iJe_k)v_k\\&=\frac{\phi}{v_j}[(\tilde e_j+iJ\tilde e_j)-(e_j+iJe_j)].\end{aligned} \tag{4.14}$$

That is,

$$\begin{aligned}\widetilde{X}-\tilde e_j&=\frac{\phi}{v_j}(X-e_j),\\ \widetilde{X}-J\tilde e_j&=\frac{\phi}{v_j}(X-Je_j).\end{aligned}$$

This means that the condition (a) in Definition 4.1 holds. So P is a Ribaucour transformation.

We call the transformation in Theorem 4.2(4) a generalized Ribaucour transformation for the flat Lagrangian submanifold in C^n. Furthermore, from [2, Proposition 8], we get that $X(M^n)\subset S^{2n-1}$ if and only if $\sum_j b_j^2=1$. Hence, we have

Theorem 4.3. *Let $X:M\to S^{2n-1}$ be a flat submanifold of S^{2n-1} that is Lagrangian in R^{2n} with the first fundamental form $I=\sum_j b_j^2dx_j^2$, where $\{e_1,\cdots,e_n\}$ is a tangent frame on M. Φ, Φ_λ, L, V, s are as in Theorem 4.1. Then*

(1) Let $\widetilde{X}+iJ\widetilde{X}=(X+iJX)-\frac{2s(s+i)}{1+s^2}(\Phi+iJ\Phi)V^tW\gamma$, where $W=(VV^t)^{-1}$ and $\gamma:M\to R^k$ is defined by $(Vb)^t(W\gamma)+(W\gamma)^t(Vb)+2s(W\gamma)^t\gamma=0$. Then $\widetilde{X}$ identified with $\widetilde{X}+iJ\widetilde{X}$ is a local flat submanifold of S^{2n-1} that is Lagrangian in R^{2n} with the first fundamental form $\widetilde{I}=\sum_j\tilde b_j^2d{x_j}^2$, where $(\tilde e_1,\cdots,\tilde e_n,J\tilde e_1,\cdots,J\tilde e_n)$ is identified with $(\tilde e_1+iJ\tilde e_1,\cdots,\tilde e_n+iJ\tilde e_n)=(e_1+iJe_1,\cdots,e_n+iJe_n)\big(I-\frac{2s^2+2is}{1+s^2}V^tWV\big)$, $\tilde b=b+2sV^tW\gamma$.

(2) The solution $\widetilde{F}$ of (2.6) corresponding to $\widetilde{X}$ is $\widetilde{F}=F+2s(V^tWV)_{off}$.

(3) $(\widetilde{X}+iJ\widetilde{X})-(\widetilde{\Phi}+iJ\widetilde{\Phi})V^tW\gamma=(X+iJX)-(\Phi+iJ\Phi)V^tW\gamma$.

Corollary 4.2. *In Theorem 4.3, let L be a nonzero constant vector in R^n and $V=L\Phi_{is}=(v_1,\cdots,v_n)$. Then*

(1) Let $\widetilde{X}+iJ\widetilde{X}=(X+iJX)+\frac{2(s+i)\sum_j v_jb_j}{(1+s^2)\sum v_j^2}\sum_j v_j(e_j+iJe_j)$. Then $\widetilde{X}$ identified with $\widetilde{X}+iJ\widetilde{X}$ is a local flat submanifold of S^{2n-1} that is Lagrangian in R^{2n} with the first fundamental form $\widetilde{I}=\sum_j\tilde b_j^2dx_j^2$, where $\tilde e_j$ is identified with $\tilde e_j+iJ\tilde e_j=e_j+iJe_j-\frac{(2s^2+2is)v_j}{(1+s^2)\sum v_j^2}\sum_k(e_k+iJe_k)v_k$ and $\tilde b_j=b_j-\frac{2\sum_j v_jb_j}{\sum v_j^2}v_j$ for $1\le j\le n$.

(2) *The solution* $\widetilde{F}=(\tilde{f}_{jk})$ *of* (2.6) *corresponding to* $\widetilde{X}$ *is* $\tilde{f}_{jk}=f_{jk}+\big(\frac{2s}{\sum v_j^2}v_jv_k\big)_{off}$.

(3) $\widetilde{X}+iJ\widetilde{X}+\frac{\sum\limits_j v_jb_j}{sv_j}(\tilde{e}_j+iJ\tilde{e}_j)=X+iJX+\frac{\sum\limits_j v_jb_j}{sv_j}(e_j+iJe_j)$ *and* $\widetilde{X}$ *is a Ribaucour transformation of* X.

If these submanifolds lie in S^{2n-1}, then they are invariant under the S^1-action of the Hopf fibration. Hence the projections of these submanifolds are flat Lagrangian submanifolds of CP^{n-1} via Hopf fibration (see [4]). Thus, from Theorem 4.3 and Corollary 4.2, we get Ribaucour transformations for flat Lagrangian submanifolds of CP^n, which develop the results in [3] obtained by geometric way and a family of new flat Lagrangian submanifolds in CP^n from a given one via a purely algebraic algorithm.

§ 5. The Permutability Formulas for Ribaucour Transformations

Terng and Uhlenbeck derived in [12] some relations among generators $h_{z,\pi}$ of G_- defined by (2.2) and proved the permutability theorem for Bäcklund transformation for surfaces in R^3. Below we find relations among $h_{is,\pi}$ and give the permutability formulas for flat Lagrangian submanifolds in C^n and CP^n similar to the proof in [12] or [1].

Proposition 5.1. *Let* L_l *be constant nonzero row vectors in* R^n *and* π_l *the real symmetric projections onto the linear subspaces spanned by* L_l*, respectively. Let* $s_1,s_2\in R\setminus\{0\}$ *be constants such that* $s_1^2\neq s_2^2$*. Denote* $Q_l=L_lh_{is_j,\pi_j}(-is_l)$ *for* $j\neq l$*, and* τ_l *the real symmetric projections onto* Q_l*, where* $l=1,2$*. Then* τ_1,τ_2 *are unique projections satisfying*

$$h_{is_2,\tau_2}(\lambda)h_{is_1,\pi_1}(\lambda)=h_{is_1,\tau_1}(\lambda)h_{is_2,\pi_2}(\lambda). \tag{5.1}$$

Proof. A direct computation gives the residue of $h_{is_1,\tau_1}h_{is_2,\pi_2}h_{is_1,\pi_1}^{-1}$ at $\lambda=is_1$

$$R_{is_1}=2is_1\tau_1h_{is_2,\pi_2}(is_1)\pi_1'.$$

Since $Q_1=L_1h_{is_2,\pi_2}(-is_1)$ is real symmetric, and $h_{is_2,\pi_2}(is_1)^th_{is_2,\pi_2}(-is_1)=I$ from Lemma 2.1,

$$\langle h_{is_2,\pi_2}(is_1)(L_1^t)^\perp,Q_1^t\rangle=\langle(L_1^t)^\perp,h_{is_2,\pi_2}(is_1)^th_{is_2,\pi_2}(-is_1)L_1^t\rangle=0,$$

where $(L_1)^\perp$ is the orthogonal complement of L_1 with respect to standard inner product in C^n. Hence $R_{is_1}=0$ and $h_{is_1,\tau_1}h_{is_2,\pi_2}h_{is_1,\pi_1}^{-1}$ is holomorphic at $\lambda=is_1$. Since $h_{is_1,\tau_1}h_{is_2,\pi_2}h_{is_1,\pi_1}^{-1}$ satisfies the reality condition (2.1), $h_{is_1,\tau_1}h_{is_2,\pi_2}h_{is_1,\pi_1}^{-1}$ is holomorphic at $\lambda=-is_1$. Thus, $h_1=:h_{is_1,\tau_1}h_{is_2,\pi_2}h_{is_1,\pi_1}^{-1}h_{is_2,\tau_2}^{-1}$ is holomorphic at $\lambda=\pm is_1$. Use a similar argument to prove $h_2=:h_{is_2,\tau_2}h_{is_1,\pi_1}h_{is_2,\pi_2}^{-1}h_{is_1,\tau_1}^{-1}$ is holomorphic at $\lambda=\pm is_2$. Because $h_1=h_2^{-1}$, and $h_2(\lambda)^{-1}=h_2(\bar\lambda)^*$, h_1 is holomorphic for all $\lambda\in S^2$. But $h_1(\infty)=I$. So $h_1=I$. This proves (5.1).

Suppose that σ_l is a projection onto the planes $\widetilde{Q}_l$ $(l=1,2)$ and $h_{is_2,\sigma_2}h_{is_1,\pi_1}=h_{is_1,\sigma_1}\cdot h_{is_2,\pi_2}$. Then $h_{is_2,\sigma_2}=h_{is_1,\sigma_1}h_{is_2,\pi_2}h_{is_1,\pi_1}^{-1}$ is holomorphic at $\lambda=is_1$ and the residue of $h_{is_1,\sigma_1}(\lambda)h_{is_2,\pi_2}h_{is_1,\pi_1}^{-1}$ is zero at $\lambda=is_1$, i.e., $\sigma_1h_{is_2,\pi_2}(is_1)\pi_1'=0$. From this and $h_{is_2,\pi_2}^t(is_1)=h_{is_2,\pi_2}^{-1}(-is_1)$, we get $\widetilde{Q}_1$ is parallel to Q_1. Similarly, $\widetilde{Q}_2$ is parallel to Q_2.

Let Φ_λ be a solution of (2.5). By Theorem 2.1, there exist $V_l = L_l\Phi_{is_l}$, $U_l = Q_l\Phi^l_{is_l}$ and $\widetilde{\pi}_l = V_l^t(V_lV_l^t)^{-1}V_l$, $\widetilde{\tau}_l = U_l^t(U_lU_l^t)^{-1}U_l$ and

$$h_{is_1,\pi_1}\Phi_\lambda = \Phi^1_\lambda h_{is_1,\widetilde{\pi}_1}, \qquad h_{is_2,\pi_2}\Phi_\lambda = \Phi^2_\lambda h_{is_2,\widetilde{\pi}_2};$$
$$h_{is_1,\tau_1}\Phi^1_\lambda = \Xi^1_\lambda h_{is_1,\widetilde{\tau}_1}, \qquad h_{is_2,\tau_2}\Phi^2_\lambda = \Xi^2_\lambda h_{is_2,\widetilde{\tau}_2},$$

such that Φ^l_λ, Ξ^l_λ are holomorphic for $\lambda \in C$ and are also solutions of (2.5), where $l = 1, 2$. Thus

$$h_{is_1,\tau_1}h_{is_2,\pi_2}\Phi = \Xi^1_\lambda h_{is_1,\widetilde{\tau}_1}h_{is_2,\widetilde{\pi}_2}, \qquad h_{is_2,\tau_2}h_{is_1,\pi_1}\Phi = \Xi^2_\lambda h_{is_2,\widetilde{\tau}_2}h_{is_1,\widetilde{\pi}_1}. \tag{5.2}$$

By the uniqueness of the Birkhoff decomposition (cf. [13]) and Proposition 5.1, we get

$$\Xi^1_\lambda = \Xi^2_\lambda, \qquad h_{is_1,\widetilde{\tau}_1}h_{is_2,\widetilde{\pi}_2} = h_{is_2,\widetilde{\tau}_2}h_{is_1,\widetilde{\pi}_1}. \tag{5.3}$$

So we have the following permutability formulas.

Theorem 5.1. *Let $\widetilde{X}_l : \widetilde{M}_l \to R^{2n} \cong C^n$ be the Ribaucour transformations for flat Lagrangian submanifolds $X : M \to R^{2n} \cong C^n$ corresponding to s_l, V_l (V_l are $1\times n$ matrices) as in Corollary 4.1, respectively, where $l = 1, 2$. If $s_1, s_2 \in R \setminus \{0\}$ and $s_1^2 \neq s_2^2$, then there exists unique flat Lagrangian submanifold $\tilde{\widetilde{X}} : M_3 \to R^{2n}$, which is identified with*

$$\begin{aligned}\tilde{\widetilde{X}} + iJ\tilde{\widetilde{X}} = X + iJx &- \frac{2s_1^2 + 2s_1 i}{1+s_1^2}(\Phi_1 + iJ\Phi_1)V_1^t(V_1V_1^t)^{-1}\phi_1\\ &- \frac{2s_2^2 + 2s_2 i}{1+s_2^2}(\Phi^1_1 + iJ\Phi^1_1)U_2^t(U_1U_2^t)^{-1}\psi_2\\ = X + iJx &- \frac{2s_2^2 + 2s_2 i}{1+s_2^2}(\Phi_1 + iJ\Phi_1)V_2^t(V_2V_2^t)^{-1}\phi_2\\ &- \frac{2s_1^2 + 2s_1 i}{1+s_1^2}(\Phi^2_1 + iJ\Phi^2_1)U_1^t(U_1U_1^t)^{-1}\psi_1,\end{aligned} \tag{5.4}$$

with the first fundamental form $I = \sum_j (\tilde{\tilde{b}}_j)^2 d\tilde{x}_j^2$, and the solution $\tilde{\widetilde{F}}$ of (2.6) associated with $\tilde{\widetilde{X}}$ is

$$\begin{aligned}\tilde{\widetilde{F}} &= F + (2s_1\widetilde{\pi}_1)_{off} + (2s_2\widetilde{\tau}_2)_{off}\\ &= F + (2s_2\widetilde{\pi}_2)_{off} + (2s_1\widetilde{\tau}_1)_{off},\end{aligned} \tag{5.5}$$

where ϕ_l, ψ_l are solutions of $d\phi_l = V_l\delta b$, $d\psi_l = U_l\delta b$, respectively,

$$\begin{aligned}\tilde{\tilde{b}} &= b + 2s_1V_1^t(V_1V_1^t)^{-1}\phi_1 + 2s_2U_2^t(U_2U_2^t)^{-1}\psi_2\\ &= b + 2s_2V_2^t(V_2V_2^t)^{-1}\phi_2 + 2s_1U_1^t(U_1U_1^t)^{-1}\psi_1,\end{aligned} \tag{5.6}$$

and $(\tilde{\tilde{e}}_1, \cdots, \tilde{\tilde{e}}_n, J\tilde{\tilde{e}}_1, \cdots, J\tilde{\tilde{e}}_n)$ is identified with

$$\begin{aligned}&(\tilde{\tilde{e}}_1 + iJ\tilde{\tilde{e}}_1, \cdots, \tilde{\tilde{e}}_n + iJ\tilde{\tilde{e}}_n)\\ &= (e_1 + iJe_1, \cdots, e_n + iJe_n)\Big(I - \frac{2s_1^2 + 2is_1}{1+s_1^2}\widetilde{\pi}_1\Big)\Big(I - \frac{2s_2^2 + 2is_2}{1+s_2^2}\widetilde{\tau}_2\Big)\\ &= (e_1 + iJe_1, \cdots, e_n + iJe_n)\Big(I - \frac{2s_2^2 + 2is_2}{1+s_2^2}\widetilde{\pi}_2\Big)\Big(I - \frac{2s_1^2 + 2is_1}{1+s_1^2}\widetilde{\tau}_1\Big).\end{aligned} \tag{5.7}$$

Similarly, we get the permutability formulas of the Ribaucour transformations for the flat submanifolds of S^{2n-1}, which are Lagrangian in R^{2n}, and CP^n. Since L_l, Q_l are vectors in R^n, it follows that $\phi_l = -\frac{\sum_j v_{lj} b_j}{s_l}$, $\psi_l = -\frac{\sum_j u_{lj} b_j}{s_l}$ from Corollary 4.2. All formulas in Theorem 5.1 are computed by purely algebraic algorithm without solving any equations.

Remark 5.1. Let $L_l = (\ell_{l1}^t, \cdots, \ell_{lk}^t)^t$ be $k \times n$ constant real matrices with $\ell_{lj_1}\ell_{lj_2}^t = 0$ for $1 \leq j_1 \neq j_2 \leq k$, $\ell = 1, 2$ and π_l the real symmetric projections onto the k-planes spanned by all row vectors of L_l. s_l, Q_l, τ_l are as in Proposition 5.1. Then the formula (5.1) still holds. But the two different row vectors in Q_l are not orthogonal. Hence we can not have the permutability formulas continuously on the new solutions as Theorem 5.1.

§ 6 . The Examples

We first take a trivial solution $F = 0$ of (2.6). Solve (4.4) to get a family of unitary frames

$$\Phi_\lambda + iJ\Phi_\lambda = \mathrm{diag}[e^{i\lambda x_1}, \cdots, e^{i\lambda x_n}]. \tag{6.1}$$

In particular, $\Phi + iJ\Phi = \mathrm{diag}[e^{ix_1}, e^{ix_2}, \cdots, e^{ix_n}]$. In a sense of equivalence,

$$\begin{aligned} e_j &= [0, \cdots, 0, \cos x_j, -\sin x_j, 0, \cdots, 0]^t, \\ Je_j &= [0, \cdots, 0, \sin x_j, \cos x_j, 0, \cdots, 0]^t. \end{aligned}$$

Thus, we get a flat Lagrangian submanifold $X : S^1(r_1) \times \cdots \times S^1(r_n) \to C^n$ up to a constant vector defined by

$$X = [r_1 \sin x_1, r_1 \cos x_1, \cdots, r_n \sin x_n, r_n \cos x_n]^t, \tag{6.2}$$

with the first fundamental form $I = \sum_j r_j^2 dx_j^2$, where $r_1, \cdots, r_n$ are positive constant numbers. X is identified with

$$X + iJX = [r_1(\sin x_1 - i \cos x_1), \cdots, r_n(\sin x_n - i \cos x_n)]^t. \tag{6.3}$$

For simplicity, let $L_{1\times n} = [\ell_1, \cdots, \ell_n] \neq 0$. Then

$$\begin{aligned} V &= L(\Phi_{is} + iJ\Phi_{is}) = [\ell_1 e^{-sx_1}, \cdots, \ell_n e^{-sx_n}], \\ \widetilde{\pi} &= V^t(VV^t)^{-1}V = \sum_j \frac{e^{2sx_j}}{\ell_j^2} \begin{bmatrix} \ell_1^2 e^{-2sx_1} & \cdots & \ell_1\ell_n e^{-s(x_1+x_n)} \\ \vdots & & \vdots \\ \ell_1\ell_n e^{-s(x_1+x_n)} & \cdots & \ell_n^2 e^{-2sx_n} \end{bmatrix}. \end{aligned}$$

Denote

$$\Gamma = \frac{2\Big(\sum_j \ell_j r_j e^{-sx_j} + C\Big)}{(1+s^2)\Big(\sum_j \ell_j^2 e^{-2sx_j}\Big)}, \qquad \Delta = \frac{2s}{(1+s^2)\Big(\sum_j \ell_j^2 e^{-2sx_j}\Big)}.$$

By Corollary 4.1, we have

$$\phi = -\frac{1}{s}\Big(\sum_j \ell_j r_j e^{-sx_j} + C\Big), \qquad C \in R, \tag{6.4}$$

$$\widetilde{X} + iJ\widetilde{X} = [-ir_1 e^{ix_1} + (s+i)\ell_1\Gamma e^{(i-s)x_1}, \cdots, -ir_n e^{ix_n} + (s+i)\ell_n\Gamma e^{(i-s)x_n}]^t \tag{6.5}$$

with $d(\widetilde{X} + iJ\widetilde{X}) = \sum_j \tilde{b}_j(\tilde{e}_j + iJ\tilde{e}_j)dx_j$, where

$$\tilde{b}_j = r_j - \frac{2\ell_j e^{-sx_j}}{\sum_j \ell_j^2 e^{-2sx_j}}\Big(\sum_k \ell_k r_k e^{-sx_k} + C\Big), \tag{6.6}$$

$$\begin{aligned}\tilde{e}_j + iJ\tilde{e}_j = {} & [0, \cdots, 0, e^{ix_j}, 0, \cdots, 0]^t + [-(s+i)\ell_1\ell_j\Delta e^{-s(x_1+x_j)}e^{ix_1}, \cdots, \\ & -(s+i)\ell_j^2\Delta e^{-2sx_j})e^{ix_j}, \cdots, -(s+i)\ell_n\ell_j\Delta e^{-s(x_n+x_j)}e^{ix_n}]^t.\end{aligned} \tag{6.7}$$

The solution $\widetilde{F}$ of (2.6) corresponding to $\widetilde{X}$ is

$$\tilde{f}_{jk} = \frac{2s\ell_j\ell_k}{\sum_j \ell_j^2 e^{-2sx_j}} e^{-s(x_j+x_k)}. \tag{6.8}$$

In a sense of equivalence, we get a new flat Lagrangian submanifold $\widetilde{X} = [\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{2n-1}, \tilde{x}_{2n}]^t : R^n \to C^n$, where

$$\begin{cases}\tilde{x}_{2j-1} = r_j \sin x_j + \ell_j\Gamma e^{-sx_j}(s\cos x_j - \sin x_j),\\ \tilde{x}_{2j} = r_j\cos x_j - s\ell_j\Gamma e^{-sx_j}(s\sin x_j + \cos x_j)\end{cases} \tag{6.9}$$

with $U(n)$-frame $\widetilde{\Phi}_1 = (\tilde{e}_1, \cdots, \tilde{e}_n, J\tilde{e}_1, \cdots, J\tilde{e}_n)$, where $\tilde{e}_j = [\tilde{a}_{j,1}, \tilde{a}_{j,2}, \cdots, \tilde{a}_{j,2n-1}, \tilde{a}_{j,2n}]^t$,

$$\begin{cases}\tilde{a}_{j,2k-1} = \ell_j\ell_k\Delta e^{-s(x_j+x_k)}(\sin x_k - s\cos x_k),\\ \tilde{a}_{j,2k} = \ell_j\ell_k\Delta e^{-s(x_j+x_k)}(\cos x_k + s\sin x_k) \qquad \text{for } k \neq j,\end{cases} \tag{6.10}$$

$$\begin{cases}\tilde{a}_{j,2j-1} = r_j + \ell_j^2\Delta e^{-2sx_j}(\sin x_j - s\cos x_j),\\ \tilde{a}_{j,2j} = \ell_j^2\Delta e^{-2sx_j}(\cos x_j + s\sin x_j).\end{cases} \tag{6.11}$$

It is evident that $\widetilde{X}$ is always non-degenerate if $L \neq 0$. Assume now that the image of X lies in S^{2n-1}, i.e., $\sum_j r_j^2 = 1$. On the other hand, the image of $\widetilde{X}$ lies in S^{2n-1} if and only if $C = 0$. Hence, when $C = 0$, $\widetilde{X}$ projects onto a flat $(n-1)$-dimension Lagrangian submanifold of CP^{n-1}. In this case, we can represent the corresponding formulas with $C = 0$.

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