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LOCAL WELL-POSEDNESS AND ILL-POSEDNESS ON THE EQUATION OF TYPE $\Box u = u^k(\partial u)^\alpha$***

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Abstract

This paper undertakes a systematic treatment of the low regularity local well-posedness and ill-posedness theory in H^s and $\dot{H}^s$ for semilinear wave equations with polynomial nonlinearity in u and ∂u. This ill-posed result concerns the focusing type equations with nonlinearity on u and $\partial_t u$.

Keywords Semilinear wave equation, Low regularity, Local well-posedness, Ill-posedness
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§ 1. Introduction

The goal of this paper is to study the low regularity local well-posedness and ill-posedness of the Cauchy problem for semi-linear wave equation with polynomial nonlinearity in u and ∂u. More precisely, we are concerned with the question to determine the smallest s (denoted by s_o) for which we have local well-posedness in H^s (or $\dot{H}^s$) of the problem

$$\begin{cases} \Box u := (\partial_t^2 - \Delta) u = G(u, \partial u), \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x), \end{cases} \tag{1.1}$$

where the nonlinearity G is polynomial with respect to its arguments and $\partial := (\partial_t, \nabla_x)$.

We say that the problem (1.1) is local well-posed (LWP) with X in H^s if, for every $(f, g) \in H^s \times H^{s-1}$, there exists a unique weak solution $u \in C([0,T]; H^s) \cap X$ to (1.1), which depends continuously on the data for some $T > 0$, where X is some reasonable additional Banach space. Moreover, if T depends only on the data's size, we say the problem is norm-LWP. When $X = C^1([0,T]; H^{s-1})$, we say it is classically LWP. Similarly, we say the problem (1.1) is ill-posed (ILP) in H^s if the problem is in contrast to the meaning of LWP. The corresponding notion for $\dot{H}^s$ is just the substitute of H^s by $\dot{H}^s$. In this paper, all LWP results will be norm-LWP and we have two senses of ILP. We say a problem is strongly ILP (s-ILP) if there is a sequence of data (f_j, g_j), which are smooth and supported in a ball B_{R_j}, for which the lifespan of the corresponding solutions u_j, the data's norm and

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R_j goes to 0, by a domain of dependence argument. The weaker ILP (w-ILP) means that the lifespan goes to zero and the data's norm stays uniformly bounded.

The question of determining the minimal regularity index s for which (1.1) is LWP in H^s or $\dot{H}^s$ has been studied in many papers, including [1, 5, 6, 8–11, 13–16]. However, to our knowledge, there is not a systematic treatment especially for ILP with G depends not only on u yet, although these results have been heuristically believed to be true since [8–10] appeared.

Now we state our main results and give several remarks, comparing them with previously known results. For an equation of the form

$$\Box u = Cu^k(\partial u)^\beta \tag{1.2}$$

with $|\beta| = l$, $k + l > 1$ and $k, l \in \mathbb{N}$, set $\alpha := \frac{l-2}{k+l-1}$ and $s_c(k, l) := \frac{n}{2} + \alpha$. Then we have the following norm-LWP results given in Section 2.

Theorem 1.1. *The equation* (1.2) *is norm-LWP with some X in H^s for*

$$s > \begin{cases} s(l,n) = \dfrac{n+5}{4} \bigvee s_c(0,l), & l \geq 2 \quad \textit{and} \quad n \geq 2, \\ \sigma(k,n) = \dfrac{n-1}{2} \bigvee s_c(k,1), & l = 1 \quad \textit{and} \quad n \geq 3. \end{cases}$$

Remark 1.1. The result in Theorem 1.1 covers the one in [11], [1] in part. In [11], Ponce and Sideris proved the special $n = 3$ case of (1.2). In [1], Beals and Bézard gave the result in case of $l = 2$ and $n \geq 3$, showing that (1.2) is LWP for $s > 2$ with $n = 3$ and $s \geq \frac{n+1}{2}$ for $n \geq 4$. For any $n \in \mathbb{N}$, the classically LWP results are well-known with $s > \frac{n+2}{2}$ for (1.1) and $s > \frac{n}{2}$ for (1.1) with G at most linear on ∂u, by classical energy arguments (e.g. in [4]). Note that $s(k, n) \to \frac{n+2}{2}$ and $\sigma(k, n) \to \frac{n}{2}$ as $k \to \infty$.

Remark 1.2. In fact, we only need to give the proof of Theorem 1.1 with $s > s(l, n) = \frac{n+5}{4} \vee \frac{n+1}{2} \vee s_c(0, l)$ for the $l \geq 2$ case, since for $l = 2$, the problem (1.2) has been extensively studied. In fact, in this case, it is LWP with some X in H^s with $s > \frac{n+5}{4} \vee \frac{n}{2}$ for any $n \in \mathbb{N}$. The $n = 3$ case was given in [11]. In contrast, by introducing some new Banach space, Tataru got the LWP result for $s > \frac{n}{2}$ with $n \geq 5$ in [15]. And for $n = 4$, it was claimed there without proof. Recently Zhou have given the proof of $n = 4$ and $n = 2$ case in [16]. Combining these observations, one can exclude the $\frac{n+1}{2}$ term in $s(l, n)$.

On the other hand, in Sections 3 and 4, we give some ILP results (mainly the s-ILP results) for focusing type equation

$$\Box u = |u|^k |\partial_t u|^{l-1} \partial_t u \tag{1.3}$$

with $k + l > 1$, $k, l \geq 0$ and $k, l \in \mathbb{R}$ (for $l = 0$, this equation is interpreted as $\Box u = |u|^{k-1}u$). Set $\tilde{s} := \frac{n+1}{4} + \frac{l-1}{k+l-1}$. Then we have the following

Theorem 1.2. *Let $n \geq 1$, $\alpha \neq 0$ or $\alpha = 0$ with $k = 0$. The model equation* (1.3) *is s-ILP in $\dot{H}^s$ for*

$$s \in \Big(1 - \frac{n}{2}, s_c\Big).$$

And, if $s_c > 1$, then it is s-ILP in H^s for the same s. Moreover, if $\alpha \neq 0$, it is w-ILP in $\dot{H}^{s_c}$; it is also w-ILP in H^{s_c} if $s_c \geq 1$. If $\alpha = 0$ with $k = 0$, then it is w-ILP in H^s for $s \in (1 - \frac{n}{2}, s_c)$.

Theorem 1.3. *Let $n \geq 3$, $\alpha \neq 0$ and $\tilde{s} > s_c$. Then we have s-ILP in $\dot{H}^s$ for (1.3) with s lying in the following set (denoted by $\mathbf{E}$) :*

- *if $l > 2$, $(s_c, \tilde{s})$;*
- *if $l < 2$ and $\tilde{s} > 0$, $(-\frac{n}{2} \vee s_c, \tilde{s} \wedge \frac{n}{2})$;*
- *if $l < 2$ and $\tilde{s} \leq 0$, $(-\frac{n}{2} \vee s_c, \frac{2n}{n+1}\tilde{s})$.*

Furthermore, we have s-ILP in H^s for $0 < s \in \mathbf{E}$ and also for $s \in (-\frac{n}{2}, 0] \cap (s_c, \tilde{s})$ if there exist $m, \widetilde{m} \in \mathbb{Z} \cap [0, \frac{n}{2})$ such that

$$\tilde{s} > \Big(\frac{n-1}{2n} m\Big) \bigvee \Big(\frac{n-1}{2n}\widetilde{m} + 1\Big) > \frac{n+1}{4} + \alpha.$$

Remark 1.3. Note that if one asks for the meaning of LWP one more condition (persistence of regularity), then we would get more than stated in Theorem 1.2. In principle, one could get s-ILP in H^s with $s < s_c$ subject to $s_c > 1$ under the condition that we have the result of formation of singularity such as in [7] and [12].

Remark 1.4. Since we give s-ILP in the sense as described above, for any "reasonable" X (here "reasonable" means that it contains all solutions with C_0^∞ data), one could not expect any result of LWP in H^s with domain of dependence. Note that for w-ILP, one could not attempt to prove norm-LWP with domain of dependence (e.g. through contraction argument), however one may show LWP with lifespan depending not only on the data's size, but also on its profile.

By applying the super-critical results in Theorem 1.2 and the sub-critical results in Theorem 1.3 to some particular forms of (1.3), we have some corollaries as follows.

Corollary 1.1. *Let $n \geq 3$, $2 < l \in \mathbb{R}$ and $\alpha = 1 - \frac{1}{l-1}$. Set $s_c = \frac{n}{2} + \alpha$ and $\tilde{s} = \frac{n+5}{4}$. Then the equation*

$$\Box u = |\partial_t u|^{l-1} \partial_t u \tag{1.4}$$

is s-ILP in $\dot{H}^s$ for

$$s \in \begin{cases} \Big(1 - \dfrac{n}{2}, s_c\Big), & l \geq \dfrac{n+3}{n-1} \quad \text{and} \quad l > 2, \\ \Big(1 - \dfrac{n}{2}, s_c\Big) \cup (s_c, \tilde{s}), & 2 < l < \dfrac{n+3}{n-1} \quad \text{and} \quad n = 3, 4. \end{cases}$$

Moreover, for $n \geq 2$, it is also s-ILP in $\dot{H}^s$ and H^s for $s \in (1 - \frac{n}{2}, s_c)$, and w-ILP in $\dot{H}^{s_c}$ and H^{s_c} .

Corollary 1.2. *Let $n \geq 3$, $1 < k \in \mathbb{R}$ and $\alpha = -\frac{1}{k-1}$. Set $s_c = \frac{n}{2} + \alpha$ and $\tilde{s} = \frac{n+1}{4}$. Then the equation*

$$\Box u = |u|^{k-1} \partial_t u \tag{1.5}$$

is w-ILP in $\dot{H}^{s_c}$ *with* $k > \frac{n}{n-1}$ *and s-ILP in* $\dot{H}^s$ *for*

$$s \in \begin{cases} \Big(1 - \dfrac{n}{2}, s_c\Big), & \text{if } \ k \geq \dfrac{n+3}{n-1}, \\ \Big(1 - \dfrac{n}{2}, s_c\Big) \cup (s_c, \tilde{s}), & \text{if } \ k \in \Big(\dfrac{n}{n-1}, \dfrac{n+3}{n-1}\Big), \\ \Big(-\dfrac{n}{2} \vee s_c, \tilde{s}\Big), & \text{if } \ k < \dfrac{n}{n-1}. \end{cases}$$

It is also w-ILP in H^{s_c} *with* $k \geq \frac{n}{n-2}$ *and s-ILP in* H^s *for*

$$s \in \begin{cases} \Big(1 - \dfrac{n}{2}, s_c\Big) & \text{with } \ k \geq \dfrac{n+3}{n-1} \text{ and } n \geq 3, \\ \Big(1 - \dfrac{n}{2}, s_c\Big) \cup (s_c, \tilde{s}) & \text{with } \ k \in \Big(\dfrac{n}{n-2}, \dfrac{n+3}{n-1}\Big) \text{ and } n \geq 4, \\ \Big(-\dfrac{n}{2} \vee s_c, \tilde{s}\Big) & \text{with } \ k < \dfrac{n}{n-2} \text{ and } n \geq 4. \end{cases}$$

In particular, the problem is s-ILP in H^s *with* $s \in ((s_c \vee \tilde{s}) - \epsilon, s_c \vee \tilde{s})$*, subject to* $n = 3$ *with* $k \geq 3$ *or* $n \geq 4$*.*

Corollary 1.3. *Let* $n \geq 3$*,* $1 < k \in \mathbb{R}$*,* $\alpha = -\frac{2}{k-1}$*, and set* $s_c = \frac{n}{2} + \alpha$*,* $\tilde{s} = \frac{n+1}{4} + \frac{\alpha}{2}$*. Then the equation*

$$\Box u = |u|^{k-1} u \tag{1.6}$$

is w-ILP in $\dot{H}^{s_c}$ *with* $k > \frac{n+1}{n-1}$ *and s-ILP in* $\dot{H}^s$ *for*

$$s \in \begin{cases} \Big(1 - \dfrac{n}{2}, s_c\Big), & \text{if } \ k \geq \dfrac{n+3}{n-1}, \\ \Big(1 - \dfrac{n}{2}, s_c\Big) \cup (s_c, \tilde{s}), & \text{if } \ k \in \Big[\dfrac{n+5}{n+1}, \dfrac{n+3}{n-1}\Big), \\ \Big(1 - \dfrac{n}{2}, s_c\Big) \cup \Big(s_c, \dfrac{2n}{n+1}\tilde{s}\Big), & \text{if } \ k \in \Big(\dfrac{n+1}{n-1}, \dfrac{n+5}{n+1}\Big) \text{ and } n \geq 4, \\ \Big(-\dfrac{n}{2} \vee s_c, \dfrac{2n}{n+1}\tilde{s}\Big), & \text{if } \ k \in \Big(\dfrac{n+3}{n+1}, \dfrac{n+1}{n-1}\Big). \end{cases}$$

It is also w-ILP in H^{s_c} *with* $k \geq \frac{n+2}{n-2}$ *and s-ILP in* H^s *for* $s \in (1 - \frac{n}{2}, s_c)$ *if* $k > \frac{n+2}{n-2}$*.*

Note that for equation

$$\Box u = (\partial_t u)^2/2, \tag{1.7}$$

if we set $v = \partial_t u$, then v satisfies (1.5) with $k = 2$. So we can give the subcritical ill-posedness for (1.7).

Theorem 1.4. *The equation* (1.7) *is s-ILP in* $\dot{H}^s$ *and* H^s *for* $s \in (1 - \frac{n}{2}, s_c)$ *with* $n \geq 3$ *and w-ILP in* $\dot{H}^s$ *and* H^s *for such* s *with* $n = 2$*. Moreover, for* $n = 3, 4$*, it is also s-ILP in* $\dot{H}^s$ *for* $s \in (s_c, \tilde{s})$*.*

Remark 1.5. In [8] and [9], Lindblad gave the ILP in $\dot{H}^s$ with $s = \tilde{s} - \epsilon$ and $n = 3$ for some types of (1.2) with $k + l = 2$ and $k = 0, 1, 2$, with small $\epsilon \geq 0$. And, in [10], it was shown that (1.3) is ILP in $\dot{H}^s$ with $(s_c \vee \tilde{s}) - \epsilon < s < s_c \vee \tilde{s}$ for

$$k > k_0 = \frac{(n+1)^2}{(n-1)^2 + 4}$$

with $n \geq 3$. However note that

$$1 + \frac{4}{n+1} \leq \frac{(n+1)^2}{(n-1)^2 + 4},$$

so Corollary 1.3 generalizes the result there.

Remark 1.6. The ILP results and LWP results above suggest that the optimal regularity should be $s_c \vee \tilde{s}$ for (1.2) in general. In particular, for (1.4) with $2 \leq l \in \mathbb{N}$, the optimal regularity in H^s is s_c when $s_c \geq \tilde{s}$ (i.e., $n \geq 5$ or $l \geq 3$ with $n = 3, 4$ or $l \geq 5$ with $n = 2$). For (1.5) with $k \geq 3$ and $n \geq 4$ or $k \geq 4$ and $n = 3$, the optimal regularity in H^s is s_c.

Remark 1.7. However, there are still some gaps, since we give ILP result mainly for $\dot{H}^s$. It is logically natural that there may be the case that a particular problem is ILP in $\dot{H}^s$ but LWP in H^s.

Our main strategy for the proof is the following. Firstly, we use Strichartz-type estimate and generalized Leibniz rule to get LWP result by usual contraction arguments. Secondly, for super-critical ILP ($s < s_c$), we are concerned with the model equation (1.3) with $k+l > 1$, $k, l \geq 0$ and $k, l \in \mathbb{R}$. Taking ODE solution in t and cutting off the data outside a ball, we get a sequence of data and show that the lifespan of solution and data's norm goes to zero simultaneously, which gives the s-ILP in $\dot{H}^s$. Then we show it also works for H^s with the same s once one can assure the data's L^2-norm goes to zero (or remains bounded for w-ILP) during the limitation. Thirdly, for sub-critical s-ILP, our strategy for the proof originates mainly in [8] and [10]. Roughly speaking, we apply scaling and Lorentz transformation on ODE solution in t, then cut-off the solution appropriately so that its lifespan decreases as the initial norm decreases.

In the sequel, we will use the following notations. As usual, $\mathcal{S}$ denotes the spaces of Schwartz classes and $\mathcal{S}'$ denotes its dual space (the tempered distribution space). We use $\mathcal{F}$ to denote the usual Fourier transform,

$$\mathcal{F}(f) = \hat{f} := \int e^{-ix\cdot\xi} f(x)dx \qquad \text{for } f \in \mathcal{S},$$

and for $f \in \mathcal{S}'$ define it by duality. Set $\mathrm{D} = (-\Delta)^{1/2}$, and use $\mathrm{U}(t)$ and $\mathrm{U}'(t)$ to denote the operator $\sin(t\mathrm{D})/\mathrm{D}$ and $\cos(t\mathrm{D})$. We use $H^{s,p}(\mathbb{R}^n)$ to denote the Sobolev space,

$$H^{s,p} := \{f \in \mathcal{S}', (1-\Delta)^{s/2} f \in L^p\} \supset \mathcal{S}$$

with norm $\|f\|_{H^{s,p}} = \|(1-\Delta)^{s/2} f\|_{L^p}$ for $s \in \mathbb{R}$ and $1 < p < \infty$. Correspondingly, the homogeneous Sobolev space $\dot{H}^{s,p}(\mathbb{R}^n)$ is defined as the closure of $\mathcal{S}$ with the norm $\|f\|_{\dot{H}^{s,p}} := \|\mathrm{D}^s f\|_{L^p}$ in $\mathcal{S}'$ for $sp + n > 0$. For $p = 2$, we use $H^s(\dot{H}^s)$ to denote $H^{s,2}(\dot{H}^{s,2})$, and the corresponding norm for f is denoted by $\|f\|_s(\|f\|_{\dot{s}})$. Moreover, we use $\|f(t,\cdot)\|_{(s)}$ to denote $\|f(t,\cdot)\|_s + \|\partial_t f(t,\cdot)\|_{s-1}$, similarly for homogeneous $\|f(t,\cdot)\|_{(\dot{s})}$. The notation $a \lesssim b$ means that there exists a constant $C > 0$ such that $a \leq Cb$, and $a \approx b$ means that there exist constants $C_1 > C_2 > 0$ such that $C_2 b \leq a \leq C_1 b$. For $x, y \in \mathbb{R}$, we use $x\pm$ to denote $x \pm \epsilon$, $x \vee y$ (or $x \wedge y$) to denote $\max\{x, y\}$ (or $\min\{x, y\}$). $[x]$ represents the integer part of x, i.e., $[x] = \max\{d \in \mathbb{Z} : d \le x\}$. Also $H(x)$ denotes the usual Heaviside function ($H(x)$ equals 1 for $x \ge 0$ and 0 for $x < 0$).

This paper is organized as follows. In Section 2, we give the proof of LWP result. In Sections 3 and 4, we prove the super-critical ILP and sub-critical s-ILP separately.

§ 2 . Local Well-Posedness

In this section, we give the proof of LWP result. We shall need some usual technical lemmas including generalized Leibniz rule and Strichartz type estimate.

Let us first recall the familiar Sobolev estimate.

Lemma 2.1. (Sobolev Inequality) *For $\frac{n}{p} - s = \frac{n}{q}$ with $1 < p \le q < \infty$, we have*

$$\|f\|_{L^q} \lesssim \|\mathrm{D}^s f\|_{L^p}.$$

In contrast, for $sp > n$, $H^{s,p} \subset L^\infty$.

The usual interpolation (cf. [2]) yields similar estimate for $H^{s,p}$, $H^{s,p} \subset L^q$ with $\frac{n}{p} - s \le \frac{n}{q}$, $q \ge p$ and $s > 0$.

Lemma 2.2. (Generalized Leibniz Rule) *If $\frac{1}{2} = \frac{1}{p_i} + \frac{1}{q_i}$ with $i = 1, 2$, $2 \le q_i \le \infty$ and $s > 0$, then we have*

$$\|\mathrm{D}^s(fg)\|_{L^2} \lesssim \|\mathrm{D}^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|\mathrm{D}^s g\|_{L^{p_2}} \|f\|_{L^{q_2}}. \tag{2.1}$$

The inhomogeneous counterpart is also valid.

One may refer to [8] for it.

Lemma 2.3. (Strichartz-Type Estimate) *Denote admissible pair set by*

$$AD = \Big\{(r,q) \,\Big|\, \frac{1}{r} \le \frac{n-1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big),\ 2 \le r \le \infty,\ 2 \le q < \infty,\ n \ge 2\Big\}.$$

Let $s = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{r}$, and $(r,q), (r_1, q_1) \in AD$. Then we have for all $T > 0$,

$$\|u\|_{L_t^r L_x^q} + \sup_{t \in [0,T]} \|u(t)\|_{(\dot{s})} \lesssim \|u(0)\|_{(\dot{s})} + \|\Box u\|_{L_t^{r_1'} L_x^{q_1'}}. \tag{2.2}$$

Moreover, if $s \ge 1$, then

$$\|U(t) g(x)\|_{L_t^r L_x^q} \lesssim \|g\|_{s-1}. \tag{2.3}$$

For the proof, see [3] and [6].

We shall also use the energy estimate

$$\|u(t)\|_{(s)} \lesssim (1+t)\Big(\|u(0)\|_{(s)} + \int_0^t \|(\Box u)(\tau, \cdot)\|_{s-1} d\tau\Big). \tag{2.4}$$

Now we are prepared to prove LWP result Theorem 1.1. We restate the result here more precisely.

Theorem 2.1. *Let $n \geq 2$, $|\beta| = l \geq 2$, $k \geq 0$ and $n, k, l \in \mathbb{Z}$. Then the equation*

$$\Box u = Cu^k(\partial u)^\beta := G$$

is LWP in H^s for $s > \frac{n+5}{4} \vee \frac{n+1}{2} \vee (\frac{n+2}{2} - \frac{1}{l-1})$. Here the solution space is

$$\mathrm{V} = \{u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1});\ \partial u \in L^r([0,T]; \dot{H}^{\gamma-1,q})\}$$

for some γ, r and q, and $T > 0$ depending only on n, s and $\|f\|_s + \|g\|_{s-1}$. Here the values of γ, r and q may be chosen to be $\gamma = s - \frac{1}{r} - n(\frac{1}{2} - \frac{1}{q})$, $1/q = 0+$,

$$\frac{1}{r} = \frac{1}{2} \bigwedge \Big(\frac{1}{l-1} -\Big) \bigwedge \frac{n-1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big).$$

Furthermore, the solution map is uniformly continuous depending on the data.

Proof. As usual, we use contraction principle. Let V be as in Theorem 2.1,

$$\|u\|_{\mathrm{V}} := \sup_{[0,T]} \|u(t)\|_s + \sup_{[0,T]} \|\partial_t u(t)\|_{s-1} + \|\partial u\|_{L^r([0,T];\dot{H}^{\gamma-1,q})}$$

and set

$$\Lambda u(t,x) := (\mathrm{U}'(t)f + \mathrm{U}(t)g) + \int_0^t \mathrm{U}(t-\tau)G(\tau,\cdot)d\tau := u^{(0)} + \mathrm{A}u.$$

Then in view of (2.2) and (2.4), we have

$$\|u^{(0)}\|_{\mathrm{V}} \lesssim (1+T)(\|f\|_s + \|g\|_{s-1}), \tag{2.5}$$

$$\|\mathrm{A}u\|_{\mathrm{V}} \lesssim (1+T)\Big(\int_0^T \|G(\tau)\|_{s-1} d\tau\Big). \tag{2.6}$$

To estimate $\|G(\tau)\|_{s-1}$, since

$$s > \frac{n}{2}, \tag{2.7}$$

we may choose p, $\tilde{p} > 2$ such that $H^s \subset H^{s-1,p}$, $H^{s-1} \subset L^{\tilde{p}}$ and $\frac{1}{p} + \frac{1}{\tilde{p}} = \frac{1}{2}$. Then by (2.1),

$$\begin{aligned}\|G\|_{s-1} &\lesssim \|u^k\|_{L^\infty} \|(\partial u)^\beta\|_{s-1} + \|u^k\|_{H^{s-1,p}} \|(\partial u)^\beta\|_{L^{\tilde{p}}} \\ &\lesssim \|u^k\|_s \|\partial u\|_{L^\infty}^{l-1} \|\partial u\|_{s-1} \lesssim \|u\|_{(s)}^{k+1} \|\partial u\|_{L^\infty}^{l-1}.\end{aligned}$$

However, if we assume

$$(\gamma - 1)q > n, \tag{2.8}$$

then

$$\|\partial u\|_{L^\infty} \lesssim \|\partial u\|_{H^{\gamma-1,q}} \lesssim \|\partial u\|_{L^2} + \|\mathrm{D}^{\gamma-1}\partial u\|_{L^q},$$

and hence

$$\|G\|_{s-1} \lesssim \|u\|_{(s)}^{k+l} + \|u\|_{(s)}^{k+1} \|\mathrm{D}^{\gamma-1}\partial u\|_{L^q}^{l-1}. \tag{2.9}$$

Combining (2.6) and (2.9), we may use Hölder inequality to get

$$\begin{aligned}\|\mathrm{A}u\|_{\mathrm{V}} &\lesssim T(1+T)\sup_{[0,T]}\|u\|_{(s)}^{k+l} + (1+T)\sup_{[0,T]}\|u\|_{(s)}^{1+k}\int_0^T \|\mathrm{D}^{\gamma-1}\partial u\|_{L^q}^{l-1}d\tau \\ &\lesssim (T+T^2)\|u\|_{\mathrm{V}}^{k+l} + (1+T)\|u\|_{\mathrm{V}}^{1+k}T^{1-(l-1)/r}\|D^{\gamma-1}\partial u\|_{L_t^rL_x^q}^{l-1} \\ &\lesssim (T^{1-(l-1)/r}+T^2)\|u\|_{\mathrm{V}}^{k+l},\end{aligned}$$

if

$$r > l-1. \tag{2.10}$$

In view of (2.5) we have

$$\|\Lambda u\|_{\mathrm{V}} \le C\{(1+T)(\|f\|_s + \|g\|_{s-1}) + (T^{1-(l-1)/r}+T^2)\|u\|_{\mathrm{V}}^{k+l}\}. \tag{2.11}$$

So, if we let $R = 2C(\|f\|_s + \|g\|_{s-1})$ and set $\mathrm{B} = \{u \in \mathrm{V} : \|u\|_{\mathrm{V}} \le R\}$, we can choose $T_0 > 0$ small, such that for any $T < T_0$, $\Lambda : \mathrm{B} \to \mathrm{B}$. Essentially in the same manner, we can show that for T small enough, Λ is a contraction mapping in B and so there is a unique solution in V with data (f, g). Furthermore, the solution map is uniformly continuous on the data.

Now the only remaining thing is to determine the maximal range of s satisfying (2.7), (2.8), (2.10) and the conditions in Strichartz-Type estimate (i.e., $\gamma = s + \frac{1}{r} - n(\frac{1}{2} - \frac{1}{q})$, $\frac{2}{r} \le (n-1)(\frac{1}{2} - \frac{1}{q})$, with $\frac{1}{r} \in [0, \frac{1}{2}]$ and $\frac{1}{q} \in (0, \frac{1}{2}]$). A direct computation yields the required

$$s > \max\Big\{\frac{n+5}{4}, \frac{n+1}{2}, \frac{n+2}{2} - \frac{1}{l-1}\Big\},$$

and one may check that the values of γ, r and q stated in Theorem 2.1 satisfy these conditions. So the result is proved.

Theorem 2.2. *Let $n \ge 3$, $|\beta| = 1$, $k \ge 2$ and $k, n \in \mathbb{N}$. Then the equation*

$$\Box u = Cu^{k-1}(\partial u)^\beta := F$$

is LWP in H^s for $s > \frac{n-1}{2} \vee (\frac{n}{2} - \frac{1}{k-1})$. Here the solution space is

$$\mathrm{W} = \{u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1});\ u \in L^r([0,T]; H^{s-1,q})\}$$

for some r, q and $T > 0$ depending only on n, s and $\|f\|_s + \|g\|_{s-1}$. Here the values of r, q may be chosen to be $\frac{1}{r} = (\frac{1}{k-1}-) \wedge \frac{1}{2}$ and $\frac{n}{q} = \frac{n}{2} - 1 - \frac{1}{r}$. Furthermore, the solution is uniformly continuous depending on the data.

Proof. Let $\|u\|_{\mathrm{W}} := \sup_{[0,T]}\|u(t)\|_s + \sup_{[0,T]}\|\partial_t u(t)\|_{s-1} + \|u\|_{L^r([0,T];H^{s-1,q})}$ with T to be determined later and set

$$\Lambda u(t,x) = \mathrm{U}'(t)f + \mathrm{U}(t)g + \int_0^t \mathrm{U}(t-\tau)F(\tau,\cdot)d\tau := u^{(0)} + \mathrm{A}u.$$

Then by (2.3) and (2.4), we have for $0 \leq t \leq T$,

$$\|u^{(0)}\|_{\mathrm{W}} \lesssim (1+T)(\|f\|_s + \|g\|_{s-1}),$$

$$\|\mathrm{A}u\|_{\mathrm{W}} \lesssim (1+T)\int_0^t \|F(\tau,\cdot)\|_{s-1} d\tau.$$

Since $(s-1)q > n$, one may choose p such that $H^{s-1} \subset L^p$ and $H^{s-1,q} \subset L^\infty$ with $\frac{1}{p} = \frac{1}{2} - \frac{1}{q}$. Then by Lemma 2.2, we have

$$\|F\|_{s-1} \lesssim \|u^{k-1}\|_{L^\infty}\|\partial u\|_{s-1} + \|u^{k-1}\|_{H^{s-1,q}}\|\partial u\|_{L^p} \lesssim \|u\|^{k-1}_{H^{s-1,q}}\|u\|_{(s)}.$$

So for $r > k-1$,

$$\|\mathrm{A}u\|_{\mathrm{W}} \lesssim (1+T)\int_0^T \|F(\tau,\cdot)\|_{s-1} d\tau \lesssim T^{1-(k-1)/r}(1+T)\|u\|^k_{\mathrm{W}}. \tag{2.12}$$

Therefore we have

$$\begin{aligned}\|\Lambda u\|_{\mathrm{W}} &\leq C(1+T)(\|f\|_s + \|g\|_{s-1}) + CT^{1-(k-1)/r}(1+T)\|u\|^k_{\mathrm{W}} \\ &= C(\|f\|_s + \|g\|_{s-1}) + CT^{1-(k-1)/r}\{(1+T)\|u\|^k_{\mathrm{W}} + T^{(k-1)/r}(\|f\|_s + \|g\|_{s-1})\} \\ &\leq 2C(\|f\|_s + \|g\|_{s-1}) := R\end{aligned}$$

with $\|u\|_{\mathrm{W}} \leq R$ and $T < T_0$ with T_0 small enough.

Hence for T small, Λ becomes a contraction map on some ball $u \in \mathrm{W} : \|u\|_{\mathrm{W}} \leq R$. And the proof is reduced to the algebraic computation of assumption on s, i.e., $(s-1)q > n$, $r > k-1$ and the constraint in Strichartz estimate. A simple computation yields the desired

$$s > \sigma(k,n) = \max\left\{\frac{n-1}{2},\ \frac{n}{2} - \frac{1}{k-1}\right\},$$

and the values of $r,\ q$ stated in Theorem 2.2 can be checked to satisfy these conditions.

§ 3. Super-critical Counterexample

In this section we are concerned with the super-critical (i.e., $s < s_c$) ill-posedness in H^s and $\dot{H}^s$, i.e., the proof of Theorem 1.2. The method is classical. First we get some ODE solutions in t which blow-up in finite time and cut off them outside a ball. Then by estimating the data's norm and the corresponding lifespan we show that the problem is ILP. The results in this and next sections are both heuristically believed to be true; however to our knowledge, there is no systematic proof of this fact yet.

In the sequel, we shall consider the model focusing problems

$$\Box u = |u|^k |\partial_t u|^{l-1}\partial_t u \tag{3.1}$$

with $k+l > 1$, $k, l \geq 0$ and $k, l \in \mathbb{R}$ (for $l = 0$, this equation is interpreted as $\Box u = |u|^{k-1}u$). A simple scaling argument shows that

$$s_c = \frac{n}{2} + \alpha \tag{3.2}$$

with $\alpha := \frac{l-2}{k+l-1} < 1$. Note that (3.1) has a simple ODE solution in t $(0 \le t < a < 1)$:

$$u_a(x,t) = \begin{cases} C_{k,l}(a-t)^\alpha, & \alpha \neq 0, \\ -\log(a-t), & l = 2 \text{ and } k = 0 \end{cases} \tag{3.3}$$

with $C_{k,l} := \{|\alpha|^{1-l}(1-\alpha)\}^{1/(k+l-1)}$.

It is shown that there are three cases to be proved: $\alpha < 0$, $\alpha > 0$ and $\alpha = 0$, i.e., $l < 2$, $l > 2$ and $l = 2$. We split this theorem into three propositions concerning the three cases respectively. For the convenience of proof, we give a definition.

Definition 3.1. *If for some given data (f,g), (3.1) has local unique distributional solution u in $\dot{H}^s$ with maximal time T_0, we define the lifespan $T^s(f,g)$ to be*

$$T^s(f,g) = \sup\{t : 0 < t < T_0,\ u \in C([0,t], \dot{H}^s) \cap C^1([0,t], \dot{H}^{s-1})\}.$$

Let $\phi \in C_0^\infty(\mathbb{R})$ with $\phi(\tau) = 1$ for $|\tau| \le 1 + d$ with some $d > 0$ and set

$$f_a = u_a(x,0)\phi(|x|/a), \qquad g_a = \partial_t u_a(x,0)\phi(|x|/a). \tag{3.4}$$

We shall need the following lemmas. Note that for $s > -\frac{n}{2}$, $C_0^\infty(\mathbb{R}^n) \subset \dot{H}^s$, i.e.,

$$\|\phi\|_{\dot{s}} < \infty. \tag{3.5}$$

Lemma 3.1. *For f_a, g_a as above, we have*

$$\|\mathrm{D}^s f_a\|_{L^2} + \|\mathrm{D}^{s-1} g_a\|_{L^2} \lesssim a^{s_c - \epsilon - s} \tag{3.6}$$

with

$$\begin{cases} \epsilon = 0, & \alpha \neq 0, \\ \epsilon > 0, & l = 2 \ \text{ and } \ k = 0, \end{cases}$$

if $s > 1 - \frac{n}{2}$ and $0 < a < 1$.

Proof. For $\alpha \neq 0$, by homogeneity and (3.5), we have

$$\|\mathrm{D}^s f_a\|_{L^2} + \|\mathrm{D}^{s-1} g_a\|_{L^2} \lesssim a^{-s+\alpha+n/2}(\|\mathrm{D}^s \phi\|_{L^2} + \|\mathrm{D}^{s-1}\phi\|_{L^2}) \lesssim a^{s_c - s}$$

subject to $s > 1 - \frac{n}{2}$. For the other case, noticing that $|\log a| \lesssim a^{-\epsilon}$ for any $\epsilon > 0$, we also have

$$\|\mathrm{D}^s f_a\|_{L^2} + \|\mathrm{D}^{s-1} g_a\|_{L^2} \lesssim a^{-s+n/2-\epsilon}\|\mathrm{D}^s\phi\|_{L^2} + a^{-s+n/2}\|\mathrm{D}^{s-1}\phi\|_{L^2} \lesssim a^{s_c-\epsilon-s}.$$

Let us first consider the case $\alpha < 0$.

Proposition 3.1. *Let $n \ge 2$. Then the problem (3.1) with $\alpha < 0$ is s-ILP in $\dot{H}^s$ for $s \in (1 - \frac{n}{2}, s_c)$. And, if $s_c > 1$, then it is s-ILP in H^s for the same s. Moreover, it is w-ILP in $\dot{H}^{s_c}$ and if $s_c \ge 1$, it is w-ILP in H^s for $s \in (1 - \frac{n}{2}, s_c]$.*

Proof. We shall show that the above data (f_a, g_a) gives the required counterexample. Without loss of generality, we assume that the solution is locally unique for Equation (3.1) with such data (since if not, the result will follow without any further proof). Then solution u of (3.1) with such data is just u_a in $|x| \le (1+d)a - t$ for $t < T^s(f_a, g_a)$. Now we claim that the lifespan $T^s(f_a, g_a)$ of u with $|s| < \frac{n}{2}$ satisfies

$$T^s(f_a, g_a) \le a. \tag{3.7}$$

We use Lemma 4.4 with $h(a,x) = F(\frac{|x|}{a})$, where $F(t) \in C_0^\infty$ vanishes for $|t| \ge 1$. Then if $T^s(f_a, g_a) > a$, we have (use Scaling, (3.5), $\text{supp}(h(a,x)) \subset \{|x| \le a\}$ and $u = u_a$ in $|x| \le (1+d)a - t$ for $t < a$)

$$\|u(t)\|_{\dot{s}} \gtrsim \|u(t)h((1+d)a-t, x)\|_{\dot{s}} \simeq (a-t)^\alpha [(1+d)a - t]^{n/2-s} \to +\infty$$

as $t \to a$ from below, which yields contradiction. Combining (3.7) with Lemma 3.1, we have proved ill-posedness in $\dot{H}^s$ by letting $a \to 0$.

For the second part, one only needs to check that $\|f_a\|_s + \|g_a\|_{s-1}$ goes to zero (or remains bounded for w-ILP) as $a \to 0$. It is true for $s_c > 1$ ($s_c \ge 1$), since

$$\|f_a\|_s \lesssim \|f_a\|_{L^2} + \|\mathrm{D}^s f_a\|_{L^2} \lesssim (a^{s_c} + a^{s_c - s})\|\phi\|_s$$

and

$$\begin{cases} \|g_a\|_{s-1} \lesssim \|g_a\|_{L^2} + \|\mathrm{D}^{s-1} g_a\|_{L^2} \lesssim (a^{s_c-1} + a^{s_c-s})\|\phi\|_{s-1}, & s \ge 1, \\ \|g_a\|_{s-1} \lesssim \|g_a\|_{L^2} \lesssim a^{s_c-1}\|\phi\|_{L^2}, & s < 1. \end{cases}$$

Similarly, for the case $\alpha > 0$, we have

Proposition 3.2. *Let $n \ge 1$. The problem* (3.1) *with $\alpha > 0$ is s-ILP in $\dot{H}^s$ for $1 - \frac{n}{2} < s < s_c$. If $s_c > 1$, it is s-ILP in H^s with the same s. Moreover, it is also w-ILP in $\dot{H}^s$ and H^s (for this case, one needs $s_c \ge 1$) for $1 - \frac{n}{2} < s \le s_c$.*

Proof. We also use the data (f_a, g_a) to give the counterexample. As in Proposition 3.1, we show that $T^s(f_a, g_a) \le a$ with $|s-1| < \frac{n}{2}$ by contradiction argument:

$$\begin{aligned} \|\mathrm{D}^{s-1}\partial_t u(t,\cdot)\|_{L^2} &\gtrsim \|\mathrm{D}^{s-1}(\partial_t u h((1+d)a-t, x))\|_{L^2} \\ &\simeq (a-t)^{\alpha-1}[(1+d)a-t]^{1+n/2-s} \to +\infty \end{aligned}$$

as $t \to a$ from below. Combining this with Lemma 3.1 we have proved ILP in $\dot{H}^s$ by letting $a \to 0$. The second part is just as in Proposition 3.1.

Proposition 3.3. *Let $n \ge 2$, $l = 2$ and $k = 0$. The model equation* (3.1) *is s-ILP in $\dot{H}^s$ for $s \in (1 - \frac{n}{2}, s_c)$. And, if $s_c > 1$ (i.e., $n \ge 3$), then it is s-ILP in H^s for the same s. Moreover, it is w-ILP in $\dot{H}^s$ and H^s for $s \in (1 - \frac{n}{2}, s_c)$.*

Proof. Just as in the proof of Proposition 3.2, we have

$$T^s(f_a, g_a) \le a \qquad \text{for} \quad |s-1| < \frac{n}{2}.$$

For any given

$$1-\frac{n}{2}<s<s_c=\frac{n}{2},$$

set

$$\epsilon=\frac{s_c-s}{2}\wedge\frac{s_c}{2}>0.$$

By Lemma 3.1, we have

$$\|f_a\|_s\lesssim\|f_a\|_{L^2}+\|\mathrm{D}^s f_a\|_{L^2}\lesssim(a^{s_c-\epsilon}+a^{s_c-s-\epsilon})\|\phi\|_s$$

and (note here one may always choose $\epsilon=0$ for the g_a part of Lemma 3.1)

$$\begin{cases}\|g_a\|_{s-1}\lesssim\|g_a\|_{L^2}+\|\mathrm{D}^{s-1}g_a\|_{L^2}\lesssim(a^{s_c-1}+a^{s_c-s})\|\phi\|_{s-1}, & s\geq 1,\\ \|g_a\|_{s-1}\lesssim\|g_a\|_{L^2}\lesssim a^{s_c-1}\|\phi\|_{L^2}, & s<1.\end{cases}$$

§ 4 . Sub-critical Counterexample

In this section we mainly prove Theorem 1.3 and Theorem 1.4. The method is the following. Applying scaling and Lorentz transformations on the solution 3.3 we yields a series of solutions which concentrate along the characteristic $t-x_1=a$ asymptotically. Then we get the ILP result by estimating the solutions' lifespan and the initial data's norm in $\dot{H}^s$ and H^s.

Applying Lorentz transformation

$$\begin{cases}\gamma=\dfrac{t-\beta x_1}{\delta}, & \delta=(1-\beta^2)^{1/2}\ \text{ and }\ \beta\in(0,1),\\ y_1=\dfrac{x_1-\beta t}{\delta}, & \delta=(1-\beta^2)^{1/2}\ \text{ and }\ \beta\in(0,1),\\ y_i=x_i & \text{for }\ 2\leq i\leq n\end{cases}$$

and scaling to the ODE solution (3.3) of (3.1), we can get some special solutions to (3.1):

$$u_{a,\beta}(t,x)=\begin{cases}C_{k,l}\delta^{\frac{2\alpha}{l-2}}(a-t+\beta x_1)^{\alpha}, & \alpha=\frac{l-2}{k+l-1}\neq 0,\\ -\delta^2\log|a-t+\beta x_1|, & l=2\ \text{ and }\ k=0\end{cases}\tag{4.1}$$

for $t-\beta x_1<a$.

For the proof of this result, we need some technical lemmas (especially Lemma 4.1 and Lemma 4.5). First let us give the initial data's estimate in $\dot{H}^s$.

Lemma 4.1. *Let $n\geq 3$, $\alpha\neq 0$, $\beta\in(\epsilon,1)$ and $h(\tau)$ be a C^∞ function with*

$$h(\tau)=\begin{cases}1, & \tau\geq 0,\\ 0, & \tau\leq-\epsilon\end{cases}$$

for some small $0<\epsilon\ll 1$ and $h_\beta(a,x):=h(\frac{1-|x|/a}{1-\beta})$. Set

$$f_{a,\beta}:=u_{a,\beta}(0,x)h_\beta(a,x)=C_{k,l}\delta^{\frac{2\alpha}{l-2}}(a+\beta x_1)^{\alpha}h_\beta(a,x)$$

and $g_{a,\beta} := -\partial_{x_1} f_{a,\beta}/\beta$. *Then*

$$\|\mathrm{D}^m f_{a,\beta}\|_{L^p} + \|\mathrm{D}^{m-1} g_{a,\beta}\|_{L^p} \le C a^{\frac{n}{p}+\alpha-m}(1-\beta)^{\frac{n+1}{2p}+\frac{l-1}{l-2}\alpha-m} \tag{4.2}$$

subject to $n+1+2p(\alpha-m)<0$, $m \in \mathbb{N}$, $1<p<\infty$.

Proof. By homogeneity, we may reduce a to 1. Since the Riesz transform $\mathrm{D}^{-1}\partial_{x_1} : L^p \to L^p$ for $1<p<\infty$, we only need to estimate the corresponding part on $f_{a,\beta}$. However (4.2) involving only $f_{a,\beta}$ is just reduced to the case we shall prove in Lemma 4.3.

Let us give a characterization of $\dot{H}^{m,p}$ in C_0^∞ here.

Lemma 4.2. *Let* $n \ge 3$, $m \in \mathbb{N}$ *and* $1<p<\infty$. *Then*

$$\|u\|_{\dot{H}^{m,p}} \approx \sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^p},$$

where u *goes over in* C_0^∞.

Proof. The inequality $\|\partial^\alpha u\|_{L^p} \lesssim \|u\|_{\dot{H}^{m,p}}$ with $|\alpha|=m$ is a direct consequence when one applies the Mihlin Multiplier Theorem (cf. [2]) with the multiplier $\frac{\xi^\alpha}{|\xi|^m}$. On the other hand, if we can show that the required inequality holds for any C_0^∞ function u with support lying in the unit ball, then we can claim that the same inequality holds for all C_0^∞ functions by scaling. If $1<p<n$ and $\mathrm{supp}(u) \subset B_1$, we have

$$\|u\|_{L^p} \lesssim \|u\|_{L^{pn/(n-p)}} \lesssim \sum_{i=1}^n \|\partial_i u\|_{L^p} \lesssim \cdots \lesssim \sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^p},$$

$$\|u\|_{\dot{H}^{m,p}} \lesssim \|u\|_{H^{m,p}} \approx \|u\|_{L^p} + \sum_{i=1}^n \|\partial_i^m u\|_{L^p} \lesssim \sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^p}.$$

Now for $n \le p < \infty$ with $1 < q = \frac{np}{(n+p)} < n$,

$$\|u\|_{L^p} \lesssim \|u\|_{\dot{H}^{1,q}} \lesssim \sum_{i=1}^n \|\partial_i u\|_{L^q} \lesssim \sum_{i=1}^n \|\partial_i u\|_{L^p} \lesssim \cdots \lesssim \sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^p},$$

and so is $\|u\|_{\dot{H}^{m,p}} \lesssim \sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^p}$ for all $1<p<\infty$.

With this lemma in mind, we can give the technical lemma needed in Lemma 4.1 now.

Lemma 4.3. *Let* $n \ge 3$, $\beta \in (\epsilon, 1)$, $v(x) = (1+\beta x_1)^{-b} h(\frac{1-r}{1-\beta})$ *with* $|x|=r$, *where* h *is as in Lemma* 4.1. *Then*

$$\|v(x)\|_{\dot{H}^{m,p}} \le C(1-\beta)^{\frac{n+1}{2p}-(b+m)}$$

with $m \in \mathbb{N}$, $n+1-2p(b+m)<0$, $1<p<\infty$.

Proof. We first prove it for $m=0$. Introducing polar coordinates ($x = rw$) on the integral yields

$$\begin{aligned}\|f(x)\|_{L^p}^p &= \int_0^\infty\int_{\mathbf{S}^{n-1}} |f(r,w_1)|^p r^{n-1} d\sigma(w) dr \\ &= 2\int_0^\infty\int_{\mathbf{B}^{n-1}(1)} |f(r,w_1)|^p r^{n-1} \frac{dw'}{\sqrt{1-|w'|^2}} dr \\ &= C\int_0^\infty\int_{-1}^1 |f(r,w_1)|^p r^{n-1}(1-w_1^2)^{\frac{n-3}{2}} dw_1 dr,\end{aligned}$$

where $w' = (w_1, \cdots, w_{n-1})$. So

$$\begin{aligned}\|v\|_{L^p}^p &= C\int_0^\infty\int_{-1}^1 (1+\beta r w_1)^{-bp}(1-w_1^2)^{\frac{n-3}{2}} h\Big(\frac{1-r}{1-\beta}\Big)^p r^{n-1} dw_1 dr \\ &\lesssim \int_0^\infty\int_{-1}^1 (1+\beta r w_1)^{\frac{n-3}{2}-bp} dw_1 h\Big(\frac{1-r}{1-\beta}\Big)^p r^{n-1} dr \\ &\lesssim \int_0^\infty h\Big(\frac{1-r}{1-\beta}\Big)^p r^{n-2}(1-\beta r)^{\frac{n-1}{2}-bp} dr \\ &\lesssim (1-\beta)^{\frac{n+1}{2}-bp},\end{aligned}$$

where we have used the assumption $n+1 < 2bp$ and the support properties of h.

Now we consider the case $m \geq 1$. Substituting $h(\frac{1-r}{1-\beta})$ by $(1-\beta)^{-l} h^{(l)}(\frac{1-r}{1-\beta})$ in v (denoted by $\tilde{v}$), and noting that $\operatorname{supp}(h^{(l)}(\frac{1-r}{1-\beta})) \subset \{1-\epsilon(1-\beta) \leq r \leq 1+\epsilon(1-\beta)\}$, we have

$$\begin{aligned}\|\tilde{v}\|_{L^p}^p &\lesssim \int_0^\infty h^{(l)}\Big(\frac{1-r}{1-\beta}\Big)^p r^{n-2}(1-\beta)^{-lp}(1-\beta r)^{\frac{n-1}{2}-bp} dr \\ &\lesssim (1-\beta)^{\frac{n+1}{2}-(b+l)p}.\end{aligned}$$

Since

$$\begin{aligned}|v^{(m)}| &\lesssim \sum_1^m |(1+\beta x_1)^{-b-(m-l)}\partial_x^l h| + |(1+\beta x_1)^{-b-m} h| \\ &\lesssim \sum_1^m |(1+\beta x_1)^{-b-(m-l)}(1-\beta)^{-l} h^{(l)}| + |(1+\beta x_1)^{-b-m} h|,\end{aligned}$$

by above arguments and Lemma 4.2 we have

$$\|v(x)\|_{\dot{H}^{m,p}}^p \lesssim (1-\beta)^{\frac{n+1}{2}-(b+m)p}$$

with $m \geq 1$, $n+1 < 2(b+m)p$.

Next we consider the estimate of the lifespan of solutions.

Lemma 4.4. *Let $h_\beta(a,x)$ be as in Lemma 4.1 and assume that $u \in C^\infty(\mathbb{R}^n)$ on the support of $h_\beta(a,x)$. Then for $|s| < \frac{n}{2}$,*

$$\|h_\beta(a,x)u(x)\|_{\dot{s}} \lesssim \|u\|_{\dot{s}}. \tag{4.3}$$

Proof. We follow the argument in [8] in principle. By homogeneity, we may assume $a = 1$. Set $f = \mathrm{D}^s u$. We need only to show that for any $g \in L^2$,

$$\int (\mathrm{D}^s h_\beta \mathrm{D}^{-s} f)\bar{g}dx = \iint K(\xi,\eta)\hat{f}(\xi)\bar{\hat{g}}(\eta)d\xi d\eta \lesssim \|f\|_{L^2}\|g\|_{L^2},$$

where $K(\xi,\eta) = C|\eta|^s|\xi|^{-s}\hat{h}_\beta(1,\eta-\xi)$. By symmetry or duality we may assume that $0 \le s < \frac{n}{2}$. It suffices to show that $\iint K(\xi,\eta)K(\xi,\alpha)d\alpha d\xi$ is bounded with a constant independent of η. Note that $\hat{h}_\beta$ is in Schwartz space, this is a standard estimate.

Lemma 4.5. *For $\beta \in (\epsilon, 1)$, then $u_{a,\beta}$ in (4.1) satisfies (3.1) when $t-\beta x_1 < a$. Moreover, if we set data $(f_{a,\beta}, g_{a,\beta})$ as in Lemma 4.1, with $h(\tau) = 1$ for $\tau \ge 0$, and assume that (3.1) has local uniqueness solution, then we have $T^s(f,g) \le a$ for $-\frac{n}{2} \vee s_c < s < \frac{n}{2} + H(\alpha)$.*

Proof. The first claim is valid by direct checking. Note that $u = u_{a,\beta}$ in $|x| < a-t$ with $0 < t < a$ under the assumption of local uniqueness time $T \ge a$ (otherwise the claim would be obviously valid). First for $\alpha < 0$, if $T^s(f,g) > a$, then applying Sobolev inequality yields

$$\begin{aligned}\|\mathrm{D}^s u(t)\|_{L^2}^p &\ge C\|u(t,\cdot)\|_{L^p(\mathbb{R}^n)}^p \ge C\|u_{a,\beta}(t,x)\|_{L^p(|x|\le a-t)}^p \\ &\ge C\delta^{2\alpha p/(l-2)}(a-t)^{n+\alpha p}\|(1+\beta x_1)^\alpha\|_{L^p(|x|\le 1)}^p \\ &\to \infty\end{aligned}$$

as $t \to a$ from below subject to the conditions $\alpha p + n < 0$ (i.e., $s > s_c$) and $0 \le s < \frac{n}{2}$, which contradicts the assumption. For $0 > s > -\frac{n}{2}$, we use Lemma 4.4 (with another h such that $h(\tau) = 0$ when $\tau \le 0$) and scaling to yield contradiction:

$$\begin{aligned}\|u(t)\|_{\dot{H}^s} &\gtrsim \|u(t)h_\beta(a-t,x)\|_{\dot{H}^s} = \|u_{a,\beta}(t,x)h_\beta(a-t,x)\|_{\dot{H}^s} \\ &\approx (a-t)^{s_c-s} \to \infty\end{aligned}$$

as $t \to a$ from below. Now for $\alpha > 0$, noting that in this case we have $\frac{n}{2} < s < \frac{n}{2}+1$, we then obtain

$$\begin{aligned}\|\mathrm{D}^s u(t)\|_{L^2}^p &\ge C\|u(t,\cdot)\|_{\dot{H}^{1,p}(\mathbb{R}^n)}^p \ge C\|\partial_1 u_{a,\beta}(t,x)\|_{L^p(|x|\le a-t)}^p \\ &\ge C\delta^{2\alpha p/(l-2)}(a-t)^{n+(\alpha-1)p}\|(1+\beta x_1)^\alpha\|_{L^p(|x|\le 1)}^p \\ &\to \infty\end{aligned}$$

as $t \to a$ from below subject to the condition $(\alpha-1)p + n < 0$ (i.e., $s > s_c$).

Now we have prepared enough to prove the subcritical ILP of Theorem 1.3 for $\alpha \ne 0$. Since Lemma 4.5 gives the estimate of the solutions' lifespan, by argument as in Section 3, the only thing left is to estimate the initial data's norm in $\dot{H}^s$ and H^s.

Proof of the Subcritical ILP of Theorem 1.3.

We use the data in Lemma 4.1 to give an ILP counterexample.

Firstly, we combine Sobolev estimate with the estimate in Lemma 4.1 to get the estimate of data in $\dot{H}^s$. For $m \in \mathbb{N}$, we have

$$\|f_{a,\beta}\|_{\dot{H}^s} \lesssim \|f_{a,\beta}\|_{\dot{H}^{m,p}} \lesssim a^{\frac{n}{p}+\alpha-m}(1-\beta)^{\frac{n+1}{2p}+\frac{l-1}{l-2}\alpha-m} \lesssim a^{s_c-s}(1-\beta)^{\tilde{s}-\lambda}, \tag{4.4}$$

where $\frac{n}{2} - s = \frac{n}{p} - m$, $s \le m$, $p > 1$, and $\lambda := \frac{n+1}{2n} s + \frac{n-1}{2n} m$. In view of Lemma 4.5, by choosing $a \to 0$ and $\beta \to 1$ appropriately, we get the ill-posedness in $\dot{H}^s$ if

$$\begin{aligned} & s < \Big(\tilde{s} + \frac{n-1}{n+1}(\tilde{s} - m)\Big) \wedge \Big(\frac{n}{2} + H(\alpha)\Big), \\ & \Big(m - \frac{n}{2}\Big) \vee s_c < s \le m. \end{aligned} \tag{4.5}$$

In particular, if $\tilde{s} \in \mathbb{N}$, then we have the $\dot{H}^s$ ILP result for $(\tilde{s} - \frac{n}{2}) \vee s_c < s < \tilde{s} \wedge (\frac{n}{2} + H(\alpha))$ by setting $m = \tilde{s}$.

Secondly, we use interpolation inequality (cf. [2, Theorem 6.4.5]) and the estimates in Lemma 4.1 to get some other estimates of data. Let $d, b \in \mathbb{N}$, $\sigma \in [0, 1]$ and $d < b$. Applying Lemma 4.1 with $m = d, b$ and $p = p, q \in (1, \infty)$ separately yields

$$\|f\|_{\dot{H}^s} \lesssim \|f\|^{\sigma}_{\dot{H}^{d,p}} \|f\|^{1-\sigma}_{\dot{H}^{b,q}} \lesssim a^{s_c - s}(1-\beta)^{\tilde{s} - s} \tag{4.6}$$

subject to

$$\begin{aligned} & 2p(d - \alpha) > n + 1, && 2q(b - \alpha) > n + 1, \\ & \frac{\sigma}{p} + \frac{1-\sigma}{q} = \frac{1}{2}, && s = \sigma d + (1 - \sigma) b. \end{aligned} \tag{4.7}$$

In view of Lemma 4.5, we want to get the maximal region of s such that $s \in \{t : s_c < t < \tilde{s} \wedge (\frac{n}{2} + H(\alpha))$ and $t \ge 0\}$ for which there exist σ, d, b, p, q satisfying (4.7). For any fixed d, b, one easily deduces that if $\alpha < d \le \frac{n+1}{4} + \alpha < b$, the ILP set will be

$$\Big(\Big(\frac{n+1}{4} + \alpha\Big) \bigvee \Big(b - \frac{(b-d)(n+1)}{2(n+1) - 4(d - \alpha)}\Big), b\Big] \bigcap \Big(s_c, \tilde{s} \wedge \Big(\frac{n}{2} + H(\alpha)\Big)\Big), \tag{4.8}$$

and if $d > \frac{n+1}{4} + \alpha$,

$$[d, \infty) \cap \Big(s_c, \tilde{s} \wedge \Big(\frac{n}{2} + H(\alpha)\Big)\Big). \tag{4.9}$$

Make unions on (4.5), (4.8), and (4.9), set $\tilde{d} = \min\{d \in \mathbb{N} : d > \frac{n+1}{4} + \alpha\}$, and note that $s_c \ge \frac{n+1}{4} + \alpha$. We get a region of homogeneous ILP (denoted by $\mathbf{E}$):

- if $l > 2$, $(s_c, \tilde{s})$;
- if $l < 2$ and $\tilde{s} > 0$, $(-\frac{n}{2} \vee s_c, \tilde{s} \wedge \frac{n}{2})$;
- if $l < 2$ and $\tilde{s} \le 0$, $(-\frac{n}{2} \vee s_c, \frac{2n}{n+1}\tilde{s})$.

In particular, for more interesting case for us (i.e., one has ILP for $\tilde{s} - \epsilon < s < \tilde{s}$ with sufficiently small ϵ), we only need to exclude the cases $\tilde{s} < 0$ or $\tilde{s} > \frac{n}{2}$ with $l < 2$.

For the inhomogeneous part, by applying Lemma 4.1, we have

$$\begin{aligned} & \|g_{a,\beta}\|_{L^2} \lesssim \|g\|_{\dot{H}^{\widetilde{m}, 2n/(n+2\widetilde{m})}} \lesssim a^{s_c - 1}(1-\beta)^{\tilde{s} - \frac{n-1}{2n}\widetilde{m} - 1}, \\ & \|f_{a,\beta}\|_{L^2} \lesssim \|f\|_{\dot{H}^{m, 2n/(n+2m)}} \lesssim a^{s_c}(1-\beta)^{\tilde{s} - \frac{n-1}{2n} m} \end{aligned}$$

with $n \geq 3$, $m, \widetilde{m} \in \mathbb{Z} \cap [0, \frac{n}{2})$ and $\frac{n+1}{4} + \alpha < (\frac{n-1}{2n}m) \vee (\frac{n-1}{2n}\widetilde{m} + 1)$. To ensure the data's inhomogeneous norm's uniformly converge to 0 in the limit procedure, we get a sufficient condition that there exist $m, \widetilde{m} \in \mathbb{Z} \cap [0, \frac{n}{2})$ such that

$$\tilde{s} > \Big(\frac{n-1}{2n}m\Big) \bigvee \Big(\frac{n-1}{2n}\widetilde{m} + 1\Big) > \frac{n+1}{4} + \alpha. \tag{4.10}$$

Then if (4.10) is satisfied, it is ILP in H^s for $0 < s \in \mathbf{E}$ and also for $s \in (-\frac{n}{2}, 0] \cap (s_c, \tilde{s})$.

Once Theorem 1.3 is proved, one gets easily the three corollaries in Section 1. Note that for equation

$$\Box u = \frac{(\partial_t u)^2}{2}, \tag{4.11}$$

if we set $v = \partial_t u$, then v satisfies (1.5) with $k = 2$. So we can give the subcritical ill-posedness for (4.11) as claimed in Theorem 1.4.

Lemma 4.6. *Let $n = 4$, $s_c = 2$, $\tilde{s} = \frac{9}{4}$, and h be as in Lemma* 4.1*. Then for* (4.11) *with data*

$$(g, f) := \Big(\mathrm{D}^{-2}\Big(\frac{1}{\beta}\partial_{x_1} f + \frac{f^2}{2}\Big), f\Big),$$

where

$$f = C\frac{1-\beta^2}{a + \beta x_1} h\Big(\frac{1 - |x|/a}{1-\beta}\Big)$$

with $C\frac{1-\beta^2}{a-t+\beta x_1}$ solves the equation (1.5) *with $k = 2$, we have*

$$\|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s-1}} \lesssim a^{s_c - s}(1-\beta)^{\tilde{s}-s}$$

and $T^s(g, f) \leq a$ with $s \in (s_c, \tilde{s})$.

Proof. Here $\alpha = 0$. Let $w := \partial_t u$. Then w satisfies

$$\begin{cases} \Box w = w\partial_t w, \\ w(0,x) = f(x), \\ \partial_t w(0,x) = -\dfrac{1}{\beta}\partial_{x_1} f. \end{cases}$$

Since the solution u in $|x| \leq a - t$ satisfies $\partial_t u(t,x) = C\frac{1-\beta^2}{a-t+\beta x_1}$, we can show $T^s(g,f) \leq a$ just as in Lemma 4.5. For $s \geq 2$, by Lemma 2.2 and Lemma 4.1,

$$\|f^2\|_{\dot{H}^{s-2}} \lesssim \|f\|_{L^4}\|f\|_{\dot{H}^{s-2,4}} \lesssim (1-\beta)^{5/8}\|f\|_{\dot{H}^{s-1}}.$$

Note that $g \in \dot{H}^s$ for $s > 0$, so we have

$$\begin{aligned} \|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s-1}} &\lesssim \|\partial_{x_1} f\|_{\dot{H}^{s-2}} + \|f\|_{\dot{H}^{s-1}} + \|f^2\|_{\dot{H}^{s-2}} \\ &\lesssim \|\partial_{x_1} f\|_{\dot{H}^{s-2}} + \|f\|_{\dot{H}^{s-1}} \\ &\lesssim a^{s_c-s}(1-\beta)^{\tilde{s}-s}, \end{aligned}$$

where the last inequality comes from interpolation of (4.2) with $m = [s-1] = 1$ and $[s] = 2$.

Combining this lemma with the known result for $n = 3$ in [8] (one can also get this result by embedding argument and using Lemma 4.1), we get the result in Theorem 1.4.

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