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ON THE ASYMPTOTIC BEHAVIOUR OF THE STEADY SUPERSONIC FLOWS AT INFINITY**

ZHANG Yongqian*

Abstract

This paper studies the asymptotic behaviour of steady supersonic flow past a piecewise smooth corner or bend. Under the hypothese that both vertex angle and the total variation of tangent along the boundary are small, it is shown that the solution can be obtained by a modified Glimm scheme, and that the asymptotic behaviour of the solution is determined by the velocity of incoming flow and the limit of the tangent of the boundary at infinity.

Keywords Asymptotic behaviour, Supersonic, Vertex angle, Tangent
2000 MR Subject Classification 35L15, 35L20

§ 1. Introduction

We are concerned with the asymptotic behaviour at infinity of steady planar potential flow of gas past a corner or a bend with a piecewise smooth boundary. That is, we study the asymptotic behaviour of the solution to the following problem

$$(\rho u)_x + (\rho v)_y = 0, \qquad \text{in } \Omega, \tag{1.1}$$

$$v_x - u_y = 0, \qquad \text{in } \Omega, \tag{1.2}$$

$$(u,v)\cdot \vec{n}\big|_{\partial\Omega} = 0, \tag{1.3}$$

$$(u,v)\big|_{x<0} = (q_\infty, 0), \tag{1.4}$$

where (u,v) and ρ are the velocity and the density, respectively, which satisfy the following Bernoulli equation

$$\frac{\gamma-1}{\gamma+1}(u^2+v^2) + \frac{2}{\gamma+1}c^2(\rho) = c_*^2 \tag{1.5}$$

with $c^2(\rho) = \gamma A \rho^{\gamma-1}$ for some constant $A > 0$ and the adiabatic exponent $\gamma > 1$. The velocity of incoming flow, $U_\infty = (q_\infty, 0)$, satisfies the following

(A1) q_∞ is a constant and $q_\infty > c_*$.

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*School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
Emial: yongqianzh@online.sh.cn
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The domain Ω is defined as follows

$$\Omega = \{(x, y) \mid y < b(x)\},$$

where the function $y = b(x)$ satisfies the following

(A2) b is a piecewise C^1 function with $b \in C(R^1)$ and $b'_+ \in BV(R^1)$, $b(x) = 0$ for $x \le 0$ where

$$b'_+(x) = \lim_{t \to 0+0} \frac{b(x+t) - b(x)}{t}.$$

Here and in the sequel we denote by $BV(E)$ the set of the functions which have finite total variation on the interval E of R, and denote by $TV\{f; E\}$ the total variation of f on E for $f \in BV(E)$. Moreover, $\vec{n}$ denotes the outer normal to $\partial\Omega$ outside the nondifferentiable points, and the boundary condition (1.3) holds outside the nondifferentiable points of $\partial\Omega$.

The problem of steady supersonic flow past a wedge, including the existence and asymptotic behaviour of the flow, had been extensively studied by many authors (for instance, see [1–4, 7, 9, 10, 13, 15, 17, 18] and references therein). This paper is a continuation of [17]. We will first establish a global weak solution to the problem (1.1)–(1.4) by using a modified Glimm Scheme as in [17]. Then we will study the asymptotic behaviour of the solution.

In the remaining part we organize the paper as follows. In Section 2 we recall some basic results on wave curves and present some estimates on the wave interactions. In Section 3 we point out that the global solution can be obtained by modifying the argument in [17] slightly. There to get the prior bounds on the total variations of approximate solutions we introduce a Glimm functional which is different from and is more simple than that used in [17] and [18], and prove the decreasing of the functional. In Section 4 we first establish some estimates on the global approximate solution, then determine the asymptotic behaviour of the solution. Main result, Theorem 4, is present there.

§ 2 . Preliminaries

2.1. Wave curves

Let us recall some basic results about the system (see [4, 17, 18]). This system is genuinely nonlinear and hyperbolic if x-direction is regarded as the time direction. It is obvious that for (u, v) close to $(q_\infty, 0)$ the system has two distinct characteristics

$$\lambda_1 = \frac{uv - c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2},$$

$$\lambda_2 = \frac{uv + c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2},$$

and the corresponding right eigenvectors

$$r_k(u, v) = e_k \begin{pmatrix} -\lambda_k \\ 1 \end{pmatrix}$$

with smooth function $e_k(u,v) > 0$ and

$$r_k \cdot \nabla\lambda_k = 1, \qquad k = 1, 2$$

for any (u,v) near $(q_\infty, 0)$ (see [17]).

Let $R_2(u_0,v_0)$ and $S_2(u_0,v_0)$ (or $R_1(u_0,v_0)$ and $S_1(u_0,v_0)$, respectively) be, respectively, the epicycloid and shock polar in the supersonic region with respect to λ_2-characteristic field (or λ_1-characteristic field, respectively) passing through (u_0,v_0), and denote

$$R_2^+(u_0,v_0) = \{(u,v) \in R_2(u_0,v_0) \mid q \le q_0\},$$
$$S_2^-(u_0,v_0) = \{(u,v) \in S_2(u_0,v_0) \mid q \ge q_0\},$$
$$R_1^+(u_0,v_0) = \{(u,v) \in R_1(u_0,v_0) \mid q \ge q_0\},$$
$$S_1^-(u_0,v_0) = \{(u,v) \in S_1(u_0,v_0) \mid q \le q_0\},$$

and

$$T_j(u_0,v_0) = R_j^+(u_0,v_0) \cup S_j^-(u_0,v_0), \qquad j = 1, 2,$$

where $q = \sqrt{u^2+v^2}$. As shown in [17], $T_j(u_l,v_l)$ can be parameterized by

$$\epsilon_j \mapsto \Phi_j(\epsilon_j, U_l)$$

in a neighbourhood of U_∞, $O_{\delta_1}(U_\infty)$, with $\Phi_j \in C^2$ and

$$\Phi_j\big|_{\epsilon_j=0} = U_l, \tag{2.1}$$

$$\frac{\partial\Phi_j}{\partial\epsilon_j}\Big|_{\epsilon_j=0} = r_j(U_l). \tag{2.2}$$

Moreover, $\epsilon_j > 0$ along $R_j^+(U_l) \cap O_{\delta_1}(U_\infty)$ while $\epsilon_j < 0$ along $S_j^-(U_l) \cap O_{\delta_1}(U_\infty)$ $(j = 1, 2)$.

Denote

$$\Phi(\epsilon_2, \epsilon_1, U_l) = \Phi_2(\epsilon_2, \Phi_1(\epsilon_1, U_l)). \tag{2.3}$$

Then for any pair of supersonic states U_r and U_l close to U_∞, the equations (1.1) and (1.2) with initial condition

$$U\big|_{x=a} = \begin{cases} U_r, & \text{if } x > 0, \\ U_l, & \text{if } x < 0, \end{cases}$$

can be reduced to the following equation

$$U_r = \Phi(\beta, \alpha, U_l). \tag{2.4}$$

As shown in [17] (see [8]), by (2.1) and (2.2), the equation (2.4) has a unique solution (α, β) in a neighbourhood of $\beta = \alpha = 0$ and $U_l = U_r = U_\infty$.

For simplicity, we will use the notation $\{U_l, U_r\} = (\alpha, \beta)$ to denote the solution of the equation (2.4) in the sequel, and call the parameters α and β the magnitude of the weak 1-wave and the magnitude of weak 2-wave respectively. It is obvious that $\alpha > 0$ along R_1^+ and $\beta > 0$ along R_2^+ while $\alpha < 0$ along S_1^- and $\beta < 0$ along S_2^-.

2.2. Estimates on the interactions and reflections

As in [17], to construct the approximate solutions to the problem (1.1)–(1.4), we need to solve a class of initial boundary value problems. Let us recall some notations and results from [17].

Let $C_k(a_k, b_k)$ $(k = 1, 2, 3)$ be points in R^2 with $a_{k+1} > a_k > 0$ $(k = 1, 2)$ and

$$\omega_1 = \arctan\frac{b_2 - b_1}{a_2 - a_1}, \quad \omega_2 = \arctan\frac{b_3 - b_2}{a_3 - a_2}, \quad \omega = \omega_2 - \omega_1,$$

$$\Omega_k = \Big\{(x, y)\,\Big|\, a_k \le x \le a_{k+1},\, y < \frac{b_{k+1} - b_k}{a_{k+1} - a_k}(x - a_k) + b_k\Big\},$$

$$\Gamma'_k = \Big\{(x, y)\,\Big|\, a_k < x < a_{k+1},\, y = \frac{b_{k+1} - b_k}{a_{k+1} - a_k}(x - a_k) + b_k\Big\},$$

$$\vec{n}_k = \frac{(-b_{k+1} + b_k, a_{k+1} - a_k)}{(-b_{k+1} + b_k)^2 + (a_{k+1} - a_k)^2}.$$

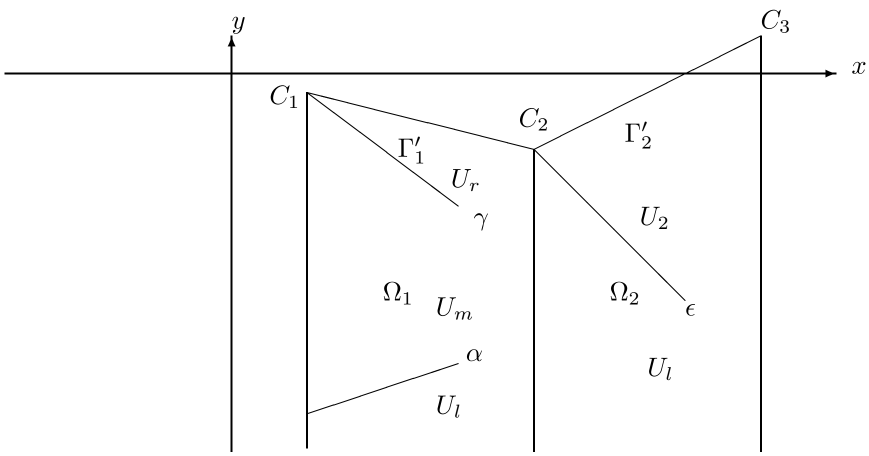

Fig. 1. Boundary interaction

We also set

$$\Delta(a, b) = \begin{cases} 0, & \text{if } a \ge 0 \text{ and } b \ge 0, \\ |a||b|, & \text{otherwise}. \end{cases} \tag{2.5}$$

Then, we consider the following problem

$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, & \text{in } \Omega_2, \\ v_x - u_y = 0, & \text{in } \Omega_2, \\ U\big|_{x=a_2} = U_l, & \\ U \cdot \vec{n_2} = 0, & \text{on } \Gamma'_2, \end{cases} \tag{2.6}$$

where $U = (u, v)$. For this problem, we have obtained the estimates on the solutions in [17] as follows.

Lemma 2.1. *There exist $\delta_i > 0$ $(i = 2, 3)$ and $\delta_2' > 0$ such that if $U_l, U_m, U_r \in O_{\delta_2}(U_\infty)$ and $\omega_1, \omega_2 \in (-\delta_3, \delta_3)$ with $\{U_l, U_m\} = (0, \alpha)$, $\{U_m, U_r\} = (\gamma, 0)$ and $U_r \cdot \vec{n}_1 = 0$, then there exists a unique $\epsilon \in (-\delta_2', \delta_2')$ and a constant state U_2 with $\{U_l, U_2\} = (\epsilon, 0)$ such that the mixed problem (2.6) in Ω_2 with the initial data $U\big|_{x=a_2} = U_l$ admits an admissible solution U consisting of a weak 1-wave with magnitude ϵ and satisfying $U = U_2$ in a neighborhood of Γ_2' (see Fig. 1). Moreover,*

$$\epsilon = \gamma + K_1\alpha + K_0\omega \tag{2.7}$$

with

$$K_j = K_j(\omega, \omega_1, \alpha, \gamma, U_l) > 0, \qquad j = 0, 1,$$

where the bounds of K_0 and K_1 depend only on the system and U_∞.

Lemma 2.2. *Let constant states U_l, U_m, U_r and U_2, and the magnitudes of waves, α, γ and ϵ, be given by Lemma 2.1. Then*

$$\big|U_r - U_2\big| = O(1)[|\alpha| + |\omega|] \tag{2.8}$$

where the bound of $O(1)$ depends only on the system.

Proof. It follows from Lemma 2.1 that

$$\Phi(0, \gamma, \Psi(\alpha, 0, U_l)) = U_r, \tag{2.9}$$

$$\Phi(0, \epsilon, U_l) = U_2 \tag{2.10}$$

with

$$\epsilon = \gamma + K_1\alpha + K_0\omega. \tag{2.11}$$

Then, by (2.9) and by Glimm interaction estimates (see [5, 6]) we have

$$U_r = \Phi\big(\alpha + O(1)|\alpha||\gamma|,\ \gamma + O(1)|\alpha||\gamma|,\ U_l\big), \tag{2.12}$$

which, together with (2.9) and (2.10), implies that

$$\begin{aligned} |U_2 - U_r| &= |\Psi(0, \epsilon, U_l) - \Psi(\alpha + O(1)|\alpha||\gamma|, \gamma + O(1)|\alpha||\gamma|, U_l)| \\ &= O(1)(|\alpha + O(1)|\alpha||\gamma|| + |\epsilon - \gamma - O(1)|\alpha||\gamma||). \end{aligned} \tag{2.13}$$

Therefore, the desired result follows from (2.11) and (2.13). The proof is complete.

§ 3. Global Solution

We use the same scheme as given in [17] to define the global approximate solutions (see [18]). For any $\Delta x > 0$ and $\Delta y > 0$ such that $\Delta y/\Delta x < 1$, let $y_k = b(k\Delta x)$ and let

$\{A_k = (k\Delta x, y_k)\}_{k=0}^{\infty}$. Denote

$$\omega(A_k) = \arctan\frac{y_{k+1}-y_k}{\Delta x} - \arctan\frac{y_k - y_{k-1}}{\Delta x}, \qquad k \geq 1,$$

$$\omega(A_0) = \arctan\frac{y_1 - y_0}{\Delta x},$$

$$\Gamma_k = \{(x,y) \mid k\Delta x < x < (k+1)\Delta x,\ y = b(x,k,\Delta x)\};$$

and $\vec{n}$ denotes the outer normal to Γ_k; denote

$$\Omega_{\Delta x,k} = \{(x,y) \mid k\Delta x \leq x < (k+1)\Delta x,\ y < b(x,k,\Delta x)\},$$

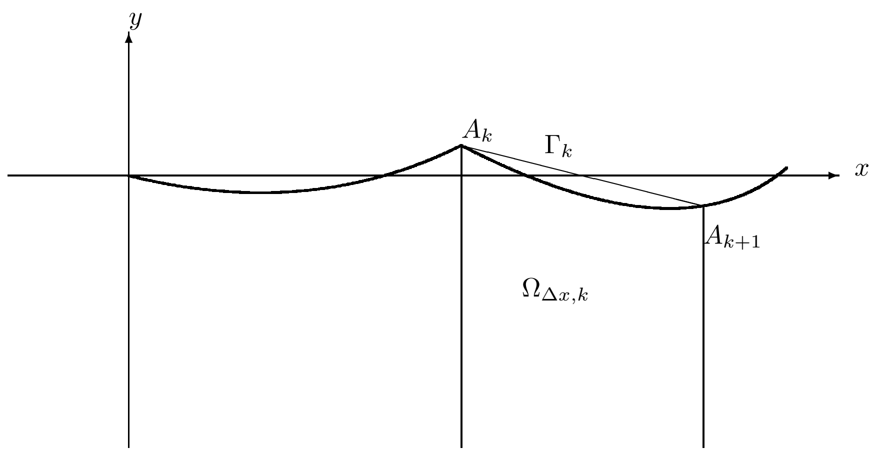

Fig. 2. Approximate domain

where $b(x,k\Delta x) = y_k + \frac{y_{k+1}-y_k}{\Delta x}(x - k\Delta x)$; and define the approximate domain as follows

$$\Omega_{\Delta x} = \bigcup_{k\geq 0} \Omega_{\Delta x,k},$$

see Fig. 2.

Choose a set of mesh points

$$\{P_{k,n} \mid P_{k,n} = (k\Delta x, a_{k,n}),\ k \geq 0,\ -\infty < n < +\infty\}$$

in R^2 where

$$a_{k,n} = (2n+1+\theta_k)\Delta y + y_k$$

and θ_k is randomly and independently chosen in $(-1,1)$. Then, as in [17], we define the global approximate solution $U_{\Delta x,\theta}$ in $\Omega_{\Delta x}$ for any Δx and $\theta = (\theta_1, \theta_2, \cdots)$ by carrying out the following steps inductively.

For $k=0$, $U_{\Delta x,\theta}$ can be defined in $\{0 \leq x < \Delta x\} \cap \Omega_{\Delta x}$ with $U_{\Delta x,\theta}\big|_{x=0,y<0} = U_\infty$ by shock polar.

Inductively assume that the approximate solution $U_{\Delta x,\theta}$ has been constructed for $\{0 \leq x < k\Delta x\}$, then for small $\Delta y/\Delta x$ we can define the $U_{\Delta x,\theta}$ in $\{k\Delta \leq x < (k+1)\Delta x\}$ by defining $U_{\Delta x,\theta}$ in $T_{k,n}$ $(n \le 0)$. Here for $n \le -1$, $T_{k,n}$ is defined to be the rhombus whose vertices are $(k\Delta x, (2n-1)\Delta y + y_k)$, $(k\Delta x, (2n+1)\Delta y + y_k)$, $((k+1)\Delta x, (2n-1)\Delta y + y_{k+1})$ and $((k+1)\Delta x, (2n+1)\Delta y + y_{k+1})$, while $T_{k,0}$ is defined to be the rhombus whose vertices are $((k+1)\Delta x, y_{k+1})$, $((k+1)\Delta x, -\Delta y + y_{k+1})$, $(k\Delta x, y_k)$ and $(k\Delta x, -\Delta y + y_k)$.

First, for $n \le -1$, in each rhombus $T_{k,n}$ the approximate solution $U_{\Delta x,\theta}$ is defined to be the solution $U_k = (u_k, v_k)$ to the Riemann problem

$$\begin{cases} (\rho_k u_k)_x + (\rho_k v_k)_y = 0, & \text{in } T_{k,n}, \\ (v_k)_x - (u_k)_y = 0, & \text{in } T_{k,n}, \\ U_k\big|_{x=k\Delta x} = U_k^0, \end{cases} \tag{3.1}$$

where $\rho_k = \rho(u_k, v_k)$ and

$$U_k^0(y) = U_{\Delta x,\theta}(k\Delta x-, a_{k,n}), \qquad y \in (y_k + 2n\Delta y, y_k + (2n+1)\Delta y).$$

Secondly, in rhombus $T_{k,0}$, the approximate solution $U_{\Delta x,\theta}$ is defined to be the solution $U_k = (u_k, v_k)$ to the following mixed problem

$$\begin{cases} (\rho_k u_k)_x + (\rho_k v_k)_y = 0, & \text{in } T_{k,0}, \\ (v_k)_x - (u_k)_y = 0, & \text{in } T_{k,0}, \\ U_k\big|_{x=k\Delta x} = U_k^0, \\ U_k \cdot \vec{n}_k\big|_{\Gamma_k} = 0, \end{cases} \tag{3.2}$$

where $\rho_k = \rho(u_k, v_k)$.

To show that $U_{\Delta x,\theta}$ can be defined globally by the above steps, we need to establish the estimates on $U_{\Delta x,\theta}$ on a class of space-like curves. We connect the mesh point $P_{k,n}$ by two line segments to the two mesh points, $P_{k-1,n-1}$ and $P_{k-1,n}$ if $\theta_k \le 0$, or connect the mesh point $P_{k,n}$ by two line segments to the two mesh points $P_{k-1,n}$ and $P_{k-1,n+1}$ if $\theta_k > 0$.

Definition 3.1. *A j-mesh curve is defined to be an unbounded space-like curves lying in the strip $\{(j-1)\Delta x \le x \le (j+1)\Delta x\}$ and consisting of the segments of the form $P_{k,n-1}N(\theta_{k+1}, n)$, $P_{k,n-1}S(\theta_k, n)$.*

It is obvious that for any $0 < k < +\infty$ each k-mesh curve I divides the R^2 into I^+ part and I^- part, the I^- being the one containing the set $\{x < 0\}$. As in [16] we also partially order these mesh curves by saying $J_1 > J_2$ if every point of the mesh curve J_1 is either on J_2 or contained in J_2^+, and call J an immediate successor to I if $J > I$ and every mesh point of J except one is on I.

For any k-mesh curve J, $U_{\Delta x,\theta}\big|_J$ consists of various weak shock and rarefaction waves.

Definition 3.2. *Define*

$$L_j(J) = \sum\{|\alpha_j| : \ \alpha_j \text{ crosses } J\}, \qquad j = 1, 2,$$

$$L_0(J) = \sum\{|\omega(A)| : A \in \Omega_J\},$$

$$Q_j(J) = \sum\{\Delta(\alpha_j, \beta_j) : \ \alpha_j \text{ and } \beta_j \text{ cross } J, \text{ and } \alpha_j \text{ lies below } \beta_j \text{ on } J\}, \qquad j = 1, 2,$$

$$Q_{21}(J) = \sum\{|\alpha_2||\beta_1| :\ \alpha_2 \textit{ and } \beta_1 \textit{ cross } J, \textit{ and } \alpha_2 \textit{ lies below } \beta_1 \textit{ on } J\}.$$

Here the summations in L_j, Q_j $(j = 1, 2)$ and Q_{21} are taken over the set $J \cap \Omega_{\Delta x}$; and Ω_J is the set of boundary points A_k that lies in J^+, that is,

$$\Omega_J = \{A_k \mid A_k \in J^+ \cap \partial\Omega_\Delta,\ A_k = (k\Delta x, y_k)\};$$

$\Delta(a, b)$ *is defined by* (2.5).

Choose a positive constant $K > 0$ such that

$$K - \max\{|K_1|, |K_0|\} > 1 \tag{3.3}$$

for any $U \in \overline{B}$ and $\omega \in \overline{B}_1$.

Then for any constant $C > 0$, we define the following

Definition 3.3.

$$L(J) = KL_0(J) + L_1(J) + KL_2(J),$$
$$Q(J) = Q_2(J) + Q_1(J) + Q_{21}(J),$$
$$F(J) = L(J) + CQ(J).$$

Next, we will estimate the functional F. To do this, let I and J be two k-mesh curves for some $k > 0$ such that J is an immediate successor to I, and suppose that Λ is the diamond between I and J. Due to the location of Λ, two cases are to be considered.

(1) If $\Lambda \subset \Omega_{\Delta x}$, then let α and β be the waves entering Λ, and as in [5, 11, 16] define

$$Q(\Lambda) = \sum |\alpha_i||\beta_j|,$$

where the sum is taken over all pairs for which the i-wave from α and j-wave from β are approaching;

(2) If $\Lambda \cap \partial\Omega_{\Delta x} \neq \emptyset$, let $\Omega_J = \Omega_I \backslash \{A_k\}$ with $A_k = (k\Delta x, y_k)$ for some $k \geq 0$, and let $I = I_0 \cup I'$ and $J = I_0 \cup J'$ such that $\partial\Lambda = I' \cup J'$, and let γ_1 and α_2 be the 1-wave and 2-wave respectively crossing I' with α_2 lying below γ_1 on I. In addition, by the construction of approximate solution, let ϵ_1 be the weak 1-wave crossing J' (see Fig. 3).

Define

$$E_{\Delta x,\theta}(\Lambda) = \begin{cases} |\omega| + |\alpha_2|, & \text{if } \Lambda \cap \partial\Omega_{\Delta x} \neq \emptyset, \\ Q(\Lambda), & \text{if } \Lambda \subset \Omega_{\Delta x}. \end{cases}$$

Then, in the same way as in [17] and [18], we can prove the following

Proposition 3.1. *If $U_{\Delta x,\theta}\big|_I \in B$, then there exist constants $\delta' > 0$ and $C > 0$, independent of $U_{\Delta x,\theta}$, k, Δx, θ, I and J, such that if $F(I) \leq \delta'$ then there hold the following*

$$U_{\Delta x,\theta}\big|_J \in B \tag{3.4}$$

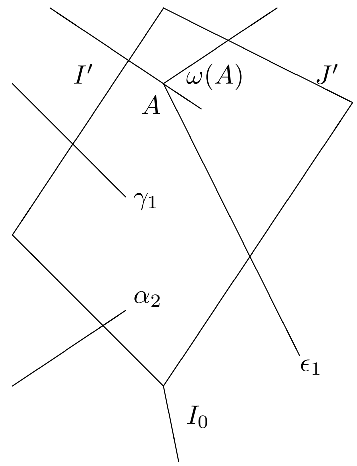

Fig. 3. Case $\Lambda \cap \partial\Omega_{\Delta x} \neq \emptyset$

and

$$F(I) - F(J) \geq \frac{1}{4} E_{\Delta x,\theta}(\Lambda). \tag{3.5}$$

Proof. If $\Lambda \subset \Omega_{\Delta x}$, then the standard argument as in [5] (see [16]) leads to the desired result.

If $\Lambda \cap \partial\Omega_{\Delta x} \neq \emptyset$, then by Lemma 2.1,

$$L(J) - L(I) \leq (K_0 - K)|\omega| + (K_1 - K)|\alpha_2| \leq -(|\omega| + |\alpha_2|),$$

which implies the desired result. Therefore, the proof is complete.

Using Proposition 3.1 and carrying out the same steps as in [5] and [11], we can deduce the following

Proposition 3.2. *If* $TV(b'_+) + |\arctan b'(0)|$ *is sufficiently small, then there is a null set* $N \subset \prod_{k=0}^{+\infty}[-1,1]$ *such that for each* $\theta \in \prod_{k=0}^{+\infty}[-1,1]\backslash N$*, there exist a sequence* $\Delta x_i \to 0$ *such that*

$$U_\theta = \lim_{\Delta x_i \to 0} U_{\Delta x_i,\theta}$$

is a weak solution to the problem (1.1)–(1.4)*, where the limit is taken in* $L^1_{\text{loc}}(\Omega)$*. Moreover,* $U_\theta \in L^\infty(\Omega)$*.*

§ 4. Asymptotic Behaviour

Let $\theta \in \prod_{k=0}^{+\infty}[-1,1]\backslash N$ and be equidistributed. To determine the asymptotic behaviour of the solution U_θ, we need further estimates on $U_{\Delta x,\theta}$. First, Proposition 3.1 implies the following

Lemma 4.1. *There exists a constant $M_1 > 0$, independent of $U_{\Delta x,\theta}$, θ and Δx, such that*

$$\sum_{\Lambda} E_{\Delta x,\theta}(\Lambda) \le M_1. \tag{4.1}$$

Here the summation is over all the diamonds.

Moreover, let $\Gamma_b = \bigcup_{k=0}^{+\infty} \overline{\Lambda}_{k,0}$, where $\Lambda_{k,0}$ is the diamond centered at A_k, and let $L_{\Delta x,\theta}(\Gamma_b)$ be the summation of the strength of waves leaving Γ_b. Then by Lemma 2.1 and Lemma 4.1, we have the following

Lemma 4.2. *There exists a constant M_2 independent of $U_{\Delta x,\theta}$, Δx, θ such that*

$$L_{\Delta x,\theta}(\Gamma_b) \le M_2 \sum_{\Lambda} E_{\Delta x,\theta}(\Lambda). \tag{4.2}$$

Let $L_2(a-)$ be the amount of all 2-waves in U_θ crossing the line $x = a$ for any $a > 0$; and $L_2^{\Delta x,\theta}(a)$ denotes the amount of 2-waves in $U_{\Delta x,\theta}$ crossing the line $x = a$ for any $a > 0$.

Lemma 4.3. $L_2(x-) \to 0$ as $x \to +\infty$.

Proof. As in [6], we denote by $dE_{\Delta x,\theta}$ the measures assigning the quantity $E_{\Delta x,\theta}(\Lambda)$ to the center of Λ. Then by Lemma 4 we can select a subsequence which we still denote by $\{E_{\Delta x_l,\theta}(\Lambda)\}_l$ so that

$$dE_{\Delta x_l,\theta} \to dE_\theta \quad \text{as} \quad \Delta x_l \to 0$$

with $E_\theta(\Omega) < \infty$, therefore for any $\epsilon > 0$ we can choose a $x_\epsilon > 0$ independent of $\{U_{\Delta_l,\theta}\}$ and $\{\Delta x\}$ such that

$$\sum_{k \ge [x_\epsilon/\Delta x]} E_{\Delta x_l,\theta}(\Lambda_{k,n}) < \epsilon \qquad \text{for sufficiently small } \Delta x_l. \tag{4.3}$$

Moreover, we can find a y_ϵ, independent of $\{\Delta x\}$ and $\{U_{\Delta_l,\theta}\}$, such that

$$U_{\Delta x,\theta}(x_\epsilon, y) = U_\infty \qquad \text{for } y < y_\epsilon. \tag{4.4}$$

Let $\chi_{\Delta x,\theta}$ be the minimum approximate 2-characteristics in $U_{\Delta x,\theta}$, issuing from the point (x_ϵ, y_ϵ). According to the construction of the approximate solutions, we have

$$|\chi_{\Delta,\theta}(x+h) - \chi_{\Delta,\theta}(x)| \le B(|h| + \Delta x)$$

for some constant B independent of Δx and θ. Then, for $\theta \in \prod_{k=0}^{+\infty}[-1,1] \backslash N$ we can select a subsequences $\{\Delta(l)\}$ of $\{\Delta x_i\}$ such that

$$\chi_{\Delta(l),\theta} \to \chi_\theta$$

uniformly on every bounded interval as $\Delta(l) \to 0$ for some $\chi_\theta \in \text{Lip}$ with χ'_θ bounded.

Let the characteristic $y = \chi_\theta(x)$ intersect $\partial\Omega$ at $(t_\epsilon, \chi_\theta(t_\epsilon))$ for some t_ϵ. Then, as in [6], applying the approximate conservation law to the domain below the characteristic $\chi_{\Delta(l),\theta}$, we have

$$L_2^{\Delta x,\theta}(x) = O(1)\epsilon \qquad \text{for } x > 2t_\epsilon$$

when $\Delta(l)$ is close enough to zero. Here the bound of $O(1)$ is independent of ϵ, $U_{\Delta_l,\theta}$ and Δx_l. This implies that

$$L_2(x) = O(1)\epsilon \qquad \text{for } x > 2t_\epsilon.$$

The proof is complete.

Next, we will study the asymptotic behaviour of the trace of U_θ on boundary. To this end, from Lemmas 2.2, 4.1 and 4.2 we can first deduce the following

Lemma 4.4. *Let*

$$W_{\Delta x,\theta}(x) = U_{\Delta x,\theta}(x, b_{\Delta x}(x)).$$

Then there exists a constant $M > 0$ depending only on the system such that

$$\mathrm{TV}\{W_{\Delta x,\theta}; [0, +\infty)\} < M. \tag{4.5}$$

Then, choose a subsequence $\{\Delta_l\}$ of $\{\Delta x_i\}$ so that

$$W_{\Delta_l,\theta} \to W_\theta \tag{4.6}$$

in $L^1_{\mathrm{loc}}([0,\infty))$ as $\Delta_l \to 0$ for some $W_\theta \in L^\infty([0,\infty))$. From the construction of the approximate solutions, we have

Lemma 4.5. *Let W_θ be given by* (4.6). *Then*

$$W_\theta \in \mathrm{BV}([0,\infty))$$

and

$$W_\theta(x-) \cdot (-b(x-), 1) = 0. \tag{4.7}$$

And we can determine the asymptotic behaviour of the trace of U_θ on $\partial\Omega$ as follows.

Lemma 4.6. *There holds the following*

$$\sup_{\hat\lambda x \le y \le b(x)} |U_\theta(x-, y) - W_\theta(x-)| \underset{x\to+\infty}{\longrightarrow} 0 \tag{4.8}$$

for any $\hat\lambda \in (\sup \lambda_1, \inf b')$.

Proof. For any $\epsilon > 0$, let x_ϵ be given as in the proof of Lemma 4.3. In the same way as in the proof of Lemma 4.3, we can have a sequence of approximate maximal 1-characteristics $g_{\Delta(l),\theta}$, issuing from $(x_\epsilon, b(x_\epsilon))$, such that $g_{\Delta(l),\theta} \to g_\theta$ uniformly on every bounded interval as $\Delta(l) \to 0$ for some $g_\theta \in \mathrm{Lip}$ with bounded derivative g'_θ.

Let $\{y = g_\theta(x)\}$ intersect the straight line $\{y = \hat{\lambda}x\}$ at some point $X_\epsilon^2 = (x'_\epsilon, \hat{\lambda}x'_\epsilon)$. Then, when $\Delta(l)$ is close enough to zero, by the approximate conservation laws and by Lemmas 4.1–4.3, we can deduce in the same way as in [6] that

$$\sup_{\hat{\lambda}x \le y \le b_{\Delta x}(x)} |U_{\Delta(l),\theta}(x-, y) - U_{\Delta(l),\theta}(x-, b_{\Delta(l)}(x))| = O(1)\epsilon \qquad \text{for } x > 2x'_\epsilon,$$

where the bound of $O(1)$ is independent of ϵ, $U_{\Delta x,\theta}$ and Δx. Therefore, passing to the limit, we have

$$\sup_{\hat{\lambda}x \le y \le b(x)} |U_\theta(x-, y) - W_\theta(x-))| = O(1)\epsilon \qquad \text{for } x > 2x'_\epsilon.$$

The proof is complete.

From Lemmas 4.3 and 4.6, we can deduce the following

Lemma 4.7. *Let*

$$W_\theta(+\infty) = \lim_{x\to+\infty} W_\theta(x-)$$

and let

$$b'(+\infty) = \lim_{x\to+\infty} b'_+(x+).$$

Then

$$\lim_{x\to+\infty} \sup_{\hat{\lambda}x \le y \le b(x)} |\lambda_1(U_\theta(x-, y) - \lambda_1(W_\theta(+\infty))| = 0$$

and

$$W_\theta(+\infty) \cdot (-b'(+\infty), 1) = 0.$$

Then, by Lemmas 4.3 and 4.7 and by Liu's result (see [12]), we have the following

Lemma 4.8. (1) *If* $\lambda_1(W_\theta(+\infty)) > \lambda_1(U_\infty)$, *then*

$$W_\theta(+\infty) \in S_1^-(U_\infty).$$

(2) *If* $\lambda_1(W_\theta(+\infty)) \le \lambda_1(U_\infty)$, *then*

$$W_\theta(+\infty) \in R_1^+(U_\infty).$$

Therefore, the equation

$$\Phi(0, \alpha_\infty, U_\infty) = W_\theta(+\infty)$$

has a unique solution α_∞.

Taking into account the geometry of the boundary, we can deduce the following

Lemma 4.9. *Suppose that* $|\arctan(b'(+\infty))| < \delta_2$. *Then*

(i) *if* $b'(+\infty) > 0$ *then* $\lambda_1(U_\infty) < \lambda_1(W_\theta(+\infty))$;

(ii) *if* $b'(+\infty) = 0$ *then* $\lambda_1(U_\infty) = \lambda_1(W_\theta(+\infty))$;

(iii) *if* $b'(+\infty) < 0$ *then* $\lambda_1(U_\infty) > \lambda_1(W_\theta(+\infty))$.

Proof. By Lemma 2.1 and Lemmas 4.7 and 4.8, we have

$$\alpha_\infty = K_0\omega_\infty,$$

where $\omega_\infty = \arctan b'(+\infty)$ and $K_0 > 0$. Then, noticing that

$$\frac{d}{d\epsilon}\lambda_1(\Psi(0,\epsilon,U_\infty))\Big|_{\epsilon=0} = 1,$$

we have

$$\lambda_1(W_\theta(+\infty)) - \lambda_1(U_\infty) = K_0\omega_\infty + O(1)|\omega_\infty|^2,$$

which implies the desired result.

Then, by carrying out same argument as in [12] (see also [14]) and by making use of Lemmas 4.6, and 4.9, we have

Theorem 4.1. *Suppose that* $\mathrm{TV}\{b'_+;[0,+\infty)\} + |\arctan b'(0)|$ *is small. Then*

(i) *if* $b'(+\infty) > 0$, *then the amount of shock waves approaches zero as* $x \to \infty$ *and* $U_\theta(x,y)$ *approaches the rarefaction* $(\alpha_\infty, 0)$*;*

(ii) *if* $b'(+\infty) < 0$, *then there exists a* 1*-shock which approaches the shock* $(\alpha_\infty, 0)$ *both in strength and speed as* $x \to +\infty$*; moreover, the total variation of* U_θ *outside this shock wave approaches zero as* $x \to +\infty$*;*

(iii) *if* $b'(+\infty) = 0$, *then* $\sup\limits_{y<b(x)} |U(x,y) - U_\infty| \to 0$ *as* $x \to +\infty$. *Here* $(\alpha_\infty, 0)$ *is given in Lemma* 4.8.

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