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LARGE SAMPLE PROPERTIES OF THE SIR IN CDMA WITH MATCHED FILTER RECEIVERS****

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Abstract

The output signal-to-interference (SIR) of conventional matched filter receiver in random environment is considered. When the number of users and the spreading factors tend to infinity with their ratio fixed, the convergence of SIR is showed to be with probability one under finite fourth moment of the spreading sequences. The asymptotic distribution of the SIR is also obtained.

Keywords Strong convergence, Asymptotic normality, Empirical distribution, Martingale
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§ 1 . Introduction

Recently, to develop more sophisticated physical-layer communication there have been a topic of great interest in developing multiuser structures, which mitigate the interference between users in code division multiple-access (CDMA) with random spreading sequences. In the area much work has already been done on the performance of multiusers receivers, however, most of which focuse on their ability to reject worst case interference, such as near-far resistance. Lately, a different point of view has been initiated in [1], where the spreading sequences are modelled as random sequences. For details one can refer to [1–5]. In this paper we study the out-put signal-to-interference ration (SIR) of the conventional match filter receiver. To keep model general, we employ the spreading sequences in [1]. We show that the limiting SIR is independent of the specific realization of the random spreading sequences. By U-statistic central limit theorem, we also obtain the asymptotic distribution of SIR, i.e., the SIR distribution is asymptotic Gaussian, which has not been set up in the literature. In order to obtain the above results, we mainly employ the tool of the martingale. One may say that the relevant results had been obtained in [1, 3, 4]. But we note that the SIR in [1] converges in probability under the condition of finite fourth moment of the spreading sequences, that the corresponding result in [3] need finite eighth moment condition, which holds with probability one, and that the result in [4] holds in probability for

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Gaussian random. The differences are that our results hold with probability one, and that we only need finite fourth moment of the spreading sequences. Particularly, we can drop the condition that the received powers are uniformly bounded in the uncorrelated cases, which is needed in [4].

Throughout this paper, the ratio of K to N is denoted by $\alpha = \frac{K}{N}$ as $K \longrightarrow \infty$ and c may denote different constant. Recent work in [1, 3, 4] showed that this method can average out the dependence on specific spreading sequences.

The paper is organized as follows. In Section 2, we introduce a discrete time model for CDMA system and the structure of the matched filter receiver. We present our results in Section 3.

§ 2 . Symbol-Synchronous Model

A sampled discrete-time model for a symbol-synchronous multiaccess CDMA system with K users, and processing gain N, is given by

$$Y = \sum_{i=1}^{K} X_i \sqrt{p_i}\, s_i + W, \tag{2.1}$$

where X_i is symbol transmitted by user i, p_i is received power, $s_i \in R^N$, is the spreading sequences of user i, $Y \in R^N$ and W is additive white gaussian noise with variance σ^2. The symbols X_i are independent, $EX_i = 0$, $EX_i^2 = 1$, and independent of the noise. To be more realistic, we assume that the received powers are random. Of course, we assume that the received powers and the symbols are independent.

We shall now focus on the demodulation of user 1. A linear receiver generates an output of the form $X_1^* = c_1^t Y$ and the output signal-to-interference ratio (SIR) is defined by

$$\mathrm{SIR}_1 \equiv \frac{p_1 (c_1^t s_1)^2}{(c_1^t c_1)\sigma^2 + \sum\limits_{i=2}^{K} p_i (c_1^t s_i)^2}.$$

(see [1, 2, 4]). For the conventional matched filter receiver (MF), c_1 is chosen to minimize

$$E(c_1^t Y - X_1)^2,$$

where expectation is taken by averaging over the spreading sequences and the received powers of all interferers besides the transmitted symbols and the noise. Thus SIR_1 is defined by

$$\beta_1 \equiv \frac{p_1 (s_1^t s_1)^2}{(s_1^t s_1)\sigma^2 + \sum\limits_{i=2}^{K} p_i (s_1^t s_i)^2}. \tag{2.2}$$

Here the performance measure for each user is considered as SIR achieved at the output of the multiuser receiver. To obtain more insight, we will assume a random signature sequence model. We assume that the spreading sequences are as follows

$$s_i = \frac{1}{\sqrt{N}} (v_{i,1}, \cdots, v_{i,N})^t, \qquad i = 1, \cdots, K.$$

The random variables $v_{i,k}$ are independent and identically distributed (i.i.d.), zero mean and variance 1. Hence, the SIR depends on the realization of the random spreading sequences as well as the received power, and is also a random variable. We assume that the spreading sequences are independent of the noise and that the received powers are independent of the spreading sequences and noise.

§ 3 . Main Results

Theorem 3.1. *Assume that the received powers are independent across the users and identically distributed,* $Ev_{11}^4 < \infty, E(p_1)^2 < \infty$*. Then as* $K \longrightarrow \infty$*,* β_1 *converges almost surely to*

$$\beta_1^* = \frac{p_1}{\sigma^2 + \alpha E p_1}.$$

Proof. Define

$$\mathcal{F}_i = \sigma(p_2, \cdots p_i, s_1, \cdots s_i) = \sigma(p_2 \cdots , p_i, v_{11}, \cdots , v_{1N}, \cdots , v_{i1}, \cdots v_{iN}).$$

Since the v_{ij}'s are independent and zero mean, if $j_1 \neq j_2$, we have

$$E(p_i v_{1j_1} v_{1j_2} v_{ij_1} v_{ij_2} | \mathcal{F}_{i-1}) = v_{1j_1} v_{1j_2} E(p_i v_{ij_1} v_{ij_2} | \mathcal{F}_{i-1}) = v_{1j_1} v_{1j_2} E p_i E v_{ij_1} E v_{ij_2} = 0.$$

It follows that

$$\begin{aligned} E(p_i (s_1^t s_i)^2 | \mathcal{F}_{i-1}) &= \frac{1}{N^2} E\Big(p_i \sum_{j=1}^N v_{1j}^2 v_{ij}^2 | \mathcal{F}_{i-1}\Big) + \frac{1}{N^2} E\Big(p_i \sum_{j_1 \neq j_2} v_{1j_1} v_{1j_2} v_{ij_1} v_{ij_2} | \mathcal{F}_{i-1}\Big) \\ &= \frac{1}{N^2} \sum_{j=1}^N v_{1j}^2 E(p_i v_{ij}^2 | \mathcal{F}_{i-1}) = \frac{1}{N^2} \sum_{j=1}^N v_{1j}^2 E p_1. \end{aligned} \tag{3.1}$$

Observe that

$$E p_i (s_1^t s_i)^2 = E p_i E[\mathrm{trace}((s_1^t s_i)^2)] = E p_i \mathrm{trace}(E s_1 s_1^t E s_i s_i^t) = \frac{1}{N} E p_1. \tag{3.2}$$

By the strong law of large number, with (3.1) and (3.2),

$$\sum_{i=2}^K E(p_i (s_1^t s_i)^2 | \mathcal{F}_{i-1}) - \sum_{i=2}^K E p_i (s_1^t s_i)^2 \xrightarrow{\text{a.s.}} 0 \tag{3.3}$$

where "$\xrightarrow{\text{a.s.}}$" denotes almost sure convergence. Next, let

$$\begin{aligned} s_i^* &= (v_{i1}, v_{i2}, \cdots , v_{iN})^t, \qquad i = 1, \cdots , K, \\ S_k &= \sum_{i=2}^k (p_i ((s_1^*)^t s_i^*)^2 - E(p_i ((s_1^*)^t s_i^*)^2 | \mathcal{F}_{i-1})). \end{aligned} \tag{3.4}$$

It is obvious that $(S_k, \mathcal{F}_k)$ is martingale, $(S_k^2, \mathcal{F}_k)$ is submartingale (see [6]). For each $m > K$, $\varepsilon > 0$, by [7, Theorem 4.18], we have

$$\varepsilon P\Big(\max_{K \le n \le m} \frac{S_n^2}{n^4} \ge \varepsilon\Big) \le \frac{E S_K^2}{K^4} + \sum_{n=K+1}^m \frac{E(S_n^2 - S_{n-1}^2)}{n^4}. \tag{3.5}$$

Expanding out the first item on the right-hand side, we obtain

$$\begin{aligned}
ES_K^2 &= \sum_{i=2}^{K} E(p_i((s_1^*)^t s_i^*)^2 - E(p_i((s_1^*)^t s_i^*)^2|\mathcal{F}_{i-1}))^2 \\
&\le c\Big(\sum_{i=2}^{K}[E(p_i((s_1^*)^t s_i^*)^2)^2 + E(E(p_i((s_1^*)^t s_i^*)^2|\mathcal{F}_{i-1}))^2]\Big) \\
&\le c\sum_{i=2}^{K} E(p_i)^2 E((s_1^*)^t s_i^*)^4 \\
&\le c\sum_{i=2}^{K}\sum_{j_1}\sum_{j_2}\sum_{j_3}\sum_{j_4} E[v_{1j_1}v_{1j_2}v_{1j_3}v_{1j_4}]E[v_{ij_1}v_{ij_2}v_{ij_3}v_{ij_4}], \qquad (3.6)
\end{aligned}$$

where we make use of martingale difference properties, C_r inequality, and Jensen inequality. Since each of these expectations is zero except when $j_1 = j_2$ and $j_3 = j_4$, or $j_2 = j_4$ and $j_1 = j_3$, or $j_1 = j_4$ and $j_2 = j_3$, we have

$$E((s_1^*)^t s_i^*)^4 = O(N^2). \qquad (3.7)$$

Expanding out the second term of relation (3.5), we get

$$\begin{aligned}
E(S_n^2 - S_{n-1}^2) &= E[p_n((s_1^*)^t s_n^*)^2 - E(p_n((s_1^*)^t s_n^*)^2|\mathcal{F}_{n-1})]^2 \\
&\quad + E[S_{n-1}(p_n((s_1^*)^t s_n^*)^2 - E(p_n((s_1^*)^t s_n^*)^2|\mathcal{F}_{n-1}))] \\
&= E[p_n((s_1^*)^t s_n^*)^2 - E(p_n((s_1^*)^t s_n^*)^2|\mathcal{F}_{n-1})]^2 \\
&\le cE((s_1^*)^t s_n^*)^4. \qquad (3.8)
\end{aligned}$$

Based on (3.5)–(3.8), for each $m > K, \varepsilon > 0$, we have

$$\begin{aligned}
\varepsilon P\Big(\max_{K\le n\le m}\frac{S_n^2}{n^4} \ge \varepsilon\Big) &\le \frac{c\Big(\sum\limits_{i=2}^{K} E(p_i((s_1^*)^t s_i^*)^2)^2\Big)}{K^4} \\
&\quad + c\,\frac{\sum\limits_{n=K+1}^{m} E[p_n((s_1^*)^t s_n^*]^2 - E[(p_n((s_1^*)^t s_n^*)^2|\mathcal{F}_{n-1})]^2}{n^4} \\
&\le c\frac{N^2}{K^3} + c\sum_{n=K+1}^{m}\frac{N^2}{n^4}.
\end{aligned}$$

Thus

$$\varepsilon P\Big(\sup_{K\le n}\frac{S_n^2}{n^4} \ge \varepsilon\Big) \le c\frac{N^2}{K^3} + c\sum_{n=K+1}^{\infty}\frac{N^2}{n^4}.$$

Hence as $K \longrightarrow \infty$,

$$P\Big(\sup_{K\le n}\frac{S_n^2}{n^4} \ge \varepsilon\Big) \longrightarrow 0,$$

which implies that

$$\frac{S_K^2}{K^4} \longrightarrow 0, \quad \text{i.e.} \quad \frac{S_K}{N^2} \longrightarrow 0, \quad \text{a.s.} \tag{3.9}$$

Combining (3.3), (3.4) and (3.9), we can conclude that

$$\begin{aligned}
&\sum_{i=2}^{K}[p_i(s_1^t s_i)^2 - E(p_i(s_1^t s_i)^2)] \\
&= \sum_{i=2}^{K}[p_i(s_1^t s_i)^2 - E((p_i(s_1^t s_i)^2)|\mathcal{F}_{i-1})] \\
&\quad + \sum_{i=2}^{K} E(p_i(s_1^t s_i)^2|\mathcal{F}_{i-1}) - \sum_{i=2}^{K} Ep_i(s_1^t s_i)^2 \xrightarrow{\text{a.s.}} 0.
\end{aligned} \tag{3.10}$$

Based on (3.3), (3.10) one can obtain

$$\sum_{i=2}^{K} p_i(s_1^t s_i)^2 - \alpha E p_1 \xrightarrow{\text{a.s.}} 0.$$

Clearly $(s_1^t s_1)^2$ converges to 1 with probability 1, by the strong law of large numbers. Therefore, we complete the proof of the theorem.

To relax the assumption that the received powers are i.i.d., and that the received powers are uniformly bounded, which is needed in [4], we need the following lemma.

Lemma 3.1. *Assume that almost surely the empirical distribution of $(p_1, \cdots, p_K)$ converges weakly to a limiting F_p as K tends to infinity and $\sup_i E(p_i)^2 < \infty$. Then*

$$\lim_{K\to\infty} E\Big(\frac{1}{K}\sum_{i=1}^{K} p_i\Big) = \int_0^\infty x dF_p(x) < \infty. \tag{3.11}$$

Further, if the received powers are uncorrelated, we have

$$\frac{\sum_{i=1}^{K} p_i}{K} \xrightarrow{a.s.} \int_0^\infty x dF_p(x), \quad a.s. \tag{3.12}$$

Proof. Let T be continuity point of $F_p(x)$. Observe that

$$\lim_{K\to\infty} E\Big(\frac{1}{K}\sum_{i=1}^{K} p_i\Big) \le \sup_K E\Big(\frac{1}{K}\sum_{i=1}^{K} p_i\Big) \le \sup_i (Ep_i^2)^{\frac{1}{2}} < \infty, \tag{3.13}$$

$$\frac{1}{K}\sum_{i=1}^{k} p_i I(p_i \le T) = \int_{x\le T} x dF_k^*(x) \xrightarrow{\text{a.s.}} \int_{x\le T} x dF_p(x), \tag{3.14}$$

where

$$F_k^*(x) = \frac{1}{k}\sum_{i=1}^{k} I(p_i \le x),$$

and (3.14) follows from Helly-Bray lemma. Applying dominated convergence theorem with (3.14) yields

$$\lim_{T\to\infty}\lim_{K\to\infty} E\Big[\frac{1}{K}\sum_{i=1}^{k} p_i I(p_i \le T)\Big] = \lim_{T\to\infty}\int_{x\le T} x dF_p(x) = \int_0^\infty x dF_p(x). \tag{3.15}$$

Further, observe that

$$E\Big[\frac{1}{K}\sum_{i=1}^{k} p_i I(p_i \ge T)\Big] \le \frac{1}{K}\sum_{i=1}^{k}\frac{Ep_i^2}{T} I(p_i \ge T) \le \frac{1}{T}\sup_i Ep_i^2,$$

which implies that

$$\lim_{T\to\infty}\lim_{K\to\infty} E\Big[\frac{1}{K}\sum_{i=1}^{k} p_i I(p_i \ge T)\Big] = 0. \tag{3.16}$$

Combining (3.13), (3.15) and (3.16), we have

$$\lim_{K\to\infty} E\Big(\frac{1}{K}\sum_{i=1}^{K} p_i\Big) = \int_0^\infty x dF_p(x) < \infty.$$

Using the properties of the uncorrelated variables, we have

$$\frac{\sum\limits_{i=1}^{K}(p_i - Ep_i)}{K} \xrightarrow{\text{a.s.}} 0, \tag{3.17}$$

which follows from the strong law of large number. Thus, based on (3.11) and (3.17), the final result follows.

Now we can set up the second main result.

Theorem 3.2. *Assume that the following conditions are satisfied:*

(1) *The empirical distribution of the received powers of the users converges we- akly to a deterministic distribution, say $F_p(x)$, with probability one.*

(2) $\sup_i E(p_i)^2 < \infty$, *and* $Ev_{11}^4 < \infty$.

(3) ξ *is integrable and* $P(p_i \le \xi) = 1$ *for* $i = 1, \cdots, k$.

Then

$$\beta_1 \xrightarrow{a.s.} \beta_1^* = \frac{p_1}{\sigma^2 + \alpha\int_0^\infty x dF_p(x)}.$$

Proof. Observe that

$$\begin{aligned} E[p_i v_{ij}^2|\mathcal{F}_{i-1}] &= E[E(p_i v_{ij}^2|\mathcal{F}_{i-1}, p_i)|\mathcal{F}_{i-1}] \\ &= E[p_i E(v_{ij}^2|\mathcal{F}_{i-1}, p_i)|\mathcal{F}_{i-1}] = E(p_i|\mathcal{F}_{i-1}) \end{aligned}$$

and

$$\begin{aligned}
&E\Big[p_i \sum_{j_1\neq j_2} v_{1j_1}v_{1j_2}v_{ij_1}v_{ij_2}|\mathcal{F}_{i-1}\Big]\\
&= E\Big[E\Big(p_i \sum_{j_1\neq j_2} v_{1j_1}v_{1j_2}v_{ij_1}v_{ij_2}|\mathcal{F}_{i-1},p_i\Big)\Big|\mathcal{F}_{i-1}\Big]\\
&= E\Big[p_i \sum_{j_1\neq j_2} v_{1j_1}v_{1j_2}E(v_{ij_1}v_{ij_2})|\mathcal{F}_{i-1}\Big]=0,
\end{aligned}$$

where $i \neq 1$, $\mathcal{F}_i$ is defined as before.

It follows that

$$\begin{aligned}
\sum_{i=2}^{K}[E(p_i(s_1^t s_i)^2|\mathcal{F}_{i-1})] &= \sum_{i=2}^{K}\Big[\frac{1}{N^2}E\Big(p_i\sum_{j=1}^{N}v_{1j}^2v_{ij}^2|\mathcal{F}_{i-1}\Big)\\
&\quad+\frac{1}{N^2}E\Big(p_i\sum_{j_1\neq j_2}v_{1j_1}v_{1j_2}v_{ij_1}v_{ij_2}|\mathcal{F}_{i-1}\Big)\Big]\\
&=\frac{\sum\limits_{j=1}^{N}v_{1j}^2\sum\limits_{i=2}^{K}E(p_i|\mathcal{F}_{i-1})}{N^2}. \qquad (3.18)
\end{aligned}$$

Now define

$$S_k=\sum_{i=2}^{k}(p_i-E(p_i|\mathcal{F}_{i-1})).$$

Obviously $[S_k,\mathcal{F}_k]$ is martingale, then $[S_k^2,\mathcal{F}_k]$ is submartingale. Thus for any $m>K, \varepsilon>0$,

$$\begin{aligned}
\varepsilon P\Big(\max_{K\le n\le m}\frac{S_n^2}{n^2}\ge\varepsilon\Big) &\le E\Big(\frac{S_K^2}{K^2}\Big)+\sum_{n=K+1}^{m}E\Big(\frac{S_n^2-S_{n-1}^2}{n^2}\Big) \qquad (3.19)\\
&\le\frac{\sum\limits_{i=2}^{K}E[p_i-E(p_i|\mathcal{F}_{i-1})]^2}{K^2}+\sum_{n=K+1}^{m}\frac{E[p_n-E(p_n|\mathcal{F}_{n-1})]^2}{n^2} \qquad (3.20)\\
&\le c\left[\frac{\sum\limits_{i=2}^{K}Ep_i^2}{K^2}+\sum_{n=K+1}^{m}\frac{Ep_n^2}{n^2}\right] \qquad (3.21)\\
&\le c\Big[\frac{1}{K}+\sum_{n=K+1}^{m}\frac{1}{n^2}\Big], \qquad (3.22)
\end{aligned}$$

where the inequalities (3.19)–(3.22) follow from Hajek-Renyi-Chow inequality, martingale difference properties, Jensen inequality and the condition (2), respectively.

Hence, letting $m\longrightarrow\infty$ and then $K\longrightarrow\infty$, we have

$$p\Big(\sup_{n\ge K}\frac{S_n^2}{n^2}\ge\varepsilon\Big)\longrightarrow 0.$$

Thus

$$\frac{S_K}{K} \xrightarrow{\text{a.s.}} 0. \tag{3.23}$$

Next we will show

$$\frac{\sum_{i=1}^{K} p_i}{K} \xrightarrow{\text{a.s.}} \int_0^\infty x dF_p(x). \tag{3.24}$$

Let

$$F_k^*(x) = \frac{1}{k}\sum_{i=1}^{k} I(p_i \le x).$$

It is obvious that $\xi I(\xi > T) \xrightarrow{\text{a.s.}} 0$ $(T \longrightarrow \infty)$ by $E\xi < \infty$. Thus

$$\frac{1}{k}\sum_{i=1}^{k} P_i I(P_i > T) = \int_{x \ge T} x\, dF_k^*(x) \le \xi I(\xi > T) < \frac{\varepsilon}{3}, \quad \text{a.s.} \tag{3.25}$$

for $T > T_\varepsilon$ whatever be k.

By Helly-Bray lemma, if $T' > T$ and $T',\ T$ are continuity points of $F_p(x)$, then

$$\int_T^{T'} x\, dF_p(x) < \frac{\varepsilon}{3}, \quad \text{a.s.},$$

and letting $T' \longrightarrow \infty$, by monotone convergence theorem, we have

$$\int_{x>T} x\, dF_p(x) < \frac{\varepsilon}{3}, \quad \text{a.s.} \tag{3.26}$$

Observe that for any fixed $T > 0,\ \varepsilon > 0$, if $k \ge k_1$, we have

$$\Big| \int_0^T x dF_k^*(x) - \int_0^T x dF_p(x) \Big| < \frac{\varepsilon}{3}, \quad \text{a.s.} \tag{3.27}$$

which follows from Helly-Bray lemma. Based on (3.25)–(3.27),

$$\begin{aligned} &\Big| \int_0^\infty x\, dF_k^*(x) - \int_0^\infty x\, dF_p(x) \Big| \\ \le &\Big| \int_0^T x\, dF_k^*(x) - \int_0^T x\, dF_p(x) \Big| + \int_T^\infty x\, dF_k^*(x) + \int_T^\infty x\, dF_p(x) \xrightarrow{\text{a.s.}} 0, \end{aligned}$$

which follows on letting $T \ge T_\varepsilon, k \longrightarrow \infty$ and then $\varepsilon \longrightarrow 0$.

Combining (3.18), (3.23) and (3.24), we conclude that

$$\sum_{i=2}^{K} [E(p_i(s_1^t s_i)^2 | \mathcal{F}_{i-1})] - \alpha \int_0^\infty x dF_p(x) \xrightarrow{\text{a.s.}} 0.$$

Using the condition (2), one can easily prove the following in a similar method (the same as Theorem 3.1)

$$\sum_{i=2}^{K} [p_i(s_1^t s_i)^2 - E((p_i(s_1^t s_i)^2)|\mathcal{F}_{i-1})] \xrightarrow{\text{a.s.}} 0,$$

which completes the proof of the theorem.

Remark 3.1. Compared to [1, 3] and [4], we need the finite fourth moment of spreading sequences only and the convergence of the SIR is almost sure.

Corollary 3.1. *If the received powers are uncorrelated, the result of Theorem* 3.2 *holds without the condition* (3)*.*

Proof. By Lemma 3.1, (3.24) holds. Thus the corollary follows.

Remark 3.2. If the received powers are uncorrelated, Theorem 3.2 holds without the condition (3), i.e., we can drop the condition that the received powers are uniformly bounded, which is needed in [4].

However, for general case, in order to obtain (3.24) we believe that some kind of condition is necessary. Hence we also obtain the following result under the weaker condition.

Proposition 3.1. *Assume that the following conditions are satisfied:*

(1) *Almost surely the empirical distribution of* $(p_1, \cdots, p_K)$ *converges weakly to a limiting* F_p *as* K *tends to infinity.*

(2) $\sup_i E(p_i)^2 < \infty$.

(3) *For all sufficiently large* x*, almost surely,* $1 - F_K^*(x) \le A(1 - F_p(x))$*, where* A *is a positive constant and independent of* K*.*

Then we have

$$\frac{\sum_{i=1}^{K} p_i}{K} \xrightarrow{a.s.} \int_0^\infty x dF_p(x).$$

Proof. In the following, we use some well-known results in probability theory:

$$Ep_1 < \infty \Longrightarrow \lim_{x\to\infty} xP(p_1 > x) = 0,$$

$$Ep_1 = \int_0^\infty (1 - F_{p_1}(x))dx.$$

Let T be continuity point of $F_p(x)$, clearly,

$$\frac{\sum_{i=1}^{K} p_i}{K} = \frac{\sum_{i=1}^{K} p_i I(p_i \le T)}{K} + \frac{\sum_{i=1}^{K} p_i I(p_i \ge T)}{K}.$$

By Helly-Bray lemma and monotone convergence theorem, one can prove that

$$\lim_{T\to\infty} \lim_{K\to\infty} \frac{\sum_{i=1}^{K} p_i I(p_i \le T)}{K} = \lim_{T\to\infty} \lim_{K\to\infty} \int_{x\le T} x dF_K^*(x) \overset{\text{a.s.}}{=} \int_0^\infty x dF_p(x).$$

To obtain the result, it suffices to show that

$$\lim_{T\to\infty} \lim_{K\to\infty} \frac{\sum_{i=1}^{K} p_i I(p_i \ge T)}{K} \overset{\text{a.s.}}{=} 0. \tag{3.28}$$

Observe that

$$\frac{\sum\limits_{i=1}^{K} p_i I(p_i \geq T)}{K} = \int_{x>T} x dF_K^*(x) = -x(1-F_K^*(x))|_T^\infty + \int_{x>T} (1-F_K^*(x))dx. \tag{3.29}$$

With (3.11) and the condition (3), by dominated convergence theorem,

$$\begin{aligned} &\lim_{T\to\infty}\lim_{K\to\infty}\int_{x>T}(1-F_K^*(x))dx \\ &\overset{\text{a.s.}}{=} \lim_{T\to\infty}\int_{x>T}(1-F_p(x))dx = \lim_{T\to\infty} E\xi I(\xi > T) = 0, \end{aligned} \tag{3.30}$$

where ξ is a random variable having distribution $F_p(x)$. Further, note that

$$\lim_{T\to\infty}\lim_{K\to\infty} x(1-F_K^*(x))|_T^\infty = \lim_{K\to\infty}\lim_{x\to\infty} x(1-F_K^*(x)) - \lim_{T\to\infty}\lim_{K\to\infty} T(1-F_K^*(T)). \tag{3.31}$$

Calculating the second item of the above equality, one can obtain

$$\lim_{T\to\infty}\lim_{K\to\infty} T(1-F_K^*(T)) \overset{\text{a.s.}}{=} \lim_{T\to\infty} T(1-F_p(T)=0, \tag{3.32}$$

where the last equality follows from (3.11).

On the other hand, by $Ep_1^2 < \infty$, as $x \longrightarrow \infty$, we have $p_1 I(p_1 > x) \xrightarrow{\text{a.s.}} 0$. Thus, by induction on K, as $x \longrightarrow \infty$, one can prove that

$$\frac{\sum\limits_{i=1}^{K} p_i I(p_i > x)}{K} \xrightarrow{\text{a.s.}} 0.$$

Hence we have

$$\begin{aligned} \lim_{K\to\infty}\lim_{x\to\infty} x(1-F_K^*(x)) &= \lim_{K\to\infty}\lim_{x\to\infty} x\frac{1}{K}\sum_{i=1}^{K} I(p_i > x) \\ &\leq \lim_{K\to\infty}\lim_{x\to\infty} \frac{1}{K}\sum_{i=1}^{K} p_i I(p_i > x) \overset{\text{a.s.}}{=} 0. \end{aligned} \tag{3.33}$$

Combining (3.29)–(3.33), the equation (3.28) holds. Thus we complete the proof.

Remark 3.3. If the received powers are uniformly bounded, or dominated by some integrable random variable, the condition (3) holds.

As we know, the attained SIR in a finite system, will fluctuate around the limit. In order to characterize the performance fluctuations around the asymptotic limits, we obtain the following result.

Theorem 3.3. *If the conditions of Theorem* 3.1 *are satisfied, then we have*

$$\sqrt{N}\frac{(s_1^t s_1)^2 - E(s_1^t s_1)^2}{(s_1^t s_1)\sigma^2 + \sum\limits_{i=2}^{K} p_i (s_1^t s_i)^2} \xrightarrow{D} N(0,b), \tag{3.34}$$

where

$$b = \frac{4(Ev_{11}^4 - 1)}{\sigma^2 + \alpha E p_1}. \tag{3.35}$$

Proof. We have

$$\begin{aligned}
&\sqrt{N}((s_1^t s_1)^2 - E(s_1^t s_1)^2) \\
&= \sqrt{N}\Big(\Big(\frac{1}{N}\sum_{i=1}^N v_{1i}^2\Big)^2 - \frac{Ev_{11}^4}{N} - 1 + \frac{1}{N}\Big) \\
&= \sqrt{N}\frac{2}{N^2}\sum_{i_1<i_2}(v_{1i_1}^2 v_{1i_2}^2 - 1) + \sqrt{N}\Big(\frac{1}{N^2}\sum_{i=1}^N v_{1i}^4 - \frac{Ev_{11}^4}{N}\Big) \\
&:= S_{N_1} + S_{N_2}.
\end{aligned} \tag{3.36}$$

By appealing to the strong law of large numbers, it can be shown that

$$S_{N_2} \xrightarrow{\text{a.s.}} 0. \tag{3.37}$$

Next observe that

$$S_{N_3} := \frac{1}{\binom{N}{2}} \sum_{i_1<i_2} v_{1i_1}^2 v_{1i_2}^2 \tag{3.38}$$

is U-statistic. Thus, by Theorem 6.2.1 in [8], we have

$$S_{N_1} \xrightarrow{D} N(0, 4(Ev_{11}^4 - 1)). \tag{3.39}$$

Therefore, based on Slutsky's theorem, (3.36), (3.37), (3.39) and the result of Theorem 3.1, Theorem 3.3 holds.

Remark 3.4. It is pity that we can not prove the following

$$\sqrt{N}(\beta_1 - \beta_1^*) \xrightarrow{D} (0, b)$$

where b, β_1^* is defined as before.

However we can obtain the following result.

Theorem 3.4. *If the conditions of Theorem* 3.1 *are satisfied, then*

$$N^{\frac{1}{2}-\tau} p_1 \left(\frac{E(s_1^t s_1)^2}{(s_1^t s_1)\sigma^2 + \sum\limits_{i=2}^{K} p_i (s_1^t s_i)^2} - \frac{1}{\sigma^2 + \frac{K}{N} E p_1} \right) \xrightarrow{a.s.} 0,$$

where $\tau > 0$.

Proof. In fact, it suffices to show that

$$N^{\frac{1}{2}-\tau}\Big((s_1^t s_1 - 1)\sigma^2 + \sum_{i=2}^K p_i (s_1^t s_i)^2 - \frac{K}{N} Ep\Big) \xrightarrow{\text{a.s.}} 0. \tag{3.40}$$

Clearly

$$s_1^t s_1 - 1 = O\Big(\sqrt{\frac{\log\log N}{N}}\Big), \tag{3.41}$$

which follows from Hartman-Wintner law of the iterated logarithm. Similarly, examining the proof of Theorem 3.1, one can obtain

$$\sum_{i=2}^{K} E(p_i(s_1^t s_i)^2|\mathcal{F}_{i-1}) - \sum_{i=2}^{K} Ep_i(s_1^t s_i)^2 = O\Big(\sqrt{\frac{\log\log N}{N}}\Big). \tag{3.42}$$

Looking over the proof of Theorem 3.1 again, we have

$$\frac{S_K}{N^2} = o(N^{-(\frac{1}{2}-\tau)}), \tag{3.43}$$

where $\tau > 0$. Thus, combining (3.41)–(3.43), (3.40) holds.

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